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A note on the moduli spaces of holomorphic and logarithmic connections over a compact Riemann surface

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Abstract
Let X be a compact Riemann surface of genus $g \geq 3$. We consider the moduli space of holomorphic connections over X and the moduli space of logarithmic connections singular over a finite subset of X with fixed residues. We determine the Chow group of these moduli spaces. We compute the global sections of the sheaves of differential operators on ample line bundles and their symmetric powers over these moduli spaces and show that they are constant under certain conditions. We show the Torelli-type theorem for the moduli space of logarithmic connections. We also describe the rational connectedness of these moduli spaces.

Keywords Logarithmic connection · Moduli space · Chow group · Differential operator · Torelli theorem · Rational variety

Mathematics Subject Classification 14D20 · 14C15 · 32C38 · 14C34 · 14E05 · 14E08

1 Introduction and statements of the results

Let X be a compact Riemann surface of genus $g \geq 3$. We consider the moduli space $\mathcal{M}_h(n)$ of rank n holomorphic connections over X. In [31] and [32], Simpson constructed the moduli space of holomorphic connections over a smooth complex projective variety.

Let

$$S = \{x_1, \ldots, x_m\}$$

be a fixed subset of X such that $x_i \neq x_j$ for all $i \neq j$. We consider the moduli space $\mathcal{M}_{lc}(n, d)$ of logarithmic connections of rank n and degree d, singular over S, with fixed residues (see Sect. 2 for the definition). The moduli space of logarithmic connections over a complex projective variety singular over a smooth normal crossing divisor has been constructed in [26].

Assumption Throughout this article, we assume that the rank n and degree d are coprime, except for the case where we deal with the moduli space of holomorphic connections.

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Several algebro-geometric invariants like the Picard group, algebraic functions of the moduli space of holomorphic and logarithmic connections have been studied, see [5–7, 30, 33, 35].

In the present article, our aim is to study the Chow group, global sections of certain sheaves, Torelli-type theorems, and rational connectedness of these moduli spaces.

The structure of the article is as follows. In Sect. 2, we define the notion of holomorphic and logarithmic connections in a holomorphic vector bundle over X, and recall their moduli spaces.

In Sect. 3, we compute the Chow group of the moduli spaces which is motivated by the following result in [10]. Let $\mathcal{U}^s(2, \mathcal{O}_X(x_0))$ be the moduli space of stable vector bundles of rank 2 with determinant $\mathcal{O}_X(x_0)$, where $x_0 \in X$. Then, in [10], the Chow group of 1-cycles on $\mathcal{U}^s(2, \mathcal{O}_X(x_0))$ has been computed, and it is proved that

$$\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{U}^s(2, \mathcal{O}_X(x_0))) \cong \mathrm{CH}_0^{\mathbb{Q}}(X). \tag{1.1}$$

Fix a holomorphic line bundle L over X of degree d. Consider the moduli space of logarithmic connections $\mathcal{M}_{lc}(n, L)$ of rank n and fixed determinant L as described in (2.9). Let $\mathcal{M}'_{lc}(n, L) \subset \mathcal{M}_{lc}(n, L)$ be the moduli space of logarithmic connections (E, D) with E stable as described in (2.10). Then, we show the following (see Theorem 3.4). For every $0 \leq l \leq (n^2 - 1)(g - 1)$, we have a canonical isomorphism

$$\mathbf{CH}_{l+(n^2-1)(g-1)}(\mathcal{M}'_{lc}(n, L)) \cong \mathbf{CH}_l(\mathcal{U}^s(n, L)). \tag{1.2}$$

As a consequence for $n = 2$, we have (see Corollary 3.6),

(1) $\mathbf{CH}_{3g-3}(\mathcal{M}'_{lc}(2, L)) \cong \mathbb{Z}$.
(2) $\mathrm{CH}^{\mathbb{Q}}_{3g-2}(\mathcal{M}'_{lc}(2, L)) \cong \mathrm{CH}^{\mathbb{Q}}_0(X)$.
(3) $\mathrm{CH}^{\mathbb{Q}}_{6g-8}(\mathcal{M}'_{lc}(2, L)) \cong \mathrm{CH}^{\mathbb{Q}}_0(X) \oplus \mathbb{Q}$.

Let L_0 be a holomorphic line bundle over X of degree zero. Let $\mathcal{M}'_h(n, L_0)$ and $\mathcal{U}^s(n, L_0)$ be the moduli space defined in (3.15) and (3.14), respectively. Then, we show that (see Theorem 3.10), for every $0 \leq l \leq (n^2 - 1)(g - 1)$, we have a canonical isomorphism

$$\mathrm{CH}^{\mathbb{Q}}_{l+(n^2-1)(g-1)}(\mathcal{M}'_h(n, L_0)) \cong \mathrm{CH}^{\mathbb{Q}}_l(\mathcal{U}^s(n, L_0)). \tag{1.3}$$

In Sect. 4, we study the global sections of certain locally free sheaves. Let $\mathcal{M}'_{lc}(n, d)$ be the moduli space described in (2.8), and ζ an ample line bundle over $\mathcal{M}'_{lc}(n, d)$. For $k \geq 0$, let $\mathcal{D}^k(\zeta)$ denote the sheaf of differential operators on ζ of order k. Consider the following natural morphism

$$p_0 : \mathcal{M}'_{lc}(n, d) \to \mathcal{U}^s(n, d) \tag{1.4}$$

sending (E, D) to E. Then, we have a morphism

$$\widetilde{p_0}_\sharp : \mathrm{H}^0(T^*\mathcal{M}'_{lc}, \mathcal{O}_{T^*\mathcal{M}'_{lc}}) \to \mathrm{H}^0(T^*\mathcal{U}^s(n, d), \mathcal{O}_{T^*\mathcal{U}^s(n,d)}). \tag{1.5}$$

of vector spaces induced from

$$\widetilde{p_0} : T^*\mathcal{U}^s(n, d) \to T^*\mathcal{M}'_{lc},$$

where $T^*\mathcal{U}^s(n, d)$ and $T^*\mathcal{M}'_{lc}$ are the cotangent bundles of $\mathcal{U}^s(n, d)$ and $\mathcal{M}'_{lc}(n, d)$, respectively.

Under the assumption that $\widetilde{p_0}_\sharp$ in (1.5) is injective, we show that (see Theorem 4.1), for every $k \geq 0$,

$$\mathrm{H}^0(\mathcal{M}'_{lc}(n, d), \mathcal{S}ym^k(\mathcal{D}^1(\zeta))) = \mathbb{C}, \tag{1.6}$$

and (see Proposition 4.2)

$$\mathrm{H}^0(\mathcal{M}'_{lc}(n,d), \mathcal{D}^k(\zeta)) = \mathbb{C}. \tag{1.7}$$

Under the same assumption, the above result is true for the moduli spaces $\mathcal{M}'_{lc}(n, L)$ (see (2.10)), $\mathcal{M}'_h(n)$ (see (2.2)) and $\mathcal{M}'_h(n, L_0)$ (see (3.15)).

In Sect. 5, we prove the Torelli-type result for the moduli space of logarithmic connections and change the notation to emphasis on X, that is,

$$\mathcal{M}_{lc}(X) = \mathcal{M}_{lc}(X, S) := \mathcal{M}_{lc}(n, d)$$

and

$$\mathcal{M}_{lc}(X, L) = \mathcal{M}_{lc}(X, S, L) := \mathcal{M}_{lc}(n, L).$$

First we show that the above moduli spaces do not depend on the choice of S (see Lemmas 5.1 and 5.2), and therefore we remove S from the notation. We show the following (see Theorem 5.6).

Let (X, S) and (Y, T) be two m-pointed compact Riemann surfaces of genus $g \geq 3$. Let $\mathcal{M}_{lc}(X, L)$ and $\mathcal{M}_{lc}(Y, L')$ be the corresponding moduli spaces of logarithmic connections. Then, $\mathcal{M}_{lc}(X, L)$ is isomorphic to $\mathcal{M}_{lc}(Y, L')$ if and only if X is isomorphic to Y.

Next, we show the universal property of the morphism (see Proposition 5.7)

$$G : \mathcal{M}_{lc}(X) \longrightarrow Pic^d(X)$$

defined by sending $(E, D) \mapsto \bigwedge^n E$. Thus, $\mathcal{M}_{lc}(X)$ determines the pair $(Pic^d(X), G)$ up to an automorphism of $Pic^d(X)$. In the end of Sect. 5, we present a Torelli-type theorem for $\mathcal{M}_{lc}(X)$, that is, let (X, S) and (Y, T) be two m-pointed compact Riemann surfaces of genus $g \geq 3$. Let $\mathcal{M}_{lc}(X)$ and $\mathcal{M}_{lc}(Y)$ be the corresponding moduli spaces of logarithmic connections. Then, $\mathcal{M}_{lc}(X)$ is isomorphic to $\mathcal{M}_{lc}(Y)$ if and only if X is isomorphic to Y.

In the last Sect. 6, we talk about rational connectedness and rationality of the moduli space. This section is motivated by the results in [19]. We show that the moduli spaces $\mathcal{M}_{lc}(n, d)$ and $\mathcal{M}_h(n)$ are not rational (see Theorems 6.3 and 6.4, respectively). And finally we show that the moduli space $\mathcal{M}_{lc}(n, L)$ is rationally connected (see Corollary 6.7).

2 Preliminaries

2.1 Moduli spaces of holomorphic and logarithmic connections

We recall the notion of holomorphic and logarithmic connection in a holomorphic vector bundle over a smooth projective curve over $\mathbb{C}$, that is, over a compact Riemann surface.

Let X be a compact Riemann surface of genus $g \geq 3$. Let E be a holomorphic vector bundle over X. A **holomorphic connection** in E is a $\mathbb{C}$-linear map

$$D : E \to E \otimes \Omega^1_X$$

which satisfies the Leibniz rule

$$D(fs) = fD(s) + df \otimes s, \tag{2.1}$$

where f is a local section of $\mathcal{O}_X$ and s is a local section of E.

