ORIGINAL PAPER

Super Graphs on Groups, I

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Abstract

Let G be a finite group. A number of graphs with the vertex set G have been studied, including the power graph, enhanced power graph, and commuting graph. These graphs form a hierarchy under the inclusion of edge sets, and it is useful to study them together. In addition, several authors have considered modifying the definition of these graphs by choosing a natural equivalence relation on the group such as equality, conjugacy, or equal orders, and joining two elements if there are elements in their equivalence class that are adjacent in the original graph. In this way, we enlarge the hierarchy into a second dimension. Using the three graph types and three equivalence relations mentioned gives nine graphs, of which in general only two coincide; we find conditions on the group for some other pairs to be equal. These often define interesting classes of groups, such as EPPO groups, 2-Engel groups, and Dedekind groups. We study some properties of graphs in this new hierarchy. In particular, we characterize the groups for which the graphs are complete, and in most cases, we characterize the dominant vertices (those joined to all others). Also, we give some results about universality, perfectness, and clique number.

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1 Introduction

In this section, we describe some graphs associated with groups and discuss a convention for these graphs as well as notation and terminology. In the following sections, we prove a number of properties of the collection of graphs and show how treating them together can be helpful: we examine conditions for some pair of these graphs to be equal; in most cases, we characterize the dominant vertices; we show that some of the graphs are perfect, and examine universality properties of the others; and we calculate the clique number in some cases.

The paper concludes with some open problems and suggestions for further work.

In subsequent work [1, 17], we will look at detailed properties (including Wiener index and spectrum) of these graphs, and examine the supergraphs for an extension of the graph hierarchy to the solvability graph.

1.1 B superA Graphs on Groups

Let A be a type of graph defined on a group G. In this paper, we will consider three such types:

(a) the *power graph*, in which g is adjacent to h if either g or h is a power of the other;

(b) the *enhanced power graph*, in which g is adjacent to h if g and h are both powers of an element k (equivalently, if the group $\langle g, h \rangle$ generated by g and h is cyclic);

(c) the *commuting graph*, in which g is adjacent to h if $gh = hg$ (equivalently, if $\langle g, h \rangle$ is abelian).

Several other types of graphs can be defined, including the deep commuting graph, nilpotency graph, solvability graph, and Engel graph; these are described in the survey paper [5].

Also, let B be an equivalence relation defined on G. In this paper, we will consider three equivalence relations;

(a) *equality*, $g \sim h$ if $g = h$;

(b) *conjugacy*, $g \sim h$ if $h = x^{-1}gx$ for some $x \in G$;

(c) *same order*, $g \sim h$ if $o(g) = o(h)$, where $o(g)$ denotes the order of g.

Other relations could be considered, such as *automorphism conjugacy*, where $g \sim h$ if some automorphism of G maps g to h.

Definition 1   The B *super* A *graph* on G is defined as follows: Let $[g]$ denote the B-equivalence class of the element g. Now join g and h if and only if there exist $g' \in [g]$ and $h' \in [h]$ such that g' and h' are joined in the A-graph on G.

In principle the graph and equivalence relation are arbitrary, but there are reasons for choosing them to be preserved by the automorphism group of G, as we will see. This is the case for the examples mentioned above.

Remark 1 If we take A to be the power graph and B the relation "same order", we do obtain the definition of the superpower graph of G from [18]. For suppose that $o(g) \mid o(h)$. Then some power of h, namely $h^{o(h)/o(g)}$, has order $o(g)$; so in the definition we can take $g' = h^{o(h)/o(g)}$ and $h' = h$. Our naming convention gives this graph the name "order superpower graph" of G, which will distinguish this from the conjugacy superpower graph. Also, Herzog et al. [9] have considered the conjugacy supercommuting graph. Our aim here is to give a unified treatment of these graphs.

Our convention would also give the power graph the name "equality superpower graph", but we will simply say "power graph", with similar convention for the other basic graph types.

