Comparison between Adomian decomposition method and optimal homotopy asymptotic method for a two moving boundaries problem

6.1 Introduction

The mathematical model of a solute release from a polymer matrix is an interesting moving boundary problem (Stefan problem) that involves diffusion equation. The classical diffusion process is governed by Fick's law. If solute movement occurs in heterogeneity media (anomalous) then the process cannot be described by classical diffusion equation (Fickian) and also violates Gaussian theorem (Metzler and Klafter (2004), Xu and Tan (2006)). Therefore, the fractional diffusion equations have been widely used by the researchers in the field of one moving-boundary problems for the mathematical models of controlled drug release from polymeric matrix in the last two decades. One moving boundary problem involves only a diffusing boundary when matrix is not dissolved. Liu and Xu (2004) were the first who presented a mathematical model to the problem of one moving boundary governed with time-fractional derivative in drug release process. One moving boundary problems with fractional (space or time or space-time) diffusion equations are studied by Li et al. (2007, 2009), Yin and Li (2011), Das et al. (2011), Rajeev and Kushwaha (2013), etc. However, fractional calculus has scarcely been applied to two moving boundaries problems that involve a dissolving boundary and a diffusing boundary due to the dissolved matrix. In 2009, Yin and Xu (2009) presented an asymptotic analytical solution in term of Wright function for a problem of two moving boundaries governed with time fractional derivative operator in Caputo sense.

Moving boundary problems are nonlinear in nature and involve moving boundary/boundaries. So, exact solutions of these problems are restricted for some particular cases (Crank (1987)). Some exact solutions to the Stefan problems with fractional anomalous diffusion are discussed by Liu and Xu (2004), Junyi and Xu (2009) and Voller (2010). Some approximate methods have also been used by many researchers (Li et al. (2009), Abdekhodaie and Cheng (1997), Lin and Peng (2005)) to solve such problems. In this literature, Adomian decomposition method and optimal homotopy asymptotic method are used to find the approximate solutions. Adomian decomposition method was developed by Adomian (1988,1994, 1998) and it has been used to solve various types of differential equations (Wazwaz (2000, 2007)). Adomian decomposition method to one moving boundary problems is also discussed by Grzymkowski and Slota (2005), Das and Rajeev (2010), Hetmaniok et al. (2011) and Rajeev and Kushwaha (2013). The Adomian decomposition method provides an approximate solution for all types of differential and integral equations in the form of a rapidly convergent series whose terms are recursively determined by Adomian polynomials (Adomian (1998),Wazwaz (2000)). This method is capable of reducing the size of calculation without compromising the accuracy of the numerical solution. In particular, Adomian decomposition method provides explicit solution of moving boundary problem in visible symbolic terms without linearization or discretization (Grzymkowski and Slota (2005), Das and Rajeev (2010)).

Optimal homotopy asymptotic method was developed by Marinca and Herisanu (2008) and it has been applied to solve a wide class of non-linear differential equations (Herisanu and Marinca (2010a, 2010b), Iqbal et al. (2010), Iqbal and Javed (2011), Hashmi et al. (2012)). Ghoreishi et al. (2012) presented the comparison between homotopy analysis method and optimal homotopy asymptotic method for nonlinear age-structured population Models. In 2013, Dinarvand and Hosseini (2013) also used this technique to investigate the temperature distribution equation in a convective straight fin with temperature-dependent thermal conductivity and the convective–radiative cooling of a lumped system with variable specific heat.

In present study, Adomian decomposition method and optimal homotopy asymptotic method are used to find approximate solutions for a two moving boundaries problem (Yin and Xu (2009)) governed by fractional time derivative in Caputo sense. This problem arises during the controlled drugs release from a polymeric matrix. The aim for investigating two moving boundaries problem with fractional time derivative is to explain phenomena of anomalous (non-Fickian) solute movement through complex and/or disordered systems which occurs in the diffusion process. The obtained results by Adomian decomposition method and optimal homotopy asymptotic method are compared with the existing analytical solution and asymptotic analytical solution (Yin and Xu (2009)). It is found that solution by Adomian decomposition method and optimal homotopy asymptotic method are slightly more accurate than the asymptotic analytical solution obtained by Yin and Xu (2009).

6.2 Mathematical model

In this section, we consider the mathematical model of a two moving boundaries problem given by Yin and Xu (2009). The one-dimensional polymeric matrix dissolves slowly under perfect sink condition, diffusivity of the drug in the matrix is constant and the initial concentration C_0 of the drug is much greater than the solubility C_s of the drug are the some assumptions which are used in the formulation of the model. The governing equations and the posed conditions of the problem are as follows:

$$D_\tau^\alpha C(\xi,\tau) = \mu \frac{\partial^2}{\partial \xi^2} C(\xi,\tau), \qquad (r(\tau) < \xi < s(\tau), 0 < \alpha \le 1), \qquad (6.2.1)$$

$$C(\xi,\tau)=0, \qquad (\xi=r(\tau)), \tag{6.2.2}$$

$$C(\xi,\tau)=C_s \qquad (\xi=s(\tau)) , \tag{6.2.3}$$

$$(C_0-C_s)D_\tau^\alpha s(\tau)=\mu\left.\frac{\partial C(\xi,\tau)}{\partial \xi}\right|_{\xi=s(\tau)}, \qquad (\tau>0), \tag{6.2.4}$$

$$r(\tau)=s(\tau)=0, \qquad (\tau=0), \tag{6.2.5}$$

where $C(\xi,\tau)$ is the concentration of drug in the matrix, μ is the diffusivity of the drug in the matrix, $s(\tau)$ is the position of the diffusing boundary and $r(\tau)$ is the position of dissolving boundary at time τ.

Introducing the following dimensionless variables:

$$\begin{aligned} &U(x,t)=\frac{C(\xi,\tau)}{C_s}, \qquad t=\left(\frac{\mu}{L^2}\right)^{\frac{1}{\alpha}}\tau, \qquad \varepsilon=\frac{C_s}{C_0}, \\ &R(t)=\frac{r(\tau)}{L}, \qquad S(t)=\frac{s(\tau)}{L}, \qquad x=\frac{\xi}{L}. \end{aligned} \tag{6.2.6}$$

The equations (6.2.1-6.2.5) become

$$D_t^\alpha U(x,t)=\frac{\partial^2 U(x,t)}{\partial x^2}, \qquad (R(t)<x<S(t),\ 0<\alpha\le 1), \tag{6.2.7}$$

$$U(x,t)=0, \qquad (x=R(t)), \tag{6.2.8}$$

$$U(x,t)=1, \qquad (x=S(t)), \tag{6.2.9}$$

$$(\varepsilon^{-1}-1)D_t^\alpha S(t)=\left(\frac{\partial U}{\partial y}\right)_{y=X(t)}, \qquad (t>0), \tag{6.2.10}$$

$$R(t)=S(t)=0, \qquad (t=0). \tag{6.2.11}$$

As given by Yin and Xu (2009), considering the new dimensionless independent space-time variables

$$y=x\text{-}R(t), \qquad X(t)=S(t)\text{-}R(t), \tag{6.2.12}$$

and taking

$$R(t) = \eta t, \tag{6.2.13}$$

where η represents the dimensionless moving velocity of the dissolving boundary.