A theorem due to Atiyah [1] and Weil [38], which is known as the *Atiyah–Weil criterion*, says that a holomorphic vector bundle over a compact Riemann surface admits a holomorphic connection if and only if the degree of each indecomposable component of the holomorphic vector bundle is zero. Thus, if E admits a holomorphic connection, then

$$\deg E = 0.$$

The slope $\mu(E)$ of E is defined as

$$\mu(E) = \frac{\deg E}{\mathrm{rk}(E)}.$$

A holomorphic connection D in E is said to be **semistable** (respectively, **stable**) if for every nonzero proper subbundle F of E which is invariant under D, that is,

$$D(F) \subset F \otimes \Omega^1_X,$$

we have,

$$\mu(F) \leq 0 \text{ (resp. } \mu(F) < 0),$$

where $\mu(E)$ denotes the slope of E.

Let $\mathcal{M}_h(n)$ be the moduli space of semistable holomorphic connections of rank n. Then, $\mathcal{M}_h(n)$ is a normal quasi-projective variety of dimension $2n^2(g-1)+2$. Let

$$\mathcal{M}_h^{sm}(n) \subset \mathcal{M}_h(n)$$

be the smooth locus of the variety. Let

$$\mathcal{M}'_h(n) \subset \mathcal{M}_h^{sm}(n) \tag{2.2}$$

be the open subvariety whose underlying vector bundle is stable. Then, $\mathcal{M}'_h(n)$ is an irreducible smooth quasi-projective variety of the same dimension as of $\mathcal{M}_h(n)$.

We now define the logarithmic connection. Fix a finite subset

$$S = \{x_1, \ldots, x_m\}$$

of X such that $x_i \neq x_j$ for all $i \neq j$. Let

$$Z = x_1 + \cdots + x_m$$

denote the reduced effective divisor on X associated to the finite set S. Let $\Omega^1_X(\log Z)$ denote the sheaf of logarithmic differential 1-forms along Z, see [29] for more details. For the theory of the meromorphic and logarithmic connections, we refer to two excellent sources [11] and [9].

A **logarithmic connection** on E singular over S is a $\mathbb{C}$-linear map

$$D : E \to E \otimes \Omega^1_X(\log Z) = E \otimes \Omega^1_X \otimes \mathcal{O}_X(Z) \tag{2.3}$$

which satisfies the Leibniz identity

$$D(fs) = fD(s) + df \otimes s, \tag{2.4}$$

where f is a local section of $\mathcal{O}_X$ and s is a local section of E.

A logarithmic connection D in E is said to be **semistable** (respectively, **stable**) if for every nonzero proper subbundle F of E which is invariant under D, that is,

$$D(F) \subset F \otimes \Omega^1_X(\log Z),$$

we have,

$$\mu(F) \leq \mu(E) (\text{resp. } \mu(F) < \mu(E)),$$

where $\mu(E)$ denotes the slope of E.

We next describe the notion of residues of a logarithmic connection D in E singular over S. We will denote the fibre of E over any point $x \in X$ by $E(x)$.

Let $v \in E(x_\beta)$ be any vector in the fibre of E over x_β. Let U be an open set around x_β and $s : U \to E$ be a holomorphic section of E over U such that $s(x_\beta) = v$. Consider the following composition

$$\Gamma(U, E) \to \Gamma(U, E \otimes \Omega^1_X \otimes \mathcal{O}_X(S)) \to E \otimes \Omega^1_X \otimes \mathcal{O}_X(S)(x_\beta) = E(x_\beta),$$

where the equality is given because for any $x_\beta \in S$, the fibre $\Omega^1_X \otimes \mathcal{O}_X(S)(x_\beta)$ is canonically identified with $\mathbb{C}$ by sending a meromorphic form to its residue at x_β. Then, we have an endomorphism on $E(x_\beta)$ sending v to $D(s)(x_\beta)$. We need to check that this endomorphism is well defined. Let $s' : U \to E$ be another holomorphic section such that $s'(x_\beta) = v$. Then,

$$(s - s')(x_\beta) = v - v = 0.$$

Let t be a local coordinate at x_β on U such that $t(x_\beta) = 0$, that is, the coordinate system (U, t) is centred at x_β. Since $s - s' \in \Gamma(U, E)$ and $(s - s')(x_\beta) = 0$, $s - s' = t\sigma$ for some $\sigma \in \Gamma(U, E)$. Now,

$$\begin{aligned} D(s - s') &= D(t\sigma) = tD(\sigma) + dt \otimes \sigma \\ &= tD(\sigma) + t(\frac{dt}{t} \otimes \sigma), \end{aligned}$$

and hence $D(s - s')(x_\beta) = 0$, that is, $D(s)(x_\beta) = D(s')(x_\beta)$.

Thus, we have a well-defined endomorphism, denoted by

$$Res(D, x_\beta) \in \mathrm{End}(E)(x_\beta) = \mathrm{End}(E(x_\beta)) \tag{2.5}$$

that sends v to $D(s)(x_\beta)$. This endomorphism $Res(D, x_\beta)$ is called the **residue** of the logarithmic connection D at the point $x_\beta \in S$ (see [11] for the details).

From [27, Theorem 3], for a logarithmic connection D singular over S, we have

$$\deg E + \sum_{j=1}^{m} \mathrm{Tr}(Res(D, x_j)) = 0, \tag{2.6}$$

where $\deg E$ denotes the degree of E, and $\mathrm{Tr}(Res(D, x_j))$ denotes the trace of the endomorphism $Res(D, x_j) \in \mathrm{End}(E(x_j))$, for all $j = 1, \ldots, m$.

Let $\mathbf{LC}(E)$ denote the set of all logarithmic connections in E singular over S. Then, $\mathbf{LC}(E)$ is an affine space modelled over the vector space $\mathrm{H}^0(X, \mathrm{End}(E) \otimes \Omega^1_X(\log Z))$, that is, if D is any logarithmic connection in E singular over S, then

$$\mathbf{LC}(E) = D + \mathrm{H}^0(X, \mathrm{End}(E) \otimes \Omega^1_X(\log Z)).$$

Recall that an endomorphism $\phi \in \mathrm{End}(E(x_j))$ is said to be a rigid endomorphism if for every global endomorphism $\alpha \in \mathrm{H}^0(X, \mathrm{End}(E))$ we have

$$\phi \circ \alpha(x_j) = \alpha(x_j) \circ \phi,$$

where $\alpha(x_j) : E(x_j) \to E(x_j)$ is an endomorphism.

In what follows, we fix a rigid endomorphism $\Phi_j \in \mathrm{End}(E(x_j))$, for every $j = 1, \ldots, m$, such that for every direct summand $F \subset E$, we have

$$\deg F + \sum_{j=1}^{m} \mathrm{Tr}(\Phi_j|_{F(x_j)}) = 0. \tag{2.7}$$

Here, $\mathrm{Tr}(\Phi_j|_{F(x_j)})$ makes sense, because from [8, Theorem 1.3 (1)], for a rigid endomorphism $\Phi_j \in \mathrm{End}(E(x_j))$, and for every direct summand F of E, we have

$$\Phi_j(F(x_j)) \subset F(x_j).$$

Let $\mathbf{LC}(E; \Phi_1, \ldots, \Phi_m)$ denote the set of all logarithmic connections singular over S with fixed residues Φ_j for all $j = 1, \ldots, m$, that is,

$$\begin{aligned}&\mathbf{LC}(E; \Phi_1, \ldots, \Phi_m)\\ &\quad = \{D \mid D \text{ is a logarithmic connection in } E \text{ with } Res(D, x_j) = \Phi_j \text{ for all } j = 1, \ldots, m\}.\end{aligned}$$

Then, $\mathbf{LC}(E; \Phi_1, \ldots, \Phi_m)$ is an affine space modelled over $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$.

Notice the difference between vector spaces when residue is fixed and otherwise.

We impose some more conditions on the residues Φ_j for $1 \leq j \leq m$ to get a 'well behaved' moduli space of logarithmic connections singular over S with fixed residues.

Suppose that the residues (rigid endomorphisms) Φ_j for every $j = 1, \ldots, m$ satisfy the following condition.

(P1): For every nonzero subbundle $F \subset E$,

$$\Phi_j(F(x_j)) \subset F(x_j),$$

and

$$\frac{\mathrm{Tr}(\Phi_j|_{F(x_j)})}{\mathrm{rk}(F)} = \frac{\mathrm{Tr}(\Phi_j)}{\mathrm{rk}(E)}.$$

If we take $\Phi_j = \alpha_j \mathbf{1}_{E(x_j)}$, where $\alpha_j \in \mathbb{C}$ and $\mathbf{1}_{E(x_j)}$ is the identity morphism on $E(x_j)$, for every $1 \leq j \leq m$, then $\{\Phi_j\}_{1 \leq j \leq m}$ satisfies (P1). In what follows, we take $\Phi_j = \alpha_j \mathbf{1}_{E(x_j)}$ for every $j = 1, \ldots, m$.

We have an easy result.

Lemma 2.1 *Let* $D \in \mathbf{LC}(E; \Phi_1, \ldots, \Phi_m)$ *with* $\{\Phi_j\}_{1 \leq j \leq m}$ *satisfying (P1). Then,* D *is semistable. Moreover, if* $(\deg E, \mathrm{rk}(E)) = 1$*, then* D *is stable.*

A logarithmic connection D in a holomorphic vector bundle E is called **irreducible** if for any holomorphic subbundle F of E with $D(F) \subset F \otimes \Omega^1_X(\log Z)$, then either $F = E$ or $F = 0$.

If $D \in \mathbf{LC}(E; \Phi_1, \ldots, \Phi_m)$ satisfies (P1), and $(\deg E, \mathrm{rk}(E)) = 1$, then D is irreducible.

Let $\mathcal{M}_{lc}(n, d)$ denote the moduli space which parametrises the isomorphic class of pairs (E, D), where by a pair (E, D) we mean that

(1) E is a holomorphic vector bundle of rank n and degree d over X, such that $(n, d) = 1$.
(2) D is a logarithmic connection with fixed residues $Res(D, x_j) = \Phi_j$ satisfying (P1).