We see that any result about the power graph is in principle one of a set of nine results about related graphs.

Note that we have inclusions of the edge sets as follows: the edge set of the power graph is contained in the edge set of the conjugacy superpower graph, which is contained in the edge set of the order superpower graph; similarly for other types of graphs.

Note also that graph parameters such as clique number, chromatic number, and matching number are monotonic increasing with edge set; independence number and clique cover number are monotonic decreasing; and increasing the edge set cannot destroy properties such as being Hamiltonian.

Proposition 1

(a) *Let the equivalence relation* B *be* "*same order*". *If g and h are joined in the power graph, then for one of them, say g, every element equivalent to g is joined to some element equivalent to h.*

(b) *Let* B *be the conjugacy relation and consider any graph of type* A *on the group G. If the graph* A *is invariant under inner automorphisms of G, and g is joined to h in* A, *then every element of the conjugacy class of g is joined to some element of the conjugacy class of h in the conjugacy super* A *graph on G, and vice versa.*

(c) *More generally, let H be a subgroup of the automorphism group of G which acts on* A *for some graph type* A. *Let* B *be the equivalence relation induced by the orbit partition of this action. Then g is joined to h in* A *implies that (in* B *super* A *graph) every element equivalent to g is joined to some element equivalent to h under the equivalence relation* B.

Proof The first statement was observed in the earlier remark. We prove the third statement, from which the second follows. Note that if $\{g, h\}$ is an edge, then $\{\phi(g), \phi(h)\}$ is an edge for all $\phi \in H$ since B is the equivalence relation induced by the orbit partition. In particular, for the second statement, note that the conjugacy classes are orbits of the inner automorphism group of G; so, if $\{g, h\}$ is an edge, then $\{x^{-1}gx, x^{-1}hx\}$ is an edge for all $x \in G$. In this case, the hypothesis holds for all the three graph types we are considering. $\square$

Remark 2   Let us observe another general property. For each of the power graph, enhanced power graph and commuting graph, if H is a subgroup of G, then $\mathrm{A}(H)$ is an induced subgraph of $\mathrm{A}(G)$. This holds also for the order superA graphs, but not in general for the conjugacy superA graphs since the conjugacy relation can change when we pass from G to H.

1.2 A Convention About Equivalence Classes

Our adjacency rule is ambiguous about whether we join vertices in the same equivalence class. We now explain how we resolve this and explain the rationale.

In many of the graphs defined on a group, including all those treated here, the definition would naturally give us a loop at each vertex; any group element is a power of itself, so this holds for the power graph and enhanced power graph; any element commutes with itself, so this holds for the commuting graph; and so on. Of course, we prefer graphs not to have loops, so we silently remove these, even though they make little difference to many graph-theoretic properties (they make no change to connectivity and diameter, and simply add the identity matrix to the adjacency matrix). Adopting the convention that there is a (silent/virtual) loop at each vertex, we find that any equivalence class of the equivalence relation B will induce a complete subgraph in the B superA graph.

In fact, even without this convention, things would not be very different. Consider the order superA graph. An element g has the same order as its inverse, which is joined to it in the power graph, enhanced power graph, or commuting graph; so as long as the order of g is greater than 2, each order equivalence class will induce a complete graph. This will fail only for involutions. For the conjugacy relation, things are a bit more complicated and could be worth investigating. However, as stated, we will adopt the convention here that every equivalence class of B induces a complete subgraph in the B superA graph.

1.3 Notation

We have defined a fairly large number of graphs: what notation should we use to make it easy for the reader to recognize which graph is being discussed without being altogether too cumbersome?

In [5], the second author proposed a systematic notation for various graphs on groups: given a group G, that paper uses $\mathrm{Pow}(G)$, $\mathrm{EPow}(G)$, and $\mathrm{Com}(G)$ for the power graph, enhanced power graph, and commuting graph of G. One possibility is to modify these in an obvious way, so that $\mathrm{CSPow}(G)$ is the conjugacy superpower graph and $\mathrm{OSPow}(G)$ is the order superpower graph, with similar terminology for the other graphs.