Introducing the new dimensionless variables in the Eqs. (6.2.7-6.2.11), the equations become one moving boundary problem (Yin and Xu (2009)) as follows:

$$D_t^{\alpha} U(y,t) - \eta \frac{\partial}{\partial y}\left(D_t^{\alpha-1} U(y,t)\right) = \frac{\partial^2 U(y,t)}{\partial y^2}, \qquad (0 < y < X(t)), \tag{6.2.14}$$

$$U(y,t) = 0, \qquad (y = 0), \tag{6.2.15}$$

$$U(y,t) = 1, \qquad (y = X(t)), \tag{6.2.16}$$

$$(\varepsilon^{-1} - 1)\left[D_t^{\alpha} X(t) + \eta \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right] = \left(\frac{\partial U}{\partial y}\right)_{y=X(t)}, \qquad (t > 0), \tag{6.2.17}$$

$$X(t) = 0, \qquad (t = 0). \tag{6.2.18}$$

6.3 Solution of the problem by Adomian decomposition method

We first write the Eq. (6.2.14) in operator form (Das and Rajeev (2010)) as given below:

$$L_{yy} U(y,t) = D_t^{\alpha} U(y,t) - \eta \frac{\partial}{\partial y} D_t^{\alpha-1} U(y,t), \tag{6.3.19}$$

where

$$L_{yy} = \frac{\partial^2}{\partial y^2}.$$

Assuming that the inverse operator L_{yy}^{-1} exists and

$$L_{yy}^{-1}(.) = \int_0^y \int_0^y (.)dy\,dy\,.$$

Applying the inverse operator L_{yy}^{-1} on the both side of Eq. (6.3.19), we obtain

$$U(y,t) - U(0,t) - yU_y(0,t) = L_{yy}^{-1}\left(D_t^{\alpha} U(y,t) - \eta \frac{\partial}{\partial y} D_t^{\alpha-1} U(y,t) \right). \tag{6.3.20}$$

Choosing the following initial approximations of $U(y,t)$ and $X(t)$ as given by (Das and Rajeev (2010)):

$$U_0 = \frac{y}{X_0},$$

$$X_0 = a_0\, t^{\alpha/2},$$

where $a_0 = \left(\left(\frac{\varepsilon}{1-\varepsilon} \right) \frac{\Gamma(1-\alpha/2)}{\Gamma(1+\alpha/2)} \right)^{1/2}.$

According to the Adomian decomposition method (Wazwaz (2007), Grzymkowski and Slota (2005), Das and Rajeev (2010)), decomposing the unknown function $U(y,t)$ as follows:

$$U(y,t) = U_0 + U_1 + U_2 + \cdots. \tag{6.3.21}$$

where the components $U_0, U_1, U_2, \cdots$ can be defined as:

$$U_0 = yU_y(0,t) = \frac{y}{X_0},$$

$$U_1 = L_{yy}^{-1}\left(D_t^{\alpha} U_0(y,t) - \eta \frac{\partial}{\partial y} D_t^{\alpha-1} U_0(y,t) \right)$$
$$= \frac{\Gamma(1-\alpha/2)}{\Gamma(1-3\alpha/2)} \frac{t^{-3\alpha/2}}{a_0} \frac{y^3}{3!} - \eta \frac{\Gamma(1-\alpha/2)}{\Gamma(2-3\alpha/2)} \frac{t^{1-3\alpha/2}}{a_0} \frac{y^2}{2!},$$

$$U_2 = L_{yy}^{-1}\left(D_t^{\alpha} U_1(y,t) - \eta \frac{\partial}{\partial y} D_t^{\alpha-1} U_1(y,t) \right)$$
$$= \frac{\Gamma(1-\alpha/2)}{\Gamma(1-5\alpha/2)} \frac{t^{-5\alpha/2}}{a_0} \frac{y^5}{5!} - 2\eta \frac{\Gamma(1-\alpha/2)}{\Gamma(2-5\alpha/2)} \frac{t^{1-5\alpha/2}}{a_0} \frac{y^4}{4!} + \eta^2 \frac{\Gamma(1-\alpha/2)}{\Gamma(3-5\alpha/2)} \frac{t^{2-5\alpha/2}}{a_0} \frac{y^3}{3!},$$

$$\vdots$$

and so on.

Therefore,

$$\begin{aligned} U(y,t) &= U_0 + U_1 + U_2 + .\cdots \\ &= \frac{y}{X_0} + \frac{\Gamma(1-\alpha/2)}{\Gamma(1-3\alpha/2)} \frac{t^{-3\alpha/2}}{a_0} \frac{y^3}{3!} - \eta \frac{\Gamma(1-\alpha/2)}{\Gamma(2-3\alpha/2)} \frac{t^{1-3\alpha/2}}{a_0} \frac{y^2}{2!} \\ &+ \frac{\Gamma(1-\alpha/2)}{\Gamma(1-5\alpha/2)} \frac{t^{-5\alpha/2}}{a_0} \frac{y^5}{5!} - 2\eta \frac{\Gamma(1-\alpha/2)}{\Gamma(2-5\alpha/2)} \frac{t^{1-5\alpha/2}}{a_0} \frac{y^4}{4!} \\ &+ \eta^2 \frac{\Gamma(1-\alpha/2)}{\Gamma(3-5\alpha/2)} \frac{t^{2-5\alpha/2}}{a_0} \frac{y^3}{3!} + \cdots, \end{aligned} \tag{6.3.22}$$

which give concentration of drug in the matrix.