Two pairs (E, D) and (E', D') satisfying the above conditions (1) and (2) are said to be isomorphic if there exists an isomorphism $\Psi : E \to E'$ such that the following diagram

$$\begin{array}{ccc} E & \xrightarrow{D} & E \otimes \Omega^1_X(\log Z) \\ \downarrow{\Psi} & & \downarrow{\Psi \otimes \mathbf{1}_{\Omega^1_X(\log Z)}} \\ E' & \xrightarrow{D'} & E' \otimes \Omega^1_X(\log Z) \end{array}$$

commutes.

From [26, Theorem 3.5], the moduli space $\mathcal{M}_{lc}(n, d)$ is a separated quasi-projective scheme over $\mathbb{C}$. Since $\{\Phi_j\}_{1 \leq j \leq m}$ satisfies (2.7), from [8, Theorem 1.3 (2)] the moduli space $\mathcal{M}_{lc}(n, d)$ is non-empty.

As we have observed that every logarithmic connection (E, D) in $\mathcal{M}_{lc}(n, d)$ is irreducible, and the singular points of $\mathcal{M}_{lc}(n, d)$ correspond to reducible logarithmic connections [6, p.n. 790], the moduli space $\mathcal{M}_{lc}(n, d)$ is smooth. Since genus g of X is greater than or equal to 3, the moduli space $\mathcal{M}_{lc}(n, d)$ is irreducible [32, Theorem 11.1].

Altogether, $\mathcal{M}_{lc}(n, d)$ is an irreducible smooth quasi-projective variety of dimension $2n^2(g-1)+2$. Let

$$\mathcal{M}'_{lc}(n, d) \subset \mathcal{M}_{lc}(n, d) \tag{2.8}$$

be the moduli space of logarithmic connections whose underlying vector bundles are stable. Then, from [22, p.635, Theorem 2.8(A)] $\mathcal{M}'_{lc}(n, d)$ is an open subset of $\mathcal{M}_{lc}(n, d)$, and hence an irreducible smooth quasi-projective variety of dimension $2n^2(g-1)+2$.

Fix a holomorphic line bundle L over X of degree d, and a logarithmic connection D_L on L singular over S with residues $Res(D_L, x_j) = \mathrm{Tr}(\Phi_j)$ for all $j = 1, \ldots, m$. Let

$$\mathcal{M}_{lc}(n, L) \subset \mathcal{M}_{lc}(n, d) \tag{2.9}$$

be the moduli space parametrising isomorphism class of pairs (E, D) such that

$$\left(\bigwedge^n E, \tilde{D}\right) \cong (L, D_L),$$

where $\tilde{D}$ is the logarithmic connection on $\bigwedge^n E$ induced by D. Then, $\mathcal{M}_{lc}(n, L)$ is an irreducible smooth quasi-projective variety of dimension $2(n^2-1)(g-1)$.

Let

$$\mathcal{M}'_{lc}(n, L) \subset \mathcal{M}_{lc}(n, L) \tag{2.10}$$

be the moduli space of logarithmic connections (E, D) with E stable.

3 Chow group of the moduli spaces

In this section, we determine the Chow groups of the moduli spaces $\mathcal{M}'_{lc}(n, L)$, $\mathcal{M}'_{lc}(n, d)$, $\mathcal{M}'_h(n)$ and $\mathcal{M}'_h(n, L_0)$.

Before that, we recall the definition of Chow group of a quasi-projective scheme over a field (see [15, 37]).

Let $\mathcal{X}$ be a quasi-projective scheme over a field K. Let $Z_k(\mathcal{X})$ be the free abelian group generated by the reduced and irreducible k-dimensional closed subvarieties of $\mathcal{X}$, or we can say the free abelian group generated by k-dimensional closed integral subschemes of $\mathcal{X}$. An element of $Z_k(\mathcal{X})$ is called a k-dimensional algebraic cycle on $\mathcal{X}$.

Let $f \in K(\mathcal{X})^*$. Then, we have a divisor $\mathrm{div}(f)$ on $\mathcal{X}$ associated to the nonzero rational function f on $\mathcal{X}$.

A k-cycle α is **rationally equivalent to zero**, written $\alpha \sim 0$, if there is a finite number of $(k+1)$-dimensional subvarieties (that is, closed integral subschemes) W_i of $\mathcal{X}$ and $f_i \in K(W_i)^*$, such that

$$\alpha = \sum_i \mathrm{div}(f_i).$$

Since $0 = \operatorname{div}(1)$ and $\operatorname{div}(f^{-1}) = -\operatorname{div}(f)$, the cycles rationally equivalent to zero form a subgroup $Z_k(\mathcal{X})_{rat}$ of $Z_k(\mathcal{X})$.

We define the quotient group

$$\mathbf{CH}_k(\mathcal{X}) := Z_k(\mathcal{X})/Z_k(\mathcal{X})_{rat},$$

and call it the Chow group of k-cycles on $\mathcal{X}$. A graded sum is denoted by

$$\mathbf{CH}_*(\mathcal{X}) = \bigoplus_{k=0}^{\dim(\mathcal{X})} \mathbf{CH}_k(\mathcal{X}).$$

The Chow group of k-cycles on $\mathcal{X}$ with rational coefficients will be denoted by $\mathrm{CH}_k^{\mathbb{Q}}(\mathcal{X})$.

Let $\mathcal{U}^s(n, L)$ be the moduli space of stable vector bundles of rank n with $\bigwedge^n E \cong L$. Then, $\mathcal{U}^s(n, L)$ is a smooth projective variety of dimension $(n^2 - 1)(g - 1)$, as we have assumed n and $\deg(L) = d$ are coprime.

Let $x_0 \in X$, and $\mathcal{O}_X(x_0)$ the line bundle on X associated with the reduced effective divisor x_0. For $n = 2$, we have $\mathcal{U}^s(2, \mathcal{O}_X(x_0))$ the moduli space of stable vector bundles of rank 2 over X whose determinant is $\mathcal{O}_X(x_0)$.

In [3], it was shown that

$$\mathrm{CH}^{\mathbb{Q}}_{3g-5}(\mathcal{U}^s(2, \mathcal{O}_X(x_0))) \cong \mathrm{CH}^{\mathbb{Q}}_0(X) \bigoplus \mathbb{Q}. \tag{3.1}$$

In [10], Choe and Hwang computed the Chow group of 1-cycles on $\mathcal{U}^s(2, \mathcal{O}_X(x_0))$, and they proved that

$$\mathrm{CH}^{\mathbb{Q}}_1(\mathcal{U}^s(2, \mathcal{O}_X(x_0))) \cong \mathrm{CH}^{\mathbb{Q}}_0(X). \tag{3.2}$$

Let M be a holomorphic line bundle over X of degree d' and

$$M_0 = \mathcal{O}_X(x_0) \otimes M^{\otimes 2}.$$

Then, $\deg(M_0) = 2d' + 1$. Define a map

$$\Psi_M : \mathcal{U}^s(2, \mathcal{O}_X(x_0)) \to \mathcal{U}^s(2, M_0)$$

by

$$\Psi_M([E]) = [E \otimes M].$$

The map is well defined, and in fact, an isomorphism of varieties.

Thus, the above results (3.1) and (3.2) are true for $\mathcal{U}(2, L)$, where $\deg(L)$ is odd.

Define

$$p : \mathcal{M}'_{lc}(n, L) \to \mathcal{U}^s(n, L) \tag{3.3}$$

by sending $(E, D) \mapsto E$, that is, p is the forgetful map which forgets its logarithmic structure.

For every $E \in \mathcal{U}(n, L)$, $p^{-1}(E)$ is an affine space modelled over $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{ad}(E))$, where $\mathrm{ad}(E) \subset \mathrm{End}(E)$ is the subbundle consisting of endomorphisms of E whose trace is zero. Actually, p is a fibre bundle and using Riemann–Roch theorem and Serre duality it can be easily computed that the dimension of $p^{-1}(E)$ is $(n^2 - 1)(g - 1)$.

Let $\Omega^1_{\mathcal{U}^s(n,L)}$ denote the holomorphic cotangent bundle on $\mathcal{U}^s(n, L)$. Since n and d are coprime, there exists a universal bundle $\mathcal{E}$ on $\mathcal{U}^s(n, L) \times X$. Let $p_1 : \mathcal{U}^s(n, L) \times X \to \mathcal{U}^s(n, L)$ and $p_2 : \mathcal{U}^s(n, L) \times X \to X$ be the first and second projection, respectively. Then, from infinitesimal deformation theory and local property of moduli space, we have $R^1 p_1(\mathrm{ad}(\mathcal{E})) = T\mathcal{U}^s(n, L)$. Moreover, we have $p_{1*}(\mathrm{ad}(\mathcal{E}) \otimes p_2^*(\Omega^1_X)) = \Omega^1_{\mathcal{U}^s(n,L)}$. Now, from the fact that $\mathbf{LC}(E; \Phi_1, \ldots, \Phi_m)$ is an affine space modelled over $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$ as observed above, we have

Lemma 3.1 *$\mathcal{M}'_{lc}(n, L)$ is an $\Omega^1_{\mathcal{U}^s(n,L)}$-torsor over $\mathcal{U}^s(n, L)$.*

We state two standard lemmas from the theory of Chow groups which we will use to compute the Chow groups of moduli spaces.

Let Y be a variety over a field K. Let $i : F \to Y$ be the inclusion of a closed subscheme. Let $j : U = Y \setminus F \to Y$ be the inclusion of the complement. Since j is an open immersion, it is flat, and i is a closed immersion, it is proper. Therefore, we have morphisms $j^* : \mathbf{CH}_k(Y) \to \mathbf{CH}_k(U)$ and $i_* : \mathbf{CH}_k(F) \to \mathbf{CH}_k(Y)$ of Chow groups and $j^* \circ i_* = 0$, since the cycles supported on F do not intersect U. Thus, we have what is called **localisation sequence**.

Lemma 3.2 [37, Lemma 9.12] *The following sequence of abelian groups*

$$\mathbf{CH}_l(F) \xrightarrow{i_*} \mathbf{CH}_l(Y) \xrightarrow{j^*} \mathbf{CH}_l(U) \to 0. \tag{3.4}$$

is exact for every $l = 0, \ldots, \dim(Y)$.