This notation is a bit cumbersome but is hopefully self-explanatory.

2 Reducing Nine Graphs to Eight

We have defined nine graphs, but two of them turn out to be the same.

Theorem 1 *For any finite group G, the order superenhanced power graph of G is equal to the order supercommuting graph of G.*

Proof We use the basic fact that, if a finite abelian group G has exponent m, then it contains an element of order m. By definition, the graph OSEPow(G) is a spanning subgraph of OSCom(G). We have to prove the reverse implication. So suppose that $\{x, y\}$ is an edge of OSCom(G). By definition, there exist elements x' and y' such that $o(x) = o(x')$, $o(y) = o(y')$, and $x'y' = y'x'$. Then $A = \langle x', y' \rangle$ is abelian. Let m be its exponent, and $z \in A$ an element of order m. Then $o(x') \mid o(z)$ and $o(y') \mid o(z)$, so there exist elements x'' and y'' in A with $o(x'') = o(x')$, $o(y'') = o(y')$, and x'' and y'' are both powers of z. Then $\{x'', y''\}$ is an edge of the enhanced power graph of G, and $o(x'') = o(x)$, $o(y'') = o(y)$; so $\{x, y\}$ is an edge of the order superenhanced power graph. $\qquad\square$

Any two of the remaining eight graphs are unequal for some group G. By Theorem 3.1, the only pairs that need to be considered are the A graph and the conjugacy superA graph for each of our three graph types A; all of these are settled by the example $G = S_3$ (the dihedral/symmetric group of order 6).

One could ask: Is there a group G for which all eight graphs are different? If so, what is the smallest order of such a group?

A more challenging question would be, for each pair of graph types, to determine the groups for which the two types of graphs coincide. This has been solved for the original three graphs, and is a non-trivial exercise. The power graph and enhanced power graph are equal if and only if G contains no subgroup $C_p \times C_q$ for distinct primes p and q; this condition characterizes the so-called *EPPO groups* (elements of prime power order groups), also known as *CP groups*, which were determined by Brandl [3] using earlier work of Higman [10] and Suzuki [20]. We refer to [6] for further connections between this class and graphs defined on groups.

The enhanced power graph and the commuting graph are equal if and only if G contains no subgroup $C_p \times C_p$ for a prime p; this condition is equivalent to saying that all the Sylow subgroups are cyclic or (for the prime 2) generalized quaternion groups and it is not too difficult to list such groups. Indeed, all groups with cyclic or generalized quaternion Sylow 2-subgroups have been determined; see [2].

We give two more results along these lines. For the first, recall the definition of iterated commutators in a group: $[x, y] = x^{-1}y^{-1}xy$ and

$$[x_1, x_2, \ldots, x_{n+1}] = [[x_1, x_2, \ldots, x_n], x_{n+1}]$$

for $n \geq 2$. A group G is *nilpotent of class at most n* if $[x_1, \ldots, x_{n+1}] = 1$ for all $x_1, \ldots, x_{n+1} \in G$; and a group satisfies the nth *Engel identity*, or is *n-Engel*, if $[y, x, \ldots, x] = 1$ (with n occurrences of x) for all $x, y \in G$. Clearly a group which is nilpotent of class at most n is n-Engel; the converse is false, but it was shown by Hopkins [11] and Levi [15] independently that a 2-Engel group is nilpotent of class at most 3.

The following lemma is proved by Korhonen given in a post on Stack-Exchange [14].

Lemma 1 *The following statements are equivalent for a group G.*

(a) *Every centralizer in G is a normal subgroup.*
(b) *Any two conjugate elements in G commute, ie. $x^g x = x x^g$ for all $x, g \in G$.*
(c) *G is a 2-Engel group, ie. $[[x, g], g] = 1$ for all $x, g \in G$.*

Proof (a) $\Rightarrow$ (b): Consider x in $C_G(x)$; since a normal subgroup is a union of conjugacy classes of its elements, we have $x^g \in C_G(x)$ for all $g \in G$.