Now, using (6.3.22) and writing the interface condition (6.2.17) in the following operator form:

$$X(t) = \varphi - D_t^{-\alpha}\left(F(X)\right), \tag{6.3.23}$$

where $\varphi = X_0 = a_0 t^{\alpha/2}$ and

$$\begin{aligned} F(X) &= \left(\frac{\varepsilon}{1-\varepsilon}\right)\left(\frac{\partial}{\partial y}\left(U(X(t),t)\right)\right) - \eta \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \\ &= \left(\frac{\varepsilon}{1-\varepsilon}\right)\left(\frac{1}{X(t)} + \frac{\Gamma(1-\alpha/2)}{\Gamma(1-3\alpha/2)} \frac{(X(t))^2}{2!} \frac{t^{-3\alpha/2}}{a_0} - \eta \frac{\Gamma(1-\alpha/2)}{\Gamma(2-3\alpha/2)} \frac{t^{1-3\alpha/2}}{a_0} X(t) \right. \\ &\quad + \frac{\Gamma(1-\alpha/2)}{\Gamma(1-5\alpha/2)} \frac{(X(t))^4}{4!} \frac{t^{-5\alpha/2}}{a_0} - 2\eta \frac{\Gamma(1-\alpha/2)}{\Gamma(2-5\alpha/2)} \frac{(X(t))^3}{3!} \frac{t^{1-5\alpha/2}}{a_0} \\ &\quad \left. + \eta^2 \frac{\Gamma(1-\alpha/2)}{\Gamma(3-5\alpha/2)} \frac{(X(t))^2}{2!} \frac{t^{2-5\alpha/2}}{a_0} + \cdots \right) - \eta \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \quad . \end{aligned}$$

According to Adomian (1998) and Rajeev and Kushwaha (2013), decomposing $X(t)$ as:

$$X(t)=\sum_{n=0}^{\infty}X_n \,. \tag{6.3.24}$$

Eqs. (6.3.23) and (6.3.24) gives

$$\sum_{n=0}^{\infty}X_n = X_0 - D_t^{-\alpha}\left(\sum_{n=0}^{\infty}A_n\right), \tag{6.3.25}$$

where A_n are so-called Adomian polynomials (Wazwaz (2007), Rajeev and Kushwaha (2013)) for non-linear terms and defined as:

$$A_0 = F(X_0)\,,$$

$$A_1 = X_1\,F'(X_0)\,,$$

$$A_2 = X_2\,F'(X_0)+\frac{1}{2}X_1^2\,F''(X_0)\,,$$

$$\vdots$$

and so on.

The components of $X_n(t),\ n\geq 1,$ can be determined as:

$$X_1 = D_t^{-\alpha}\left(A_0\right), \tag{6.3.26}$$

where

$$A_0 = \left(\frac{\varepsilon}{1-\varepsilon}\right)\left(\frac{t^{-\alpha/2}}{a_0}+\frac{\Gamma(1-\alpha/2)}{\Gamma(1-3\alpha/2)}\frac{a_0}{2!}t^{-\alpha/2}-\eta\frac{\Gamma(1-\alpha/2)}{\Gamma(2-3\alpha/2)}t^{1-\alpha}+\frac{\Gamma(1-\alpha/2)}{\Gamma(1-5\alpha/2)}\frac{a_0^{\,3}}{4!}t^{-\alpha/2}\right.$$
$$\left.-2\eta\frac{\Gamma(1-\alpha/2)}{\Gamma(2-5\alpha/2)}\frac{a_0^{\,2}}{3!}t^{1-\alpha}+\eta^2\frac{\Gamma(1-\alpha/2)}{\Gamma(3-5\alpha/2)}\frac{a_0}{2!}t^{2-3\alpha/2}\cdots\right)-\eta\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\quad.$$

$$X_2 = D_t^{-\alpha}\left(A_1\right), \tag{6.3.27}$$

where

$$A_1 = X_1\left(\frac{d}{dX_0}\left(\frac{\varepsilon}{1-\varepsilon}\right)\left(\frac{1}{X_0(t)}+\frac{\Gamma(1-\alpha/2)}{\Gamma(1-3\alpha/2)}\frac{(X_0(t))^2}{2!}\frac{t^{-3\alpha/2}}{a_0}-\eta\frac{\Gamma(1-\alpha/2)}{\Gamma(2-3\alpha/2)}\frac{t^{1-3\alpha/2}}{a_0}X_0(t)\right.\right.$$
$$+\frac{\Gamma(1-\alpha/2)}{\Gamma(1-5\alpha/2)}\frac{(X(t))^4}{4!}\frac{t^{-5\alpha/2}}{a_0}-2\eta\frac{\Gamma(1-\alpha/2)}{\Gamma(2-5\alpha/2)}\frac{(X(t))^3}{3!}\frac{t^{1-5\alpha/2}}{a_0}$$
$$\left.\left.+\eta^2\frac{\Gamma(1-\alpha/2)}{\Gamma(3-5\alpha/2)}\frac{(X_0(t))^2}{2!}\frac{t^{2-5\alpha/2}}{a_0}+\cdots\right)-\eta\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right),$$

⋮

and so on.

Therefore, approximate analytical solution of $X(t)$ is given by

$$X(t) = X_0 + X_1 + X_2 + \cdots + X_m\,. \qquad (6.3.28)$$

for any integer m. This solution is the extension of previous work (Das and Rajeev (2010)).

6.4 Solution by optimal homotopy asymptotic method

First we write Eqs. (6.2.14 - 6.2.16) in operator form as follows:

$$L(U(y,t)) - N(U(y,t)) = 0\,, \qquad (6.4.29)$$

$$B\left(U, \frac{\partial U}{\partial x}\right) = 0\,, \qquad (6.4.30)$$

where $L\left(=\frac{\partial^2}{\partial y^2}\right)$ is a linear operator, $N\left(=\frac{\partial^\alpha}{\partial t^\alpha}-\eta\frac{\partial}{\partial y}\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}\right)$ is nonlinear operator.

According to optimal homotopy decomposition method (Marinca and Herisanu (2008) and Hashmi et al. (2012)), we construct an optimal

$$U(y,t,p):[0,X(t)]\times[0,1]\rightarrow\Re$$

which satisfies

$$(1-p)[L(U(y,t,p))] = H(p)[L(U(y,t,p)) - N(U(y,t,p))], \quad (6.4.31)$$

$$U(0,t) = 0, \quad (6.4.32)$$

$$U(X(t),t) = 1, \quad (6.4.33)$$

where $p \in [0,1]$ is an embedding parameter, $U(y,t;p)$ is an unknown function, $H(p)$ is a nonzero auxiliary function for $p \neq 0$ and $H(0) = 0$.

Obviously, if $p = 0$,

$$U(y,t;0) = U_0(y,t), \quad (6.4.34)$$

and when $p = 1$ then

$$U(y,t;1) = U(y,t). \quad (6.4.35)$$

Therefore, as p increase from 0 to1, the unknown function $U(y,t,p)$ varies from $U_0(y,t)$ to the solution $U(y,t)$.

Now, the following auxiliary function $H(p)$ is considered:

$$H(p) = m_1 p + m_2 p^2 + m_3 p^3 + \cdots, \quad (6.4.36)$$

where $m_1, m_2, m_3, \cdots$ are constants to be determined later.

Considering the solution of (6.2.14) in the following series form:

$$U(y,t,p,m_i) = \sum_{n=0}^{\infty} U_n(y,t,m_i)p^n, \quad i = 0,1,2,\cdots l, \quad (6.4.37)$$

and taking

$$X(t) = \sum_{n=0}^{\infty} p^n X_n(t), \tag{6.4.38}$$

where $U_0(y,t,0) = U_0(y,t)$.