Theorem 3.3 [37, Theorem 9.25] *Let $\pi : \mathbb{P}(E) \to Y$ be a projective bundle with rank $\mathrm{rk}(E) = r$. Then, the map*

$$\bigoplus_{k=0}^{r-1} h^k \pi^* : \bigoplus_{k=0}^{r-1} \mathbf{CH}_{l-r+1+k}(Y) \to \mathbf{CH}_l(\mathbb{P}(E)) \tag{3.5}$$

is an isomorphism, where $h \in \mathrm{Pic}(\mathbb{P}(E))$ denotes the class of the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$.

Theorem 3.4 *For every $0 \le l \le (n^2-1)(g-1)$, we have a canonical isomorphism*

$$\mathbf{CH}_{l+(n^2-1)(g-1)}(\mathcal{M}'_{lc}(n, L)) \cong \mathbf{CH}_l(\mathcal{U}^s(n, L)). \tag{3.6}$$

Proof Let $\mathcal{G}$ be an affine bundle modelled on a vector bundle $\mathcal{H}$ of rank $r = (n^2-1)(g-1)$ over $\mathcal{U}^s(n, L)$. Now, using the standard inclusion of the affine group in $\mathrm{GL}(r+1, \mathbb{C})$, we obtain a vector bundle $\mathcal{F}$ of rank $r+1$ together with an embedding of $\mathcal{G}$ in $\mathbb{P}(\mathcal{F})$ as an open subset with complement $\mathbb{P}(\mathcal{H}^\vee)$.

Since $\mathcal{M}'_{lc}(n, L)$ is an $\Omega^1_{\mathcal{U}^s(n,L)}$-torsor over $\mathcal{U}^s(n, L)$ (see Lemma 3.1), the above construction gives an algebraic vector bundle $\mathcal{F}$ over $\mathcal{U}^s(n, L)$ with $\mathcal{M}'_{lc}(n, L)$ embedded in $\mathbb{P}(\mathcal{F})$ such that the complement $\mathbb{P}(\mathcal{F}) \setminus \mathcal{M}'_{lc}(n, L)$ is a hyperplane $\mathbf{H}$ at infinity. Now, the hyperplane at infinity $\mathbf{H}$ is canonically identified with the total space of the projective bundle $\mathbb{P}(T\mathcal{U}^s(n, L))$, the space of all hyperplanes in the fibre of the tangent bundle $T\mathcal{U}^s(n, L)$.

Putting $F = \mathbb{P}(T\mathcal{U}^s(n, L))$, $Y = \mathbb{P}(\mathcal{F})$ and $U = \mathcal{M}'_{lc}(n, L)$ in Lemma 3.2, we get an exact sequence

$$\mathbf{CH}_l(\mathbb{P}(T\mathcal{U}^s(n, L))) \xrightarrow{i_*} \mathbf{CH}_l(\mathbb{P}(\mathcal{F})) \xrightarrow{j^*} \mathbf{CH}_l(\mathcal{M}'_{lc}(n, L)) \to 0. \tag{3.7}$$

of abelian groups for every $l = 0, \ldots, \dim(\mathbb{P}(\mathcal{F})) = 2(n^2-1)(g-1)$.

Since $\mathrm{rk}(\mathcal{F}) = (n^2-1)(g-1)+1$, and $\mathrm{rk}(T\mathcal{U}^s(n, L)) = (n^2-1)(g-1)$, from Theorem 3.3, we have following isomorphisms

$$\mathbf{CH}_l(\mathbb{P}(\mathcal{F})) \cong \bigoplus_{k=0}^{(n^2-1)(g-1)} \mathbf{CH}_{l-(n^2-1)(g-1)+k}(\mathcal{U}^s(n, L)) \tag{3.8}$$

and

$$\mathbf{CH}_l(\mathbb{P}(T\mathcal{U}^s(n, L))) \cong \bigoplus_{k=0}^{(n^2-1)(g-1)-1} \mathbf{CH}_{l-(n^2-1)(g-1)+1+k}(\mathcal{U}^s(n, L)). \tag{3.9}$$

From (3.7), (3.8) and (3.9), we get an exact sequence

$$\begin{aligned} &\bigoplus_{k=0}^{(n^2-1)(g-1)-1} \mathbf{CH}_{l-(n^2-1)(g-1)+1+k}(\mathcal{U}^s(n,L)) \xrightarrow{i_*} \\ &\bigoplus_{k=0}^{(n^2-1)(g-1)} \mathbf{CH}_{l-(n^2-1)(g-1)+k}(\mathcal{U}^s(n,L)) \xrightarrow{j^*} \mathbf{CH}_l(\mathcal{M}'_{lc}(n,L)) \to 0, \end{aligned} \tag{3.10}$$

which is actually a short exact sequence, because i_* is injective. Thus, we have

$$\mathbf{CH}_l(\mathcal{M}'_{lc}(n,L)) \cong \mathbf{CH}_{l-(n^2-1)(g-1)}(\mathcal{U}^s(n,L)), \tag{3.11}$$

for every $(n^2-1)(g-1) \leq l \leq 2(n^2-1)(g-1)$. Now, rescaling l, we will get the desired result, and this completes the proof. □

Corollary 3.5 *For $l = 2(n^2-1)(g-1)-1$, we have*

$$\mathbf{CH}_l(\mathcal{M}'_{lc}(n,L)) \cong \mathbb{Z}.$$

Proof See [35, Proposition 5.3]. □

Corollary 3.6 *For $n = 2$, we have*

(1) $\mathbf{CH}_{3g-3}(\mathcal{M}'_{lc}(2,L)) \cong \mathbb{Z}$.
(2) $\mathrm{CH}^{\mathbb{Q}}_{3g-2}(\mathcal{M}'_{lc}(2,L)) \cong \mathrm{CH}^{\mathbb{Q}}_0(X)$.
(3) $\mathrm{CH}^{\mathbb{Q}}_{6g-8}(\mathcal{M}'_{lc}(2,L)) \cong \mathrm{CH}^{\mathbb{Q}}_0(X) \oplus \mathbb{Q}$.

Proof From Theorem 3.4, and Eqs. (3.1) and (3.2), we conclude the corollary. □

Next, let $\mathcal{U}^s(n,d)$ be the moduli space of stable vector bundle of rank n and degree d. Consider the following natural morphism

$$p_0 : \mathcal{M}'_{lc}(n,d) \to \mathcal{U}^s(n,d) \tag{3.12}$$

sending (E,D) to E. Then, $p_0^{-1}(E)$ is an affine space modelled over the vector space $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$. Since E is stable, the dimension of the vector space $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$ is $n^2(g-1)+1$. In view of [35, Theorem 1.1], we can show a result similar to Theorem 3.4, which interprets the Chow groups of $\mathcal{M}'_{lc}(n,d)$ in terms of Chow groups of $\mathcal{U}^s(n,d)$.

Theorem 3.7 *For every $0 \leq l \leq n^2(g-1)+1$, we have canonical isomorphism*

$$\mathbf{CH}_{l+n^2(g-1)+1}(\mathcal{M}'_{lc}(n,d)) \cong \mathbf{CH}_l(\mathcal{U}^s(n,d)). \tag{3.13}$$

Now, we compute the same for the moduli space of holomorphic connections. Fix a holomorphic line bundle L_0 of degree 0 on X. Let $\mathcal{U}(n,L_0)$ denote the moduli space of S-equivalence classes of semistable vector bundles of rank n and determinant $\bigwedge^n E \cong L_0$. Then, the moduli space $\mathcal{U}(n,L_0)$ is known to be an irreducible normal projective variety of dimension $(n^2-1)(g-1)$.

Let

$$\mathcal{U}^s(n,L_0) \subset \mathcal{U}(n,L_0) \tag{3.14}$$

be the open subvariety parametrising the stable bundles on X. This open subvariety coincides with the smooth locus of $\mathcal{U}(n,L_0)$ follows from [23, p. 20, Theorem 1].

Fix a holomorphic connection D^{L_0} on L_0. Let $\mathcal{M}_h(n, L_0)$ be the moduli space of holomorphic connections parametrising the isomorphism classes of the pairs (E, D) where E is a holomorphic vector bundle of rank n with

$$\left(\bigwedge^n E, \tilde{D}\right) \cong (L_0, D^{L_0}),$$

and $\tilde{D}$ is a holomorphic connection on $\bigwedge^n E$ induced from D. Then, $\mathcal{M}_h(n, L_0)$ is an irreducible normal quasi-projective variety of dimension $2(n^2-1)(g-1)$. Let

$$\mathcal{M}_h^{sm}(n, L_0) \subset \mathcal{M}_h(n, L_0)$$

be the smooth locus of $\mathcal{M}_h(n, L_0)$. Let

$$\mathcal{M}'_h(n, L_0) \subset \mathcal{M}_h^{sm}(n, L_0) \tag{3.15}$$

be the subset consisting of holomorphic connections whose underlying vector bundle is stable. Then, $\mathcal{M}'_h(n, L_0)$ is an irreducible smooth quasi-projective variety of dimension $2(n^2-1)(g-1)$.

Let

$$q : \mathcal{M}'_h(n, L_0) \to \mathcal{U}^s(n, L_0) \tag{3.16}$$

be the forgetful map which forgets the holomorphic connection. Then, for every $E \in \mathcal{U}^s(n, L_0)$, $q^{-1}(E)$ is an affine space modelled over $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{ad}(E))$. In fact, $\mathcal{M}'_h(n, L_0)$ is an $\Omega^1_{\mathcal{U}(n,L_0)}$-torsor on $\mathcal{U}^s(n, L_0)$.

Let Y be an N-dimensional smooth quasi-projective variety. Then, the Picard group $\mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ can be identified with $\mathrm{CH}^{\mathbb{Q}}_{N-1}(Y)$. Thus, it is enough to compute $\mathrm{Pic}(Y)$.

The morphism q as defined in (3.16) induces a homomorphism

$$q^* : \mathrm{Pic}(\mathcal{U}^s(n, L_0)) \to \mathrm{Pic}(\mathcal{M}'_h(n, L_0)) \tag{3.17}$$

of Picard groups given by sending a line bundle M to its pull-back q^*M. Using the similar techniques as in [35, Theorem 1.2], we can show the following.