(b) $\Rightarrow$ (c): Since $x^g = x[x, g]$, if x^g commutes with x, $[x, g]$ also commutes with x.

(c) $\Rightarrow$ (a): If $[[x, g], g] = 1$ for all $g \in G$, then according to [13, Lemma 2.2] we have $[x, [g, h]] = [[x, g], h]^2$. Therefore $[C_G(x), G] \leq C_G(x)$, which means that $C_G(x)$ is a normal subgroup.   □

Theorem 2 *Let G be a finite group. Then the following conditions are equivalent:*

(a) *the commuting graph of G is equal to the conjugacy supercommuting graph;*
(b) *the centralizer of every element of G is a normal subgroup of G;*
(c) *G is a 2-Engel group.*

Proof First, we show the equivalence of (a) and (b). Suppose the commuting graph and conjugacy supercommuting graph are equal. If x and y commute, they are joined in the commuting graph, and so every conjugate of y is joined to x in the conjugacy supercommuting graph, and hence also in the commuting graph; thus x commutes with every conjugate of y. Hence the centralizer of x is a union of conjugacy classes, so it is a normal subgroup of G. The argument reverses. The equivalence of (b) and (c) follows from the above lemma.   □

For the second result, recall that a *Dedekind group* is a group in which every subgroup is normal. Dedekind [7] showed that such a group is either abelian or of the form $Q_8 \times E \times F$, where Q_8 is the quaternion group of order 8, E an elementary abelian 2-group, and F an abelian group of odd order.

Theorem 3 *For a finite group G, the following conditions are equivalent:*

(a) *the power graph of G is equal to the conjugacy superpower graph;*
(b) *the enhanced power graph of G is equal to the conjugacy superenhanced power graph;*
(c) *G is a Dedekind group.*

Proof We use the fact that conjugate elements have the same order. Let x^G denote the conjugacy class of x in G.

In either the power graph or the enhanced power graph, elements of the same order which are adjacent generate the same cyclic subgroup. So, if either (a) or (b) holds, then all elements of x^G generate $\langle x \rangle$. So every cyclic subgroup of G, and hence every subgroup, is normal; that is, G is a Dedekind group.

For the converse, if x and y are joined in either the power graph or the enhanced power graph, then there is a cyclic group C containing x and y. If G is a Dedekind group, then C is normal; so $x^G \cup y^G \subseteq C$. In the case of the enhanced power graph, C

is a clique, so all vertices in x^G are joined to all vertices in y^G, and (b) holds. In the case of the power graph, (a) holds because either $o(x)$ divides $o(y)$ or *vice versa*. $\square$

Remark 3 There are some other elementary observations. The group G has the property "two elements are conjugate if and only if they are equal" exactly when G is abelian. So we have that the conjugacy superA graph is equal to the A graph if G is abelian, for any graph type A. In a similar way, the conjugacy superA graph is equal to the order superA graph if G is a group in which any two elements of the same order are conjugate. (There are only three finite groups with this property, the symmetric groups of degrees 1, 2 and 3: see Fitzpatrick [8, Theorem 3.6].)

3 Completeness and Dominant Vertices

In this section, we begin a study of how the properties of the graphs relate to properties of the groups they are built on.

3.1 When is the B superA Graph Complete?

The theorem below summarises the answer to this question "When is the graph complete?" for our three types of graph and three types of partition, and is intended as an example of treating the hierarchy uniformly. In the table, $(*)$ means that the group G has an element whose order is the exponent m of G; equivalently, the *spectrum* of G (the set $\pi^*(G)$ of orders of elements of G, sometimes denoted by $\pi_e(G)$) is the set of all divisors of m. Such groups are not so rare. Any nilpotent group has this property; and, for any finite group G, there is a positive integer r such that G^r has property $(*)$. For example, $(A_5)^3$ contains elements of order 30, which is the exponent of the group.