Substituting (6.4.37) into (6.4.31) and equating the coefficients of like powers of p, the following problems are obtained:

$$p^0: \quad L(U_0(y,t)) = 0\ , \tag{6.4.39}$$

$$p^1: \quad L(U_1(y,t)) = -m_1 N_0(U_0(y,t)), \tag{6.4.40}$$

$$p^2: \quad L(U_2(y,t)) - L(U_1(y,t)) = m_1 L(U_1(y,t)) - m_1 N_1(U_0(y,t), U_1(y,t)), \tag{6.4.41}$$

and so on.

The general equation for $U_k(y,t)$ is given as:

$$\begin{aligned} L(U_k(y,t)) &= L(U_{k-1}(y,t)) - m_k N_0(U_0(y,t)) \\ &+ \sum_{i=1}^{k-1} m_i \left[L(U_{k-i}(y,t)) - N_{k-i}(U_0(y,t), U_1(y,t), \cdots U_{k-1}(y,t)) \right], \end{aligned} \tag{6.4.42}$$

where $k = 2,3,\cdots$ and $N_s(U_0(y,t), U_1(y,t), \cdots U_s(y,t))$ is the coefficient of p^s that can be obtained from the following series:

$$N_s(U(y,t;p,m_j)) = N_0(U_0) + \sum_{s\geq 1} N_s(U_0, U_1, U_2, \cdots, U_s) p^s, \quad j = 1, 2, \cdots. \tag{6.4.43}$$

Substituting (6.4.37) and (6.4.38) in the boundary conditions (6.2.15) and (6.2.16), respectively which give

$$\sum_{n=0}^{\infty} U_n(y=0, t, m_i) p^n = 0, \tag{6.4.44}$$

and

$$\sum_{n=0}^{\infty} U_n\left(\sum_{n=0}^{\infty} p^n X_n, t, m_i\right) p^n = 1, \tag{6.4.45}$$

where $i = 0, 1, 2, \cdots l$.

In order to compare the coefficients of various powers of p in interface condition, expending $U_i(y,t,m_i)$ in Taylor's series form (as given in Li et al. (2009)) about a point (X_0, t) as:

$$U_l(x,t,m_i) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n U_l(X_0, t, m_i)}{\partial x^n} (x - X_0)^n, \tag{6.4.46}$$

where $l = 0, 1, 2, 3 \cdots$ and $i = 0, 1, 2, 3 \cdots, l$.

From (6.4.37), (6.4.38) and (6.4.46), the interface condition (6.2.17) becomes

$$\sum_{m=0}^{\infty} p^m (D_t^{\alpha} X_m(t)) = \left(\frac{\varepsilon}{1-\varepsilon}\right)\left(\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{p^l}{m!}\left(\sum_{n=1}^{\infty} p^n X_n(t)\right)^m \frac{\partial^{m+1}}{\partial x^{m+1}} U_l(X_0, t, m_i)\right) - \eta \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}. \tag{6.4.47}$$

Comparing the coefficients of various powers of p from (6.4.47), we have

$$p^0 : D_t^{\alpha} X_0 - \frac{1}{X_0}\left(\frac{\varepsilon}{1-\varepsilon}\right) + \frac{\eta\, t^{1-\alpha}}{\Gamma(2-\alpha)} = 0, \tag{6.4.48}$$

$$p^1 : D_t^{\alpha} X_1 = \left(\frac{\varepsilon}{1-\varepsilon}\right)\left(\frac{\partial}{\partial y} U_1 + X_1 \frac{\partial^2}{\partial y^2} U_0\right), \tag{6.4.49}$$

$$p^2 : D_t^{\alpha} X_2 = \left(\frac{\varepsilon}{1-\varepsilon}\right)\left(\frac{\partial}{\partial y} U_2 + X_1 \frac{\partial^2}{\partial y^2} U_1 + X_2 \frac{\partial^2}{\partial y^2} U_0 + X_1^2 \frac{\partial^3}{\partial y^3} U_0\right), \tag{6.4.50}$$

$\vdots$

and so on.

Considering (6.4.39) and taking the coefficient of p^0 from (6.4.44) and (6.4.45), we have following system:

$$L(U_0(y,t)) = 0, \tag{6.4.51}$$

$$U_0(0,t) = 0, \tag{6.4.52}$$

$$U_0(X_0,t) = 1. \tag{6.4.53}$$

Eqs. (6.4.51-6.4.53) gives

$$U_0(y,t) = \frac{y}{X_0}, \tag{6.4.54}$$

According to Li et al. (2009) and Rajeev and Kushwaha (2013), we construct a homotopy for (6.4.48) as:

$$(1-p)\left(X_0 D_t^\alpha X_0 - \left(\frac{\varepsilon}{1-\varepsilon}\right)\right) + p\left(X_0 D_t^\alpha X_0 - \left(\frac{\varepsilon}{1-\varepsilon}\right) + \frac{\eta\, t^{1-\alpha}}{\Gamma(2-\alpha)} X_0\right) = 0. \tag{6.4.55}$$

Considering the solution of (6.4.48) in the following series form:

$$X_0 = \sum_{i=0}^{\infty} \phi_i(t) p^i. \tag{6.4.56}$$

Substituting (6.4.56) into (6.4.55), we get

$$\sum_{i=0}^{\infty} \phi_i(t) p^i \left(D_t^\alpha \left(\sum_{i=0}^{\infty} \phi_i(t) p^i\right)\right) - \left(\frac{\varepsilon}{1-\varepsilon}\right) + p\left(\frac{\eta\, t^{1-\alpha}}{\Gamma(2-\alpha)}\right) \sum_{i=0}^{\infty} \phi_i(t) p^i = 0. \tag{6.4.57}$$

Comparing the coefficients of various powers of p from (6.4.57), we have

$$p^0:\ \phi_0 D_t^\alpha \phi_0 - \left(\frac{\varepsilon}{1-\varepsilon}\right) = 0, \tag{6.4.58}$$

$$p^1: \quad \phi_0 D_t^{\alpha}\phi_1 + \phi_1 D_t^{\alpha}\phi_0 = -\frac{\eta\, t^{1-\alpha}}{\Gamma(2-\alpha)}\phi_0, \tag{6.4.59}$$

$$p^2: \quad \phi_1 D_t^{\alpha}\phi_2 + \phi_2 D_t^{\alpha}\phi_1 = -\frac{\eta\, t^{1-\alpha}}{\Gamma(2-\alpha)}\phi_1, \tag{6.4.60}$$

$\vdots$

and so on.