Proposition 3.8 *The homomorphism* $q^* : \mathrm{Pic}(\mathcal{U}^s(n, L_0)) \to \mathrm{Pic}(\mathcal{M}'_h(n, L_0))$ *is an isomorphism of groups.*

Since $\mathrm{Pic}(\mathcal{U}^s(n, L_0)) \cong \mathbb{Z}$ (see [14]), we have

Corollary 3.9 *For* $l = 2(n^2-1)(g-1) - 1$, *we have*

$$\mathbf{CH}_l(\mathcal{M}'_h(n, L_0)) \cong \mathbb{Z}.$$

Using the exactly similar steps as in Theorem 3.4, we can prove the following.

Theorem 3.10 *For every* $0 \le l \le (n^2-1)(g-1)$, *we have canonical isomorphisms*

$$\mathbf{CH}_{l+(n^2-1)(g-1)}(\mathcal{M}'_h(n, L_0)) \cong \mathbf{CH}_l(\mathcal{U}^s(n, L_0)). \tag{3.18}$$

Next, let $\mathcal{U}^s(n) := \mathcal{U}^s(n, 0)$ be the moduli space of stable bundles of rank n and degree zero. Then, $\mathcal{U}^s(n)$ is an irreducible smooth projective variety of dimension $n^2(g-1)+1$. Again, we have a natural morphism

$$q_0 : \mathcal{M}'_h(n) \to \mathcal{U}^s(n) \tag{3.19}$$

of varieties which forgets the holomorphic connection. Using the same method as above, we have the following theorem similar to Theorem 3.7.

Theorem 3.11 *For every* $0 \le l \le n^2(g-1)+1$*, we have canonical isomorphism*

$$\mathbf{CH}_{l+n^2(g-1)+1}(\mathcal{M}'_h(n)) \cong \mathbf{CH}_l(\mathcal{U}^s(n)). \tag{3.20}$$

4 Differential operators on the moduli spaces

In [4], Biswas studied the global sections of sheaves of differential operators on an ample line bundle over a polarised abelian variety. Also, in [34], Hitchin variety is defined and global sections of the sheaf of k-th-order differential operators, and symmetric powers of the sheaf of first-order differential operators on a line bundle over a Hitchin variety have been studied. The moduli space of stable vector bundles over a compact Riemann surface is an example of Hitchin variety. The moduli spaces of holomorphic and logarithmic connections are not Hitchin varieties. In this section, we study the global sections of certain sheaves over the four moduli spaces $\mathcal{M}'_{lc}(n,d)$, $\mathcal{M}'_h(n)$, $\mathcal{M}'_{lc}(n,L)$ and $\mathcal{M}'_h(n,L_0)$ which we have defined in previous sections.

Let ζ be an ample line bundle over $\mathcal{M}'_{lc}(n,d)$. Let $k \ge 0$ be an integer. Recall that a differential operator of order k on ζ is a $\mathbb{C}$-linear map

$$\theta : \zeta \to \zeta \tag{4.1}$$

such that for every open subset U of $\mathcal{M}'_{lc}(n,d)$ and for every $f \in \mathcal{O}_{\mathcal{M}'_{lc}(n,d)}(U)$, the bracket

$$[\theta|_U, f] : \zeta|_U \to \zeta|_U$$

defined as

$$[\theta|_U, f]_V(s) = \theta(f|_V s) - \theta|_V P_V(s)$$

is a differential operator of order $k-1$, for every open subset V of U, and for all $s \in \zeta(V)$, where a differential operator of order zero on ζ is just a $\mathcal{O}_{\mathcal{M}'_{lc}(n,d)}$-module homomorphism (see [17] and [28] for the definition and properties of differential operators).

For $k \ge 0$, let $\mathcal{D}^k(\zeta)$ denote the sheaf of differential operators on ζ of order k. In fact, $\mathcal{D}^k(\zeta)$ is a locally free sheaf with $\mathcal{D}^0(\zeta) = \mathcal{O}_{\mathcal{M}'_{lc}(n,d)}$. Given a first-order differential operator θ on ζ, we get a section of the tangent bundle $T\mathcal{M}'_{lc}(n,d)$ denoted by $\sigma_1(\theta)$, where σ_1 is called the symbol of a first-order differential operator. For simplicity, we shall denote $T\mathcal{M}'_{lc}(n,d)$ by $T\mathcal{M}'_{lc}$. Thus, consider the symbol operator $\sigma_1 : \mathcal{D}^1(\zeta) \to T\mathcal{M}'_{lc}$. This induces a morphism

$$\mathcal{S}ym^k(\sigma_1) : \mathcal{S}ym^k(\mathcal{D}^1(\zeta)) \to \mathcal{S}ym^k(T\mathcal{M}'_{lc})$$

of k-th symmetric powers. Now, because of the following composition

$$\mathcal{O}_{\mathcal{M}'_{lc}(n,d)} \otimes \mathcal{S}ym^{k-1}(\mathcal{D}^1(\zeta)) \hookrightarrow \mathcal{D}^1(\zeta) \otimes \mathcal{S}ym^{k-1}(\mathcal{D}^1(\zeta)) \to \mathcal{S}ym^k(\mathcal{D}^1(\zeta)),$$

we have

$$\mathcal{S}ym^{k-1}(\mathcal{D}^1(\zeta)) \subset \mathcal{S}ym^k(\mathcal{D}^1(\zeta)) \quad \text{for all } k \ge 1. \tag{4.2}$$

Thus, we get a short exact sequence

$$0 \to \mathcal{S}ym^{k-1}(\mathcal{D}^1(\zeta)) \to \mathcal{S}ym^k(\mathcal{D}^1(\zeta)) \xrightarrow{\mathcal{S}ym^k(\sigma_1)} \mathcal{S}ym^k(T\mathcal{M}'_{lc}) \to 0. \tag{4.3}$$

Thus, we have

$$\mathcal{S}ym^k(\mathcal{D}^1(\zeta))/\mathcal{S}ym^{k-1}(\mathcal{D}^1(\zeta)) \cong \mathcal{S}ym^k(T\mathcal{M}'_{lc}) \quad \text{for all } k \ge 1. \tag{4.4}$$

From (4.2), we have the following chain of $\mathbb{C}$-vector spaces

$$\mathrm{H}^0(\mathcal{M}'_{lc}(n,d), \mathcal{O}_{\mathcal{M}'_{lc}(n,d)}) \subset \mathrm{H}^0(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^1(\mathcal{D}^1(\zeta))) \subset \dots \tag{4.5}$$

Consider the following commutative diagram,

$$\begin{array}{ccc} T^*\mathcal{M}'_{lc} & \xleftarrow{\widetilde{p_0}} & T^*\mathcal{U}(n,d) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \mathcal{M}'_{lc}(n,d) & \xrightarrow{p_0} & \mathcal{U}(n,d) \end{array} \tag{4.6}$$

where π, π' are the canonical projections and $\widetilde{p_0}$ is induced from p_0 as defined in (3.12). Thus, we have a morphism

$$\widetilde{p_0}_\sharp : \mathrm{H}^0(T^*\mathcal{M}'_{lc}, \mathcal{O}_{T^*\mathcal{M}'_{lc}}) \to \mathrm{H}^0(T^*\mathcal{U}^s(n,d), \mathcal{O}_{T^*\mathcal{U}^s(n,d)}). \tag{4.7}$$

of vector spaces induced from $\widetilde{p_0}$.

Theorem 4.1 *Suppose that $\widetilde{p_0}_\sharp$ in (4.7) is an injective morphism. Then, for every $k \geq 0$, we have*

$$\mathrm{H}^0(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^k(\mathcal{D}^1(\zeta))) = \mathbb{C}. \tag{4.8}$$

Proof Let

$$\mathcal{M}_X^{lc} := \mathcal{M}_{lc}(1,d) \tag{4.9}$$

be the moduli space of rank one logarithmic connections singular over S, with fixed residues $\mathrm{Tr}(\Phi_j)$ for every $j = 1, \dots, m$, for more details, see [30] and [33]. Then, there is a natural morphism of varieties

$$\det : \mathcal{M}'_{lc}(n,d) \longrightarrow \mathcal{M}_X^{lc} \tag{4.10}$$

sending $(E, D) \mapsto (\bigwedge^n E, \widetilde{D})$, where $\widetilde{D}$ is the induced logarithmic connection on $\bigwedge^n E$. For any pair $(L, \nabla) \in \mathcal{M}_X^{lc}$,

$$\det{}^{-1}((L, \nabla)) = \mathcal{M}'_{lc}(n, L).$$

From [30, Theorem 2], we have

$$\mathrm{H}^0(\mathcal{M}_X^{lc}, \mathcal{O}_{\mathcal{M}_X^{lc}}) = \mathbb{C},$$

and from [35, Theorem 1.4], we have

$$\mathrm{H}^0(\mathcal{M}'_{lc}(n,L), \mathcal{O}_{\mathcal{M}'_{lc}(n,L)}) = \mathbb{C}.$$

Combining both the results and using (4.10), we have

$$\mathrm{H}^0(\mathcal{M}'_{lc}(n,d), \mathcal{O}_{\mathcal{M}'_{lc}(n,d)}) = \mathbb{C}.$$

Thus, from (4.5), it is enough to show that for every $k \geq 0$, the inclusion

$$\mathrm{H}^0(\mathcal{M}'_{lc}(n,d), \mathcal{O}_{\mathcal{M}'_{lc}(n,d)}) \to \mathrm{H}^0(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^k(\mathcal{D}^1(\zeta)))$$

is an isomorphism. From the isomorphism in (4.4), we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}ym^{k-1}(\mathcal{D}^1(\zeta)) & \longrightarrow & \mathcal{S}ym^{k}(\mathcal{D}^1(\zeta)) & \xrightarrow{\mathcal{S}ym^k(\sigma_1)} & \mathcal{S}ym^{k}(T\mathcal{M}'_{lc}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{S}ym^{k-1}(T\mathcal{M}'_{lc}) & \longrightarrow & \frac{\mathcal{S}ym^{k}(\mathcal{D}^1(\zeta))}{\mathcal{S}ym^{k-2}(\mathcal{D}^1(\zeta))} & \longrightarrow & \mathcal{S}ym^{k}(T\mathcal{M}'_{lc}) & \longrightarrow & 0 \end{array} \tag{4.11}$$

which gives rise to the following commutative diagram of long exact sequences

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{H}^0(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^kT\mathcal{M}'_{lc}) & \xrightarrow{\delta'_k} & \mathrm{H}^1(\mathcal{M}_{lc}(n,d)', \mathcal{S}ym^{k-1}(\mathcal{D}^1(\zeta))) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{H}^0(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^kT\mathcal{M}'_{lc}) & \xrightarrow{\delta_k} & \mathrm{H}^1(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^{k-1}T\mathcal{M}'_{lc}) & \longrightarrow & \cdots \end{array} \tag{4.12}$$

In order to prove the theorem, it is enough to show that the connecting homomorphism δ'_k, depicted in the above commutative diagram (4.12), is injective for all $k \geq 1$. Again from the above commutative diagram (4.12), δ'_k is injective for every $k \geq 1$ if and only if the connecting homomorphism

$$\delta_k : \mathrm{H}^0(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^kT\mathcal{M}'_{lc}) \to \mathrm{H}^1(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^{k-1}T\mathcal{M}'_{lc}) \tag{4.13}$$

is injective for every $k \geq 1$.