Theorem 4 *The following table describes groups whose power graph, enhanced power graph, commuting graph, or their conjugacy or order supergraph is complete.*

	Power graph	Enhanced Power graph	Commuting graph
Equality	Cyclic p-group	Cyclic	Abelian
Conjugacy	Cyclic p-group	Cyclic	Abelian
Order	p-group	$(*)$	$(*)$

Proof The results for the power graph, enhanced power graph, and commuting graph are well-known [5].

To prove that the conjugacy supercommuting graph is complete if and only if G is

abelian, we use a result which goes back to Jordan [12]: if G is a finite group and H a proper subgroup of G, then there is a conjugacy class in G which is disjoint from H. (For the reader's convenience we sketch a proof. Let H have index n in G, and consider the action of G by right multiplication on the set of right cosets of H. This action is transitive, so by the Orbit-counting Lemma (sometimes called Burnside's Lemma), the average number of fixed points of the elements of G is 1. But the identity fixes n points, and $n > 1$; so some element g fixes no point. This means that g lies in no conjugate of H; equivalently, no conjugate of g lies in H. For a modern take on Jordan's theorem, we strongly recommend a paper of Serre [19]).

Now, if G is abelian, then the conjugacy supercommuting graph coincides with the commuting graph, and is complete. So suppose that G is a finite group whose conjugacy supercommuting graph is complete, and take any element $g \in G$. If $C_G(g) \neq G$ then, by Jordan's theorem, there is an element h such that the conjugacy class of h is disjoint from $C_G(g)$; thus no conjugate of h commutes with g, and so g and h are non-adjacent, a contradiction. Thus $C_G(g) = G$, or $g \in Z(G)$. Since this holds for all $g \in G$, we see that G is abelian.

If G is not a p-group, then it contains elements of distinct prime orders. These elements are non-adjacent in the power graph and both of its supergraphs. So if any of these graphs are complete, then G must be a p-group. Conversely, if G is a p-group, its order superpower graph is complete, as shown in [18].

If G is a cyclic p-group, then its power graph, and hence its conjugacy superpower graph, is complete. Suppose conversely that G is a group whose conjugacy superpower graph is complete. Then G cannot have elements of distinct prime order, so G is a p-group. Let g be an element of order p in $Z(G)$. Then G is conjugate only to itself, so cannot be joined to any element of order p outside $\langle g \rangle$; so there can be no such elements. Thus G has a unique subgroup of order p, and by a result of Burnside [4, Sections 104–105] it is cyclic or generalized quaternion. But generalized quaternion groups contain non-conjugate subgroups of order 4, so cannot arise here.

If G is cyclic, then its enhanced power graph, and hence its conjugacy superenhanced power graph, is complete. Suppose conversely that G is a group whose conjugacy superenhanced power graph is complete. Then the conjugacy supercommuting graph of G is complete, so G is abelian. Then the conjugacy superenhanced power graph coincides with the enhanced power graph, so G is cyclic.

Finally, let G be a group whose order supercommuting graph is complete. Take two elements of G, with orders (say) g and h. Since g and h are adjacent, we can replace them with elements of the same orders which commute, and so the order of their product is the least common multiple of k and l. Thus the set $\pi_e(G)$ is closed under taking least common multiples (as well as under taking divisors), and so $(*)$ holds. Conversely, if $g \in G$ has order equal to the exponent of G, then every element in $\pi_e(G)$ is the order of some power of g, and all these powers are joined in the enhanced power graph and in the commuting graph. So the order supercommuting graph is complete. $\square$

3.2 Dominant Vertices

A graph is complete if and only if every vertex is *dominant* (or *universal*), that is, joined to all other vertices. So, as a generalization of Theorem 3.1, we could ask: for each of the nine graphs, which elements of a group G are dominant vertices? The answers are known for the basic graphs, and can be found in [5, Section 9.1]; we summarise the results here.