Therefore,

$$\begin{aligned} X_0 &= \phi_0(t) + \phi_1(t) + \phi_2(t) + \cdots, \\ &= a_0 t^{\alpha/2} - \frac{\eta\, t\, \Gamma(2+\alpha/2)}{\Gamma(2-\alpha)\,\Gamma(1+3\alpha/2)} + \cdots, \end{aligned} \tag{6.4.61}$$

where $\phi_0(t) = a_0 t^{\alpha/2}$

and $\quad a_0 = \left(\frac{\varepsilon}{1-\varepsilon}\right)^{1/2}\left(\frac{\Gamma(1-\alpha/2)}{\Gamma(1+\alpha/2)}\right)^{1/2}.$

In order to calculate other components of $U(y,t,m_i)$, i.e. $U_1, U_2, U_3 \cdots$, taking

$$U_0(y,t) \approx \frac{y}{\phi_0} \quad \text{(for small value of } \eta \text{ and } t\text{)}. \tag{6.4.62}$$

From Eq. (6.4.40) and taking the coefficient of p^1 from Eq. (6.4.44) and using $U_1(y,0) = 0$, we get

$$U_1(y,t,m_1) \approx -\frac{m_1}{3!a_0}\frac{\Gamma(1-\alpha/2)}{\Gamma(1-3\alpha/2)}y^3 t^{-3\alpha/2} + \frac{m_1\eta}{2!a_0}\frac{\Gamma(1-\alpha/2)}{\Gamma(2-3\alpha/2)}y^2 t^{1-3\alpha/2}. \tag{6.4.63}$$

Using boundary conditions $U_2(0,t)=0$ and $U_2(y,0)=0$ in the Eq. (6.4.41), we have

$$
\begin{aligned}
U_2(y,t,m_1,m_2) \approx & -\frac{m_2}{3!a_0}\frac{\Gamma(1-\alpha/2)}{\Gamma(1-3\alpha/2)}y^3t^{-3\alpha/2}+\frac{m_2\eta}{2!a_0}\frac{\Gamma(1-\alpha/2)}{\Gamma(2-3\alpha/2)}y^2t^{1-3\alpha/2} \\
& +(1+m_1)\left(-\frac{m_1}{3!a_0}\frac{\Gamma(1-\alpha/2)}{\Gamma(1-3\alpha/2)}y^3t^{-3\alpha/2}+\frac{m_1\eta}{2!a_0}\frac{\Gamma(1-\alpha/2)}{\Gamma(2-3\alpha/2)}y^2t^{1-3\alpha/2}\right) \\
& -m_1\left(-\frac{m_1}{5!a_0}\frac{\Gamma(1-\alpha/2)}{\Gamma(1-5\alpha/2)}y^5t^{-5\alpha/2}+\frac{m_1\eta}{4!a_0}\frac{\Gamma(1-\alpha/2)}{\Gamma(2-5\alpha/2)}y^4t^{1-5\alpha/2}\right) \\
& +m_1\eta\left(-\frac{m_1}{4!a_0}\frac{\Gamma(1-\alpha/2)}{\Gamma(2-5\alpha/2)}y^4t^{1-5\alpha/2}+\frac{m_1\eta}{3!a_0}\frac{\Gamma(1-\alpha/2)}{\Gamma(2-3\alpha/2)}y^3t^{2-5\alpha/2}\right),
\end{aligned}
\qquad (6.4.64)
$$

$\vdots$

and so on.

Therefore, approximate analytical solution of $U(y,t,m_i)$ is given by

$$U(y,t,m_i)=U_0+U_1+U_2+\cdots. \qquad (6.4.65)$$

Eqs. (6.4.49) and (6.4.50) give

$$X_1=\left(\frac{\varepsilon}{1-\varepsilon}\right)D_t^{-\alpha}\left(\frac{\partial}{\partial y}U_1(X_0,t,m_1)\right), \qquad (6.4.66)$$

and

$$X_2=\left(\frac{\varepsilon}{1-\varepsilon}\right)D_t^{-\alpha}\left(\frac{\partial}{\partial y}U_2(X_0,t,m_1,m_2)+X_1\frac{\partial^2}{\partial y^2}U_1(X_0,t,m_1)\right), \qquad (6.4.67)$$

respectively.

From (6.4.66), we have

$$
\begin{aligned}
X_1 \approx m_1\left(\frac{\varepsilon}{1-\varepsilon}\right)\Bigg(& \frac{a_0(\Gamma(1-\alpha/2))^2}{2\Gamma(1+\alpha/2)\Gamma(1-3\alpha/2)}t^{\alpha/2}-\frac{\eta\Gamma(1-\alpha/2)\Gamma(2+\alpha/2)}{\Gamma(1+3\alpha/2)\Gamma(1-3\alpha/2)(\Gamma(2-\alpha))^2}t^{1-\alpha} \\
& +\frac{\eta\Gamma(1-\alpha/2)\Gamma(2-\alpha)}{\Gamma(2-3\alpha/2)}t-\frac{\eta^2\Gamma(1-\alpha/2)\Gamma(3-\alpha/2)\Gamma(2+\alpha/2)}{a_0\Gamma(2-\alpha)\Gamma(1+3\alpha/2)\Gamma(3+\alpha/2)\Gamma(2-3\alpha/2)}t^{2+\alpha/2} \\
& +\frac{\eta^2\Gamma(1-\alpha/2)\Gamma(3+\alpha/2)(\Gamma(2+\alpha/2))^2}{\Gamma(3+3\alpha/2)\Gamma(1-3\alpha/2)(\Gamma(1+3\alpha/2))^2(\Gamma(2-\alpha))^2}t^{2+3\alpha/2}\Bigg),
\end{aligned}
\qquad (6.4.68)
$$

and similarly $X_{2,}X_3, X_4, \cdots$ can be computed.

Approximate analytical solution of $X(t)$ is given by

$$X(t) = X_0 + X_1 + X_2 + \cdots. \tag{6.4.69}$$

In order to get the constants involved in the expression of $U(x,t)$, Least Square Method (as given in Ghoreishi et al. (2012)) is used. For this purpose, we define the residual for $U(x,t)$ as:

$$R(x,t;m_1,m_2,\cdots m_l) = L\left(\widetilde{U}(x,t,m_1,m_2,\cdots m_l)\right) - \mu N\left(\widetilde{U}(x,t,m_1,m_2,\cdots m_l)\right). \tag{6.4.70}$$

where $\widetilde{U}(x,t,m_1,m_2,\cdots m_l)$ is an approximate value of $U(x,t)$ which can be found from Eq. (6.4.65).