Let $\mathrm{at}(\zeta) \in \mathrm{H}^1(\mathcal{M}'_{lc}(n,d), T^*\mathcal{M}'_{lc})$ be the Atiyah class of the line bundle ζ, which is nothing but the extension class of the Atiyah exact sequence (see [1])

$$0 \to \mathcal{O}_{\mathcal{M}'_{lc}} \to \mathcal{D}^1(\zeta) \xrightarrow{\sigma_1} T\mathcal{M}'_{lc} \to 0. \tag{4.14}$$

The Atiyah class $\mathrm{at}(\zeta)$ determines the first Chern class $c_1(\zeta)$ of the line bundle ζ. Let γ_k be the extension class of the short exact sequence (4.3). Since the short exact sequence (4.3) is the symmetric power of (4.14), the extension class γ_k can be expressed in terms of the first Chern class $c_1(\zeta)$. Further, let α_k denote the extension class of the following short exact sequence

$$0 \to \mathcal{S}ym^{k-1}(T\mathcal{M}'_{lc}) \to \frac{\mathcal{S}ym^{k}(\mathcal{D}^1(\zeta))}{\mathcal{S}ym^{k-2}(\mathcal{D}^1(\zeta))} \to \mathcal{S}ym^{k}(T\mathcal{M}'_{lc}) \to 0, \tag{4.15}$$

which is the bottom short exact sequence in the commutative diagram (4.11). Then, γ_k maps to α_k. Thus, α_k can also be described in terms of the first Chern class $c_1(\zeta)$.

Since a connecting homomorphism can be expressed as the cup product by the extension class of the corresponding short exact sequence, the connecting homomorphism δ_k in (4.13) can be described using the first Chern class $c_1(\zeta)$ of the line bundle ζ. Indeed, the cup product with $c_1(\zeta)$ gives rise to a homomorphism

$$\tau : \mathrm{H}^0(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^kT\mathcal{M}'_{lc}) \to \mathrm{H}^1(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^kT\mathcal{M}'_{lc} \otimes T^*\mathcal{M}'_{lc}). \tag{4.16}$$

The canonical homomorphism

$$\upsilon : \mathcal{S}ym^kT\mathcal{M}'_{lc} \otimes T^*\mathcal{M}'_{lc} \to \mathcal{S}^{k-1}T\mathcal{M}'_{lc}$$

induces a morphism of $\mathbb{C}$-vector spaces

$$\upsilon^* : \mathrm{H}^1(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^k T\mathcal{M}'_{lc} \otimes T^*\mathcal{M}'_{lc}) \to \mathrm{H}^1(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^{k-1} T\mathcal{M}'_{lc}). \quad (4.17)$$

Thus, we get a morphism

$$\tilde{\tau} = \upsilon^* \circ \tau : \mathrm{H}^0(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^k T\mathcal{M}'_{lc}) \to \mathrm{H}^1(\mathcal{M}'_{lc}(n,d), \mathcal{S}ym^{k-1} T\mathcal{M}'_{lc}), \quad (4.18)$$

Then, from the above observation we have $\tilde{\tau} = \delta_k$. It is sufficient to show that $\tilde{\tau}$ is injective. In view of the assumption that $\widetilde{p}_{0_\sharp}$ in (4.7) is an injective morphism, now using the similar technique as in the proof of [35, Theorem 1.4], we can show that $\tilde{\tau}$ is injective. □

Proposition 4.2 *Under the hypothesis of Theorem 4.1, for* $k \geq 0$, *we have*

$$\mathrm{H}^0(\mathcal{M}'_{lc}(n,d), \mathcal{D}^k(\zeta)) = \mathbb{C}. \quad (4.19)$$

Proof Proof follows from the similar steps as in Theorem 4.1. □

Under the same hypothesis of Theorem 4.1 for the corresponding moduli spaces, we have

Theorem 4.3 *Suppose that the hypothesis of Theorem 4.1 holds for the moduli space* $\mathcal{X}$, *where* $\mathcal{X}$ *denote* $\mathcal{M}'_{lc}(n,L)$, $\mathcal{M}'_h(n)$ *or* $\mathcal{M}'_h(n,L_0)$. *Let* ζ *be a line bundle over* $\mathcal{X}$. *Then, for every* $k \geq 0$, *we have*

(1) $\mathrm{H}^0(\mathcal{X}, \mathcal{S}ym^k(\mathcal{D}^1(\zeta))) = \mathbb{C}$.
(2) $\mathrm{H}^0(\mathcal{X}, \mathcal{D}^k(\zeta)) = \mathbb{C}$.

In [6], global sections of a line bundle on the moduli space of logarithmic connections singular exactly over one point of a compact Riemann surface have been studied. In this section, we study global sections of line bundles over $\mathcal{M}'_{lc}(n,L)$.

Let $\mathcal{L}$ be a line bundle over $\mathcal{M}'_{lc}(n,L)$. Then,

$$\mathcal{L} = q^*\Theta^l \quad (4.20)$$

for some $l \in \mathbb{Z}$, where p is the morphism defined in (3.3) and Θ is the generalised theta line bundle over $\mathcal{U}(n,L)$. Then, we have a natural generalisation of [6, p.797, Theorem 4.3], and the same ideas can be used to prove the following.

Theorem 4.4 *For every* $l < 0$, *we have*

$$\mathrm{H}^0(\mathcal{M}'_{lc}(n,L), q^*\Theta^l) = 0. \quad (4.21)$$

5 Torelli-type theorem for the moduli spaces

In [5, Theorem 5.2], a Torelli-type theorem has been proved for the moduli space of holomorphic connections over compact Riemann surface, and in [7], Torelli-type theorems have been proved for the moduli space of logarithmic connections singular exactly over one point with fixed residue. In this section, we prove Torelli-type theorems for the moduli spaces $\mathcal{M}_{lc}(n,d)$ and $\mathcal{M}_{lc}(n,L)$. We assume that $\Phi_j = \alpha_j \mathbf{1}_{E(x_j)}$, for every $j = 1, \ldots, m$, where $\alpha_j \in \mathbb{C}$.

We show that the isomorphism classes of the moduli spaces $\mathcal{M}_{lc}(n,d)$ and $\mathcal{M}_{lc}(n,L)$ do not depend on the choice of S. Let $T = \{y_1, \ldots, y_m\}$ be a finite subset of X such that $y_i \neq y_j$ for $i \neq j$. Note that $\sharp S = \sharp T$.

In this section, we use the following notations

$$\mathcal{M}_{lc}(X, S) := \mathcal{M}_{lc}(n, d),$$

and

$$\mathcal{M}_{lc}(X, S, L) := \mathcal{M}_{lc}(n, L)$$

to emphasise S and T. Let $\mathcal{M}_{lc}(X, T)$ and $\mathcal{M}_{lc}(X, T, L)$ denote the moduli spaces corresponding to T.

Lemma 5.1 *There is an isomorphism between $\mathcal{M}_{lc}(X, S)$ and $\mathcal{M}_{lc}(X, T)$.*

Proof Depending on the sets S and T, we have two cases

(1) $S \cap T = \emptyset$.
(2) $S \cap T \neq \emptyset$.

Suppose $S \cap T = \emptyset$. For every $i = 1, \ldots, m$, let $L_i = \mathcal{O}_X(y_i - x_i)$ be a line bundle of degree zero. Let D^i be the de Rham logarithmic connection on the line bundle L_i singular over x_i and y_i, defined by sending a local section s_i of L_i to ds_i. Then, $Res(D^i, x_i) = -1$ and $Res(D^i, y_i) = 1$. Define a line bundle

$$L_0 = \bigotimes_{i=1}^{m} L_i.$$

Then, L_0 admits a logarithmic connection induced from $\{D^i\}_{i=1}^m$, which can be expressed as follows

$$D_0 = \sum_{i=1}^{m} \mathbf{1}_{L_1} \otimes \cdots \otimes \alpha_i D^i \otimes \cdots \otimes \mathbf{1}_{L_m}.$$

Moreover, $Res(D_0, x_i) = -\alpha_i$ and $Res(D_0, y_i) = \alpha_i$ for every $i = 1, \ldots, m$. Let $(E, D) \in \mathcal{M}(X, S)$. Then, $E \otimes L_0$ admits a logarithmic connection given by

$$D \otimes \mathbf{1}_{L_0} + \mathbf{1}_E \otimes D_0.$$

Note that for every $i = 1, \ldots, m$, we have

$$Res(D \otimes \mathbf{1}_{L_0} + \mathbf{1}_E \otimes D_0, x_i) = 0,$$

and

$$Res(D \otimes \mathbf{1}_{L_0} + \mathbf{1}_E \otimes D_0, y_i) = \alpha_i.$$

Thus, we have a morphism

$$\Psi_{(L_0, D_0)} : \mathcal{M}(X, S) \longrightarrow \mathcal{M}(X, T)$$

of algebraic varieties sending (E, D) to $(E \otimes L_0, D \otimes \mathbf{1}_{L_0} + \mathbf{1}_E \otimes D_0)$, which is an isomorphism.