(a)　The set of dominant vertices of the power graph of G is the whole of G, if G is a cyclic p-group; the identity and the generators of G, if G is cyclic but not a p-group; the centre, if G is a generalized quaternion group; and only the identity in all other cases.

(b)　The set of dominant vertices in the enhanced power graph is a cyclic subgroup of $Z(G)$ called the *cyclicizer* of G; it is the product of the Sylow p-subgroups of $Z(G)$ for those primes p for which a Sylow p-subgroup of G is cyclic or generalized quaternion.

(c)　The set of dominant vertices in the commuting graph is the centre $Z(G)$.

Now we solve the problem for the conjugacy supergraphs.

Theorem 5 *If* A *is the power graph, enhanced power graph, or commuting graph, then the set of dominant vertices in the conjugacy super* A *graph of G is the same as the set of dominant vertices in the* A *graph.*

Proof We show this first for the commuting graph. Suppose that $g \in G$ and g is joined to all other vertices of G in the conjugacy supercommuting graph. By Proposition 1, g is joined to an element of every conjugacy class of G; in other words, its centralizer $C_G(g)$ meets every conjugacy class. Hence $C_G(g) = G$, so $g \in Z(G)$. Thus g is joined to all other vertices in the commuting graph.

　　Now suppose that $A(G)$ is the power graph or enhanced power graph of G, and let g be a dominant vertex in the conjugacy superA graph of G. Since the conjugacy superA graph is a spanning subgraph of the conjugacy supercommuting graph, $g \in Z(G)$. So, for any $h \in G$, h is joined to a conjugate of g. But the only conjugate is g itself; so g is joined to all other vertices in $A(G)$.　　□

　　We have also solved the problem for the order superpower graph. If G has prime power order then its order superpower graph is complete; so we can suppose not.

Proposition 2 *Let G be a group not of prime power order, having exponent m. Then the set of dominant vertices in the order superpower graph of G consists of the identity and the elements of order m (if any).*

Proof Let $p_1, \ldots, p_r$ be the prime divisors of $|G|$ with $r > 1$, and let $p_i^{a_i}$ be the largest power of p^i dividing the order of an element of G; then there are elements of order $p_i^{a_i}$ in G. Suppose that n has the property that elements of order n are dominant, and that $n > 1$. Then, for each i, either $p_i^{a_i}$ divides n, or n divides $p_i^{a_i}$. Since $r > 1$, the second cannot hold. (If, say, n divides $p_1^{a_1}$, then n is a proper power of p_1; then neither $p_2^{a_2} \mid n$ or $n \mid p_2^{a_2}$ can hold.) Thus n is the product of the prime powers $p_i^{a_i}$, which is equal to

the exponent of G.

Conversely, if g has order m, the exponent of G, then $o(h) \mid o(g)$ for all $h \in G$, so g is dominant. □

We have not found a characterisation of the dominant vertices in the (one) remaining case, the order supercommuting graph.

Remark 4 If a graph has a dominant vertex, then it is connected, with diameter at most 2. So it is customary to remove the dominant vertices in order to get non-trivial questions about connectedness and diameter. This is one reason why it is important to know such vertices. This remark suggests that a next step in the investigation of these graphs would be to decide about the connectedness of the "reduced" graphs.

4 Some Graph Properties and Parameters

In this section, we discuss several further graph properties (perfectness, universality) and parameters (clique number) for our graphs.

4.1 Perfectness and Universality

It is known that power graphs of finite groups are perfect, but enhanced power graphs and commuting graphs are not necessarily perfect; indeed, any finite graph can be embedded as an induced subgraph in the enhanced power graph (or the commuting graph) of a finite group (see [5]). In this section we give some similar results for supergraphs.