If $R(x,t;m_i)=0$ then $U(x,t;m_i)$ will be exact solution. Generally, optimal homotopy asymptotic gives an approximate solution. Therefore, $R(x,t;m_i) \neq 0$ in such a case, but we can minimize the functional

$$J(m_i) = \int_0^t \int_0^{s(t)} R^2(x,t;m_i)\, dxdt, \tag{6.4.71}$$

where *R* is the residual. The constants $m_i\,(i=1,2\cdots,\, l)$ can be optimally obtained from the following conditions:

$$\frac{\partial J}{\partial m_1} = \frac{\partial J}{\partial m_2} = \cdots = \frac{\partial J}{\partial m_l} = 0. \tag{6.4.72}$$

6.5 Numerical discussion and comparison

In this section, all numerical results for diffusing boundary $S(t)$ are calculated by ADM and OHAM. In Adomian decomposition method, all the numerical calculations have been done by taking only three terms of the series of the $U(x,t)$ and $X(t)$ which are used in the calculation of $S(t)$. In case of optimal homotopy asymptotic method, only two terms of $U(x,t)$ and $X(t)$ are considered for numerical calculations. According to Liu and Xu (2004) and Yin and Xu (2009), the exact solutions to (6.2.14 – 6.2.18) at $\eta = 0$ are

$$U(x,t) = \frac{1 - W\left(-xt^{-\frac{\alpha}{2}}; -\frac{\alpha}{2}, 1\right)}{1 - W(-p; \frac{-\alpha}{2,}, 1)}, \tag{6.5.73}$$

and

$$S(t) = p t^{\frac{\alpha}{2}}, \tag{6.5.74}$$

where $W(-x; -\rho, 1-\rho) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma[-n\rho + (1-\rho)]}$ is Wright function and p is a constant that will be determined by following transcendental equation:

$$\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} p \frac{(\varepsilon^{-1} - 1)\left(1 - W(-p; -\frac{\alpha}{2}, 1)\right)}{W(-p; -\frac{\alpha}{2}, 1 - \frac{\alpha}{2})} = 1. \tag{6.5.75}$$

Table 6.1 shows the comparisons among exact solution, proposed approximate solutions (OHAM and ADM) and asymptotic solution for diffusing boundary $S(t)$ at the fixed value of $\varepsilon = 0.05$, $\alpha = 1.0$ and $\eta = 0$. From the table, it is confirmed that our approximate results are in good agreement with the exact results for $\alpha = 1.0$ (standard motion).

Figure 6.1 and Figure 6.2 show the comparisons among exact solution, proposed approximate solutions (ADM and OHAM) and asymptotic solution of diffusing boundary $S(t)$ for the fixed values of $\varepsilon = 0.05$ at $\alpha = 0.75$ and $\alpha = 0.5$, respectively. Figures (6.2-6.4) depict the comparisons among exact solution, proposed approximate solutions and asymptotic solution for diffusing boundary $S(t)$ for different value of ε $(\varepsilon = 0.05, 0.1, 0.15)$ at $\alpha = 0.5$ and $\eta = 0$. From figures (6.1-6.4), it is clear that our approximate results are more near to the exact solution than asymptotic solution of Yin and Xu (2009) for $\eta = 0$. Moreover, it can be seen that results of the problem obtained by optimal homotopy asymptotic method are more accurate than Adomian decomposition method.

Fig. 6.5 and fig. 6.6 depict the dependence of dimensionless diffusing boundary $S(t)$ on dimensionless time t for various value of the parameter ε at $\eta = 0$ and $\eta = 0.5$, respectively. It can be seen from these figures that the movement of diffusing boundary increases with the increase of the parameter ε $(\varepsilon = 0.01, 0.05, 0.1)$. Hence, the diffusion process becomes fast as the values of ε increases.

Figures (6.7-6.8) show the dependence of dimensionless diffusing boundary $S(t)$ on dimensionless time t for different values of α $(\alpha = 1/3, 1/2, 2/3, 1.0)$. From figures (6.7-6.8), it can be seen that the movement of diffusing boundary decreases with the increase of α. Hence, the diffusion process becomes slow as the values of α increases. Fig. 6.9 and fig. 6.10 represent the dependence of dimensionless diffusing boundary $S(t)$ on dimensionless time t for different value of η at $\varepsilon = 0.3$ for $\alpha = 0.5$ and $\alpha = 1.0$, respectively. From figures (6.9-6.10), it is clear that the diffusion process increases slowly as the value of η $(\eta = 0, 0.25, 0.5)$ increases.

Time	Exact solution	Asymptotic solution	ADM solution	OHAM solution
1	0.32161	0.3136396724404 (0.007970328)	0.3161265997764 (0.0054834)	0.3244428397 (0.002832842)
2	0.45482522379	0.443553478463 (0.011271745)	0.4470705248307 (0.007754698)	0.4588314641 (0.004006077)
3	0.55704486022	0.543239847936 (0.01380502)	0.5475473324368 (0.009497528)	0.5619514825 (0.00490614)
4	0.64322	0.6272793448808 (0.015940655)	0.6322531995529 (0.010966801)	0.6488856794 (0.005666)
5	0.71914182224	0.7013196280176 (0.017822194)	0.706880566596 (0.012261256)	0.725476244 (0.006334178)
6	0.78778039617	0.7682571605727 (0.01952323)	0.774348863573 (0.013431527)	0.7947194079 (0.006334178)
7	0.85090007915	0.82981257 (0.02108743)	0.83639236582 (0.014507634)	0.8583950685 (0.007495)
8	0.90965044759	0.887106956927 (0.02254305)	0.89414104966 (0.01550895)	0.9176629282 (0.008013)
9	0.9648300	0.9409190173213 (0.023910983)	0.948379799329 (0.016450201)	0.9733285191 (0.008499)
10	1.017020118	0.991815729501 (0.025204398)	0.999680084258 (0.017340034)	1.0259783440 (0.008958232)

Table 6.1. Comparison among Asymptotic solution, ADM solution and OHAM solution of $S(t)$ at $\alpha = 1.0$, $\varepsilon = 0.05$ and $\eta = 0$.