Next suppose that $S \cap T \neq \emptyset$. Without loss of generality, we assume that $x_1 = y_1, x_2 = y_2, \ldots, x_r = y_r$ for $r \leq m$. In this case, we consider the line bundle

$$L_0 = \bigotimes_{j=r+1}^{m} \mathcal{O}_X(y_j - x_j).$$

Now, using the above steps, we can get a logarithmic connection D_0 in L_0. A morphism similar to $\Psi_{(L_0, D_0)}$ can be defined, which turns out to be an isomorphism. □

A similar result is true for the moduli space $\mathcal{M}_{lc}(X, S, L)$.

Lemma 5.2 *There is an isomorphism between* $\mathcal{M}_{lc}(X, S, L)$ *and* $\mathcal{M}_{lc}(X, T, L')$.

Thus, for the simplicity of the notations, we write $\mathcal{M}_{lc}(X)$ in place of $\mathcal{M}_{lc}(X, S)$ and $\mathcal{M}_{lc}(X, L)$ in place of $\mathcal{M}_{lc}(X, S, L)$.

Now, we shall compute the cohomology group of $\mathcal{M}_{lc}(X)$ and $\mathcal{M}_{lc}(X, L)$.

Let $\mathcal{M}'_{lc}(X){:=}\mathcal{M}'_{lc}(n, d)$, and $\mathcal{M}'_{lc}(X, L){:=}\mathcal{M}'_{lc}(n, L)$. Then, let

$$p_0 : \mathcal{M}'_{lc}(X) \longrightarrow \mathcal{U}^s(n, d)$$

be the morphism defined in (3.12). Since a fibre of p_0 is an affine space modelled over a vector space, which is contractible, we get an isomorphism

$$p_0^* : \mathrm{H}^i(\mathcal{U}^s(n, d), \mathbb{Q}) \longrightarrow \mathrm{H}^i(\mathcal{M}'_{lc}(X), \mathbb{Q}) \tag{5.1}$$

of rational cohomology groups for every $i \geq 0$. For the cohomology of $\mathcal{U}^s(n, d)$ see [2] and [20].

Let $Z{:=}\mathcal{M}_{lc}(n, L) \setminus \mathcal{M}'_{lc}(n, L)$. Then, from [7, Lemma 3.1], we have

Lemma 5.3 *The codimension of the Zariski closed set* Z *in* $\mathcal{M}_{lc}(n, L)$ *is at least* $(n-1)(g-2)+1$. *In particular, if* $n \geq 2$, $g \geq 3$, *then* $\mathrm{codim}(Z, \mathcal{M}_{lc}(n, L)) \geq 2$.

Similarly, let $p : \mathcal{M}'_{lc}(X, L) \to \mathcal{U}^s(n, L)$ be the morphism defined in (3.3). Then, p is a fibre bundle with fibres as affine spaces modelled over vector spaces, and since affine spaces with the usual topology are contractible, the induced homomorphism

$$p^* : \mathrm{H}^i(\mathcal{U}^s(n, L), \mathbb{Z}) \longrightarrow \mathrm{H}^i(\mathcal{M}'_{lc}(X, L), \mathbb{Z}) \tag{5.2}$$

of cohomology groups, is an isomorphism for all $i \geq 0$.

Let $\mathcal{Y}$ be a complex algebraic variety. For every $i \geq 0$, there is a **mixed Hodge structure** on the cohomology group $\mathrm{H}^i(\mathcal{Y}, \mathbb{Z})$. This result is due to Deligne, for more details see [12, 13].

The isomorphism p^* in (5.2) is an isomorphism of mixed Hodge structures. Moreover, the cohomology group $\mathrm{H}^i(\mathcal{M}'_{lc}(X, L), \mathbb{Z})$ is equipped with **pure Hodge structure** of weight i for every $i \geq 0$, because $\mathcal{U}^s(n, L)$ is a smooth projective variety over $\mathbb{C}$, and from [12] the cohomology group $\mathrm{H}^i(\mathcal{U}^s(n, L), \mathbb{Z})$ is endowed with a pure Hodge structure of weight i, for every $i \geq 0$.

Let $\mathcal{A}$ be a smooth complex analytic space. For every integer $k \geq 0$, the $(k+1)$-th **intermediate Jacobian variety** $J^{k+1}(\mathcal{A})$ of $\mathcal{Y}$ is defined as follows.

$$J^{k+1}(\mathcal{A}){:=}\mathrm{H}^{2k+1}(\mathcal{A}, \mathbb{R})/\mathrm{H}^{2k+1}(\mathcal{A}, \mathbb{Z}) \tag{5.3}$$

The space $J^{k+1}(\mathcal{A})$ carries a canonical structure of complex manifold. We consider that the moduli space $\mathcal{M}_{lc}(X, L)$ is equipped with the complex analytic topology.

Proposition 5.4 *The second intermediate* $J^2(\mathcal{M}_{lc}(X, L))$ *is isomorphic to the Jacobian* $J(X){:=}Pic^0(X)$ *of* X.

Proof First we show that the mixed Hodge structure on $\mathcal{M}_{lc}(X, L)$ is in fact a pure Hodge structure. Let $Z{:=}\mathcal{M}_{lc}(X, L) \setminus \mathcal{M}'_{lc}(X, L)$ as in Lemma 5.3. Then, we have a long exact sequence of relative cohomology groups,

$$\mathrm{H}^3_Z(\mathcal{M}_{lc}(X, L), \mathbb{Z}) \to \mathrm{H}^3(\mathcal{M}_{lc}(X, L), \mathbb{Z}) \xrightarrow{\iota^*} \mathrm{H}^3(\mathcal{M}'_{lc}(X, L), \mathbb{Z}) \xrightarrow{\partial} \mathrm{H}^4_Z(\mathcal{M}_{lc}(X, L), \mathbb{Z})$$

where ι^* is induced by the inclusion map $\iota : \mathcal{M}'_{lc}(X, L) \hookrightarrow \mathcal{M}_{lc}(X, L)$ and ∂ is the boundary operator. We show that ι^* is an isomorphism. From Alexander duality [18, Theorem 4.7, p.381], we have an isomorphism

$$\mathrm{H}^i_Z(\mathcal{M}_{lc}(X, L), \mathbb{Z}) \longrightarrow \mathrm{H}^{BM}_{2N-i}(Z, \mathbb{Z}),$$

where $N = 2(n^2 - 1)(g - 1)$ is the complex dimension of the moduli space $\mathcal{M}_{lc}(X, L)$ and H^{BM}_* is the Borel–Moore homology. In view of Lemma 5.3, we have

$$\operatorname{codim}(Z, \mathcal{M}_{lc}(X, L)) \geq 2,$$

therefore the real dimension of Z is at most $2N - 4$. Thus,

$$\mathrm{H}^{BM}_{2N-i}(Z, \mathbb{Z}) = 0, \quad \text{for } i = 0, 1, 2, 3,$$

and hence

$$\mathrm{H}^3_Z(\mathcal{M}_{lc}(X, L), \mathbb{Z}) = 0.$$

This implies that ι^* is an injective morphism. Let Γ be a smooth compactification of $\mathcal{M}_{lc}(X, L)$, and

$$Z' = \Gamma \setminus \mathcal{M}'_{lc}(X, L).$$

Then, from [12, Corollaire 3.2.17], we have a surjective morphism

$$\mathrm{H}^3(\Gamma, \mathbb{Q}) \longrightarrow \mathrm{H}^3(\mathcal{M}_{lc}(X, L), \mathbb{Q})$$

of mixed Hodge structures, and since

$$\operatorname{codim}(Z', \Gamma) \geq 3$$

from [36, p.269, Lemma 11.13], we have an isomorphism

$$\mathrm{H}^3(\Gamma, \mathbb{Z}) \longrightarrow \mathrm{H}^3(\mathcal{M}'_{lc}(X, L), \mathbb{Z})$$

of Hodge structures. Then, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^3(\Gamma, \mathbb{Q}) & & \\ \downarrow & \searrow & \\ \mathrm{H}^3(\mathcal{M}_{lc}(X, L), \mathbb{Q}) & \xrightarrow{\iota^*_{\mathbb{Q}}} & \mathrm{H}^3(\mathcal{M}'_{lc}(X, L), \mathbb{Q}) \end{array}$$

and from the above facts, the vertical and diagonal arrows are the surjective morphisms of mixed Hodge structures induced from their respective inclusion maps. Now, because of the commutativity of the diagram, $\iota^*_{\mathbb{Q}}$ is a surjective morphism of mixed Hodge structures. Since $\mathrm{H}^3(\mathcal{M}'_{lc}(X, L), \mathbb{Z})$ (being isomorphic to $\mathrm{H}^3(\mathcal{U}(n, L), \mathbb{Z})$) is torsion-free $\mathbb{Z}$-module of finite rank [24, Theorem 3] and ι^* is an injective morphism, $\mathrm{H}^3(\mathcal{M}_{lc}(X, L), \mathbb{Z})$ is torsion free. In order to show that ι^* is surjective, we need to show that $\mathrm{H}^4_Z(\mathcal{M}_{lc}(X, L), \mathbb{Z})$ is torsion free. In fact, if Z has codimension ≥ 3, this group is zero; if Z has codimension 2, it is isomorphic to $\mathrm{H}^{BM}_{2N-4}(Z, \mathbb{Z})$, which is the top homology group and necessarily torsion free. Thus, ι^* is a surjective morphism, and hence the mixed Hodge structure on $\mathrm{H}^3(\mathcal{M}_{lc}(X, L), \mathbb{Z})$ is a pure Hodge structure of weight 3. Therefore, the second intermediate Jacobian $J^2(\mathcal{M}_{lc}(X, L))$ is isomorphic to $J^2(\mathcal{M}'_{lc}(X, L))$, and the latter is isomorphic to $J^2(\mathcal{U}(n, L))$. Thus, from [24, Theorem 3], $J^2(\mathcal{M}_{lc}(X, L))$ is isomorphic to $J(X)$. This completes the proof. □

Let Θ be the theta divisor on the Jacobian $J(X)$. The pair $(J(X), \Theta)$ is called a principally polarised Jacobian. Then, the classical Torelli theorem says that the pair $(J(X), \Theta)$ determines the compact Riemann surface X up to isomorphism.

In view of Proposition 5.4, the moduli space $\mathcal{M}_{lc}(X, L)$ determines the Jacobian $J(X)$ of the compact Riemann surface X. But this does not qualify for the determination of X, because two non-isomorphic compact Riemann surfaces can have isomorphic Jacobian.

Nevertheless, from [25, p.125, Corollary 1.2], there are, up to isomorphism, only finitely many compact Riemann surfaces having a given abelian variety as the Jacobian. Thus, there are, up to isomorphism, only finitely many compact Riemann surface Y such that $\mathcal{M}_{lc}(Y, L)$ is isomorphic to $\mathcal{M}_{lc}(X, L)$.

Remark 5.5 Let $\widetilde{\Theta}$ be the canonical polarisation on the second intermediate Jacobian $J^2(\mathcal{U}(n, L))$. Then, from [24, Theorem 3], we have

$$(J^2(\mathcal{U}(n, L)), \widetilde{\Theta}) \cong (J(X), \Theta).$$

In [7, Section 4], Biswas and Muñoz constructed the principal polarisation $\widehat{\Theta}$ on the second intermediate Jacobian of the moduli space $\mathcal{M}_{lc}^{x_0}(X)$ of logarithmic connections singular exactly over one point x_0 of the compact Riemann surface X with fixed determinant such that the principally polarised abelian variety $(J^2(\mathcal{M}_{lc}^{x_0}(X)), \widehat{\Theta})$ is isomorphic to the principally polarised abelian variety $(J^2(\mathcal{U}(n, L)), \widetilde{\Theta})$. Imitating the similar technique as in [7, Section 4], a principal polarisation can be constructed on $\mathcal{M}_{lc}(X, L)$.

From Lemma 5.2, the moduli space $\mathcal{M}_{lc}(X, L)$ does not depend on the choice of S. Thus, we have

Theorem 5.6 *Let (X, S) and (Y, T) be two m-pointed compact Riemann surfaces of genus $g \geq 3$. Let $\mathcal{M}_{lc}(X, L)$ and $\mathcal{M}_{lc}(Y, L')$ be the corresponding moduli spaces of logarithmic connections. Then, $\mathcal{M}_{lc}(X, L)$ is isomorphic to $\mathcal{M}_{lc}(Y, L')$ if and only if X is isomorphic to Y.*

Next, we show the Torelli-type theorem for the moduli space $\mathcal{M}_{lc}(X)$. Let

$$G : \mathcal{M}_{lc}(X) \longrightarrow Pic^d(X) \tag{5.4}$$

be the map sending $(E, D) \mapsto \bigwedge^n E$. Note that the morphism G is surjective. Since d, n and $\Phi_j = \alpha_j \mathbf{1}_{E(x_j)}$ for $j = 1, \ldots, m$ satisfy (2.7), from [8, Theorem 1.3 (2)] E admits a logarithmic connection with residues $\alpha_j \mathbf{1}_{E(x_j)}$ at $x_j \in S$.

Now, we have a natural generalisation of [5, p.431, Lemma 5.1] and [7, p.313, Proposition 5.1].

Proposition 5.7 *Let A be a complex abelian variety, and*

$$f : \mathcal{M}_{lc}(X) \longrightarrow A \tag{5.5}$$

a regular morphism. Then, there exists a unique regular morphism

$$f_0 : Pic^d(X) \longrightarrow A$$

such that

$$f_0 \circ G = f, \tag{5.6}$$

where G is defined in (5.4).

Proof Consider $\mathcal{M}'_{lc}(X):=\mathcal{M}'_{lc}(n,d) \subset \mathcal{M}_{lc}(X)$ as in (2.8). Let

$$p_0 : \mathcal{M}'_{lc}(X) \longrightarrow \mathcal{U}(n,d)$$

be the morphism defined in (3.12). For $E \in \mathcal{U}(n,d)$, it has been observed that $p_0^{-1}(E)$ is an affine space modelled over the vector space $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$, and hence $p_0^{-1}(E)$ is a rational variety. Restricting f to $p_0^{-1}(E)$, we get a map

$$f|_{p_0^{-1}(E)} : p_0^{-1}(E) \longrightarrow A,$$

which is a constant map, because any regular morphism from a rational variety to an abelian variety is constant.

Now, consider the determinant map

$$F : \mathcal{U}(n,d) \longrightarrow Pic^d(X)$$

defined by sending E to $\bigwedge^n E$. Then, F is a surjective map. For any $L \in Pic^d(X)$, $F^{-1}(L)$ is nothing but the moduli space $\mathcal{U}(n,L)$. Thus, we get a regular morphism

$$\psi_0|_{F^{-1}(L)} : \mathcal{U}(n,L) = F^{-1}(L) \longrightarrow A.$$

on each of the fibres of F. From [19, Theorem 1.2], $\mathcal{U}(n,L)$ is a rational variety, and hence the regular morphism $\psi_0|_{F^{-1}(L)}$ is constant. This completes the proof. □

Let $\mathcal{M}^X_{lc}$ be the moduli space defined in (4.9). Then, we have a morphism

$$\delta : \mathcal{M}^X_{lc} \longrightarrow Pic^d(X) \tag{5.7}$$

defined by $(L, D) \mapsto L$. Then, $\delta^{-1}(L)$ is an affine space modelled over $\mathrm{H}^0(X, \Omega^1_X)$. Then,

$$G = \delta \circ \det,$$

where G is defined in (5.4), and $\det : \mathcal{M}_{lc}(X) \to \mathcal{M}^X_{lc}$ defined in (4.10). Thus, we have a morphism

$$\eta : G^{-1}(L) \longrightarrow \mathcal{M}_{lc}(X, L) \tag{5.8}$$

which is a fibration and each fibre is an affine space modelled over $\mathrm{H}^0(X, \Omega^1_X)$. Since the fibre of η is contractible, we have an isomorphism

$$\eta^* : \mathrm{H}^i(\mathcal{M}_{lc}(X,L), \mathbb{Z}) \longrightarrow \mathrm{H}^i(G^{-1}(L), \mathbb{Z})$$

of cohomology groups for all $i \geq 0$. Therefore, we have

$$J^2(\mathcal{M}_{lc}(X,L)) \cong J^2(G^{-1}(L)).$$

As mentioned in Remark 5.5, similar steps give a principal polarisation $\widehat{\widehat{\Theta}}$ on $J^2(G^{-1}(L))$ such that

$$(J^2(\mathcal{M}_{lc}(X,L)), \widehat{\Theta}) \cong (J^2(G^{-1}(L)), \widehat{\widehat{\Theta}}).$$

Thus, in view of Lemma 5.1 and using Theorem 5.6, we get

Theorem 5.8 *Let (X, S) and (Y, T) be two m-pointed compact Riemann surfaces of genus $g \geq 3$. Let $\mathcal{M}_{lc}(X)$ and $\mathcal{M}_{lc}(Y)$ be the corresponding moduli spaces of logarithmic connections. Then, $\mathcal{M}_{lc}(X)$ is isomorphic to $\mathcal{M}_{lc}(Y)$ if and only if X is isomorphic to Y.*

6 Rational connectedness of the moduli spaces

In [33], we have shown that the moduli space of rank one logarithmic connections with fixed residues is not rational. In this section, we show that the moduli space $\mathcal{M}_{lc}(n,d)$ is not rational. For the theory of rational varieties, we refer to [21].

Recall that a smooth complex variety V is said to be rationally connected if any two general points on V can be connected by a rational curve in V. The following lemma is an easy consequence of the definition.

Lemma 6.1 *Let $f : \mathcal{Y} \to \mathcal{X}$ be a dominant rational map of complex algebraic varieties with $\mathcal{Y}$ rationally connected. Then, $\mathcal{X}$ is rationally connected.*

Theorem 6.2 ([19], Theorem 1.1) *The moduli space $\mathcal{U}(n,d)$ is birational to $J(X) \times \mathbb{A}^{(n^2-1)(g-1)}$, where $J(X)$ is the Jacobian of X.*

Note that $J(X)$ is not rationally connected, because it does not contain any rational curve. Therefore, $\mathcal{U}(n,d)$ is not rationally connected.

Proposition 6.3 *The moduli space $\mathcal{M}_{lc}(n,d)$ is not rational.*

Proof It is enough to show that the moduli space $\mathcal{M}_{lc}(n,d)$ is not rationally connected. Let

$$p_0 : \mathcal{M}'_{lc}(n,d) \longrightarrow \mathcal{U}(n,d)$$

be the morphism of varieties defined in (3.12). Suppose that $\mathcal{M}'_{lc}(n,d)$ is rationally connected. Then, from Lemma 6.1, $\mathcal{U}(n,d)$ is rationally connected, which is not true. Thus, $\mathcal{M}'_{lc}(n,d)$ is not rationally connected and hence not rational. Since $\mathcal{M}'_{lc}(n,d)$ is an open dense subset of $\mathcal{M}_{lc}(n,d)$, $\mathcal{M}_{lc}(n,d)$ is not rational. □

A similar argument gives the following.

Proposition 6.4 *The moduli space $\mathcal{M}_h(n)$ is not rational.*

Lemma 6.5 ([16], Corollary 1.3) *Let $f : \mathcal{X} \to \mathcal{Y}$ be any dominant morphism of complex varieties. If $\mathcal{Y}$ and the general fibre of f are rationally connected, then $\mathcal{X}$ is rationally connected.*

Proposition 6.6 *The moduli space $\mathcal{M}'_{lc}(n,L)$ is rationally connected.*

Proof Consider the dominant morphism

$$p : \mathcal{M}'_{lc}(X,L) \longrightarrow \mathcal{U}(n,L)$$

defined in (3.3). As observed earlier every fibre of p is an affine space and hence rationally connected. Since $\mathcal{U}(n,L)$ is rationally connected, from Lemma 6.5, $\mathcal{M}'_{lc}(n,L)$ is rationally connected. □

Corollary 6.7 *$\mathcal{M}_{lc}(n,L)$ is rationally connected.*

Proof It follows from the fact that rationally connectedness is a birational invariant