Theorem 6 *The conjugacy or order superpower graph of a finite group is the comparability graph of a finite partial preorder and hence is perfect.*

Proof For this, we define directed versions of these graphs and show that they are comparability graphs. For conjugacy, we put an arc from x to y if some conjugate of y is a power of x (or equivalently if y is a power of some conjugate of x); for order, we put an arc from x to y if $o(y) \mid o(x)$. Both are reflexive (if we add loops) and transitive. □

Theorem 7 *Every finite graph Γ is embeddable as an induced subgraph in the conjugacy superenhanced power graph, and in the conjugacy supercommuting graph, of some finite group.*

Proof The proof in [5, Theorem 5.5] of the analogous result for the enhanced power graph constructs an abelian group, where conjugacy coincides with equality, proving the result for this case. Here, we give a different proof, which works for both graph types. We use the fact that two elements of distinct prime orders are joined in the enhanced power graph if and only if they are joined in the commuting graph. (One way round is trivial since the enhanced power graph is a subgraph of the commuting graph. In the other direction, if g and h have distinct prime orders and commute, then both are powers of gh.) Now, if Γ is a complete graph on n vertices, then we can take G to be the direct product of cyclic groups of distinct prime orders $p_1, \ldots, p_n$; if X

consists of one element of each prime order, then the induced subgraph of the commuting graph of G on X is Γ. So we may assume that Γ is not complete.

First, we observe that it is possible to find a set of n prime numbers $p_1, \ldots, p_n$ such that, if $i \neq j$ and $p_i < p_j$, then $p_i \mid p_j - 1$. This is proved by induction. Suppose that $p_1, \ldots, p_{n-1}$ have been chosen. Then we choose p_n to be congruent to 1 mod p_i for $i = 1, \ldots, n-1$ (this is possible by the Chinese remainder theorem) and to be prime (this is possible by Dirichlet's theorem on primes in arithmetic progression).

Now for $i, j \in \{1, \ldots, n\}$, let G_{ij} be the direct product of the non-abelian group of order $p_i p_j$ and the cyclic group of order p_k for every $k \notin \{i, j\}$; let x_{ijk} be an element of order p_k in G_{ij} for $k = 1, \ldots, n$. We note that the induced subgraph of the commuting graph of G_{ij} on $\{x_{ij1}, \ldots, x_{ijn}\}$ is the complete graph K_n with the edge $\{i, j\}$ deleted.

Given a graph Γ with vertex set $\{1, \ldots, n\}$, let G be the direct product of the groups G_{ij} over all *nonedges* $\{i, j\}$ of G; let x_k be the element of G which projects onto x_{ijk} in the factor G_{ij} for all such pairs $\{i, j\}$. Then the element x_k has order p_k, and x_k and x_l commute if and only if x_{ijk} and x_{ijl} commute for all $\{i, j\}$, that is, $\{k, l\}$ is an edge of Γ.

Finally, we note that, if two elements commute, they are joined in the conjugacy supercommuting graph. Conversely, if x_j and x_k do not commute, then they project onto non-commuting elements in G_{jk}; the structure of the non-abelian group of order $p_j p_k$ shows that conjugates of these elements also do not commute. □

4.2 Clique Number

We describe the maximal cliques in the order superpower and superenhanced power graphs.

We begin with some definitions. The *spectrum* $\pi^*(G)$ of a finite group G (cf. Sect. 3.1) is closed under divisibility (if $k \in \pi^*(G)$ and $l \mid k$ then $l \in \pi^*(G)$), so $\pi^*(G)$ is determined by the set $\pi^*_{\max}(G)$ of its elements which are maximal in the divisibility partial order.

A sequence $(m_1, m_2, \ldots, m_r)$ of distinct positive integers is a *chain* in the divisibility partial order if $m_i \mid m_{i+1}$ for $i = 1, \ldots, r-1$. It is a *maximal chain* if $m_1 = 1$ and m_{i+1}/m_i is prime for $i = 1, \ldots, r-1$. The *top* of the chain is m_r.