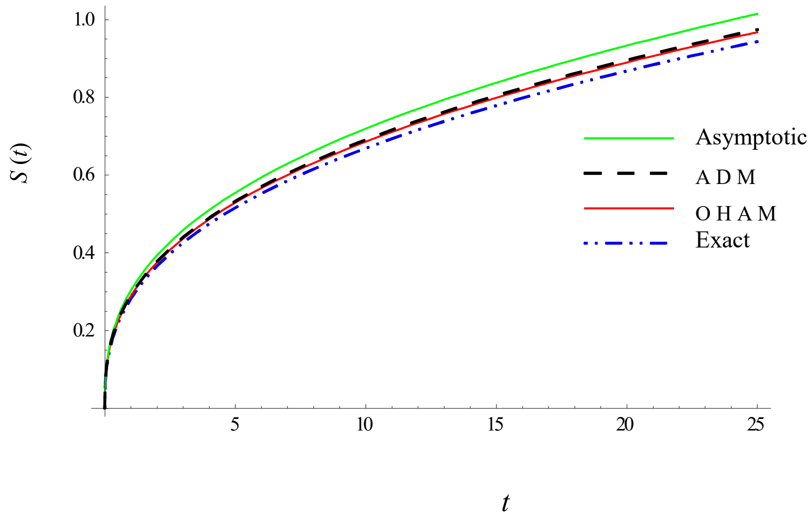

Fig.6.1. Plot of $S(t)$ vs. t at $\alpha = 0.75$, ε = 0.05 and $\eta = 0$

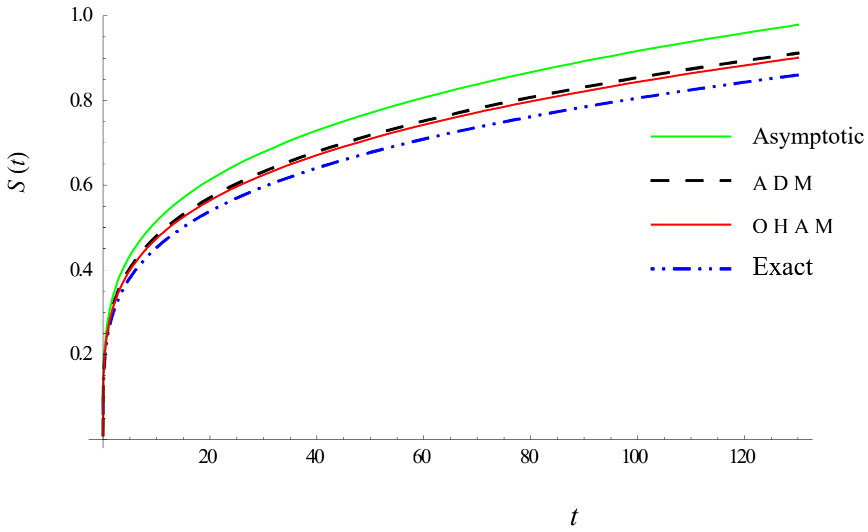

Fig.6.2. Plot of $S(t)$ vs. t at $\alpha = 0.5$, ε = 0.05 and $\eta = 0$

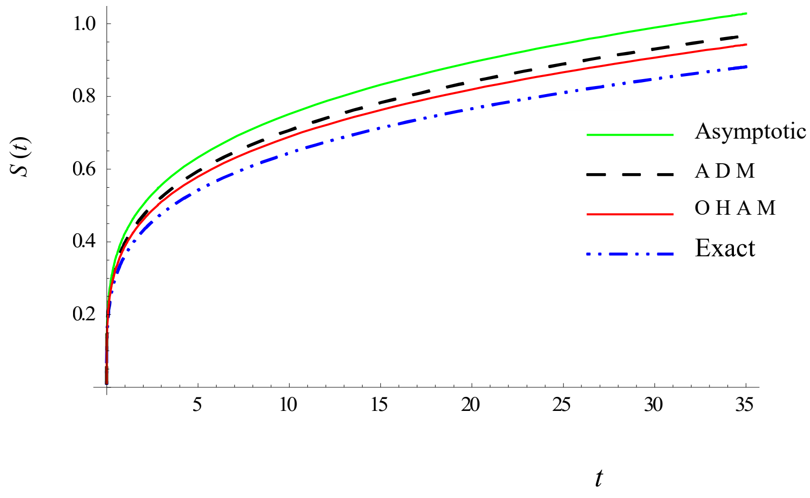

Fig.6.3. Plot of $S(t)$ vs. t at $\alpha = 0.5$, $\varepsilon = 0.1$ and $\eta = 0$

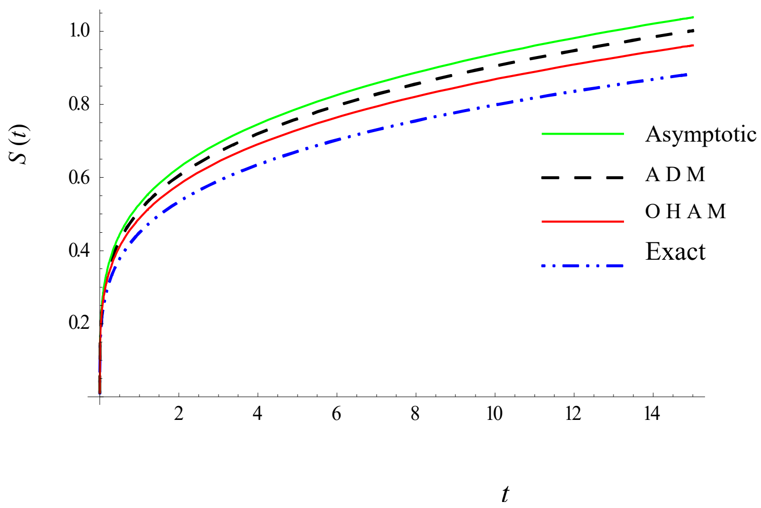

Fig.6.4. Plot of $S(t)$ vs. t at $\alpha = 0.5$, $\varepsilon = 0.15$ and $\eta = 0$

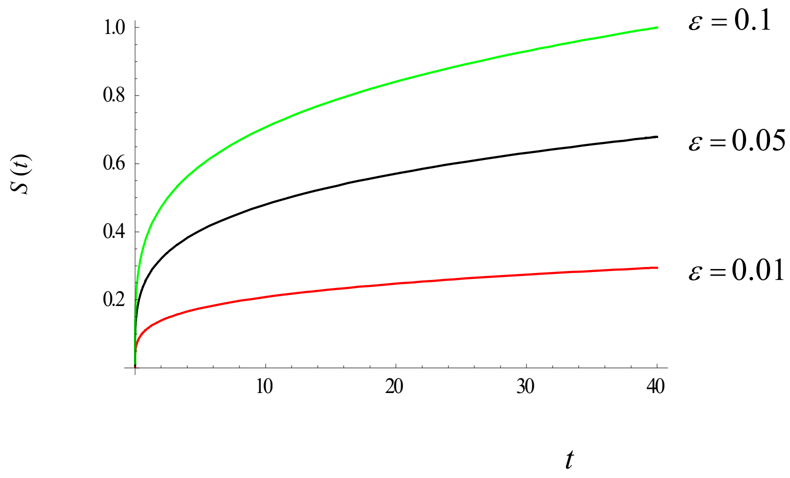

Fig.6.5. Plot of $S(t)$ vs. t for $\alpha = 0.5$ and $\eta = 0$

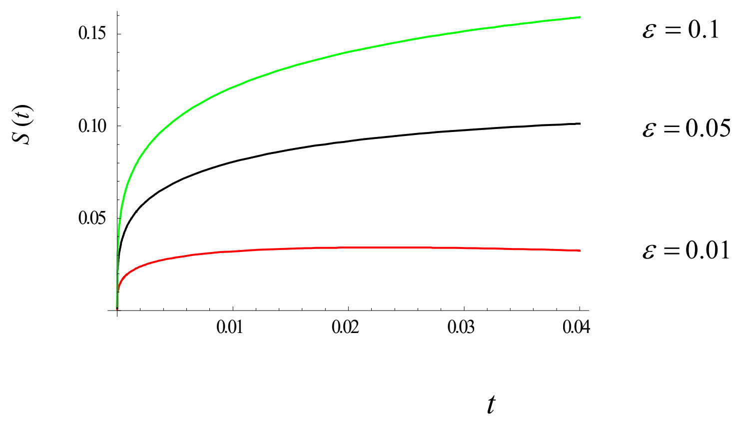

Fig.6.6. Plot of $S(t)$ vs. t for $\alpha = 0.5$ and $\eta = 0.5$

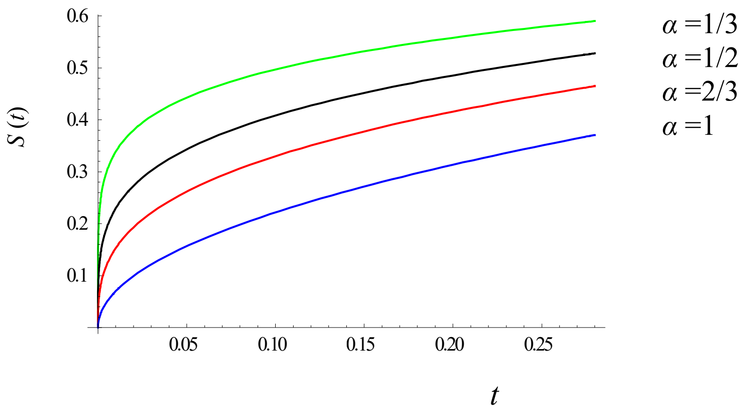

Fig.6.7. Plot of $S(t)$ vs. t at ε = 0.25 and $\eta = 0$

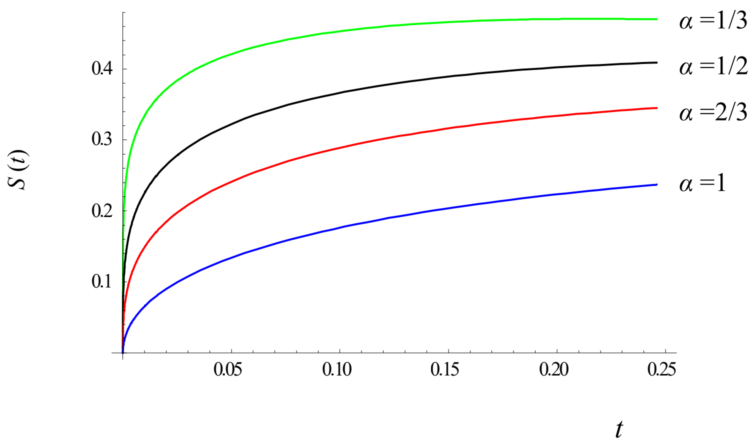

Fig.6.8. Plot of $S(t)$ vs. t at $\varepsilon = 0.25$ and $\eta = 0.5$

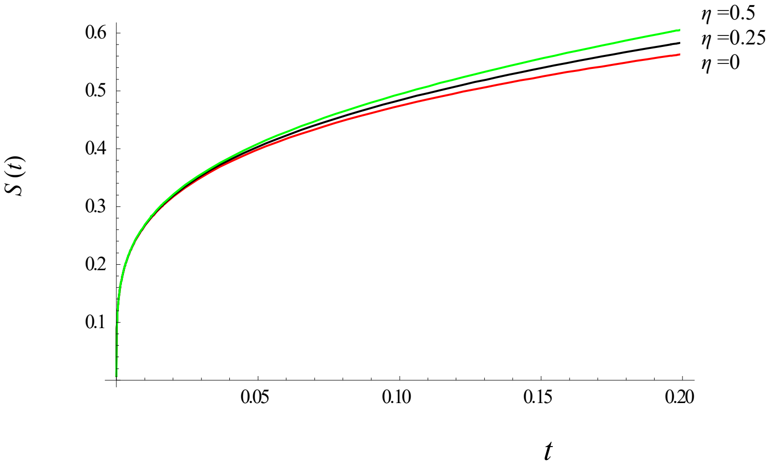

Fig.6.9. Dependence of $S(t)$ on t at $\alpha = 0.50$ and $\varepsilon = 0.3$

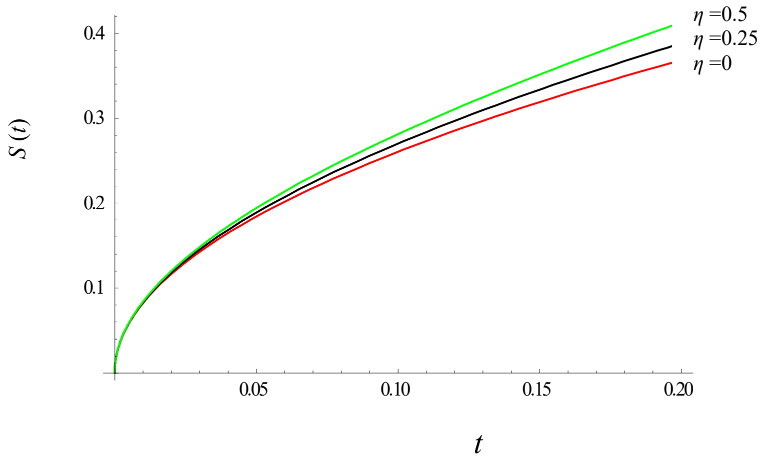

Fig. 6.10. Dependence of $S(t)$ on t at $\alpha = 1.0$ and $\varepsilon = 0.3$

6.6 Conclusion

In this study, we discussed approximate solutions of the two moving boundaries problem by Adomian decomposition method and optimal homotopy asymptotic method. It is found that velocity of diffusing boundary becomes fast as the values of parameter ε increases and movement of diffusing boundary becomes slow with increase in the values of the α as well as η. It is seen that Adomian decomposition method and optimal homotopy asymptotic method both are straight forward and sufficient accurate method for finding the solution of moving boundary problems. Every method has its advantages and disadvantages. In present study, it is found that the approximate solution by optimal homotopy asymptotic method is more close to exact result than Adomian decomposition method. But, in optimal homotopy asymptotic method, an additional work is required each time to calculate constants after changing the value of the parameters in the numerical computation. It is also possible that optimal homotopy asymptotic method may not be applicable at large value of η and/or time for this problem.