For a finite group G and $m \in \pi^*(G)$, we let $G(m)$ denote the set of all elements of order m in G.

Theorem 8

(a) *A maximal clique in the order superpower graph has the form* $G(m_1) \cup G(m_2) \cup \cdots \cup G(m_r)$ *for some maximal chain whose top belongs to* $\pi^*_{\max}(G)$.

(b) *A maximal clique in the order superenhanced power graph has the form* $\bigcup_{r \mid m} G(r)$, *for some* $m \in \pi^*_{\max}(G)$.

Given this theorem, the clique numbers of these graphs are obtained by maximizing over all maximal chains with top in $\pi^*_{\max}(G)$ (in the first case) or elements of $\pi^*_{\max}(G)$ (in the second).

Proof (a) Given a clique C in the order superpower graph, let m be the largest element of C. Then the orders of all other elements of C divide m, and so they form a chain with top m. So C is contained in the union given in part (a) of the theorem, and maximality implies that the chain is maximal, its top is in $\pi^*_{\max}(G)$, and that every element of $G(k)$ for k in the chain belongs to C.

(b) Given k and l, elements of $G(k)$ and $G(l)$ are joined in the order superenhanced power graph if and only if $\mathrm{lcm}(k, l) \in \pi^*(G)$. So the set of orders of elements in a clique has a unique maximal element m. As in the preceding argument, if C is maximal, then the orders include every divisor of m and C contains $G(k)$ whenever $k \mid n$. □

For the superpower and superenhanced power graphs, we do not give a formula, but explain what maximal cliques look like.

Proposition 3 *Let G be a finite group.*

(a) *Let C be a maximal clique in the conjugacy superenhanced power graph of G. Then there exists $m \in \pi^*(G)$ such that C is the union of a conjugacy class of cyclic subgroups of order m.*

(b) *Let C' be a maximal clique in the conjugacy superpower graph of G. Then there exists $m \in \pi^*(G)$ and a maximal chain $(m_1, \ldots, m_r)$ of divisors of m and a conjugacy class of cyclic subgroups of G of order m (with union C) such that C' consists of all elements of C which have order m_i for some i with $1 \leq i \leq r$.*

Proof In either case, let m be the largest order of an element of the clique; then the order of any element of the clique divides m. (This is clear for the power graph. For the enhanced power graph, let g be an element of order m. If there are elements of order q not dividing m, then there is one (say h) joined to g in the enhanced power graph; but then there is an element of larger order to which both g and h are joined, and so it is joined to all elements of the clique. Now the result follows as in the preceding theorem. □

Note that we cannot conclude in this case that $m \in \pi^*_{\max}(G)$. For example, in the dihedral group D_4 of order 8, the cyclic group C_4 consisting of rotations is a maximal clique in either graph; the reflections fall into two conjugacy classes, each of which (together with the identity) forms a maximal clique. In general, it seems not an easy task to decide which clique (as given in the Proposition) is largest, or to give a formula for its size.

5 Open Problems and Further Directions

We mention here some questions which have arisen in this investigation which we have not been able to answer, and some further directions for research.

Problem 1 Extend these investigations to other graphs defined on groups (such as the nilpotency and solvability graphs) and other equivalence relations (such as automorphism conjugacy). This question for the conjugacy supergraphs has been studied in several papers, for example, [16, 17].

Problem 2 Complete the characterization of the classes of groups G for which a given pair of the super graphs on G coincide, especially for classes that are adjacent in a row or column of the 3×3 table (as in Theorem 3.1): see Theorems 2 and 3.

Problem 3 Characterize the dominant vertices in the order supercommuting graph.

Problem 4 What can be said about the connectedness of super graphs when dominant vertices are deleted?

Problem 5 Characterize the cliques of maximum size in the conjugacy supergraphs.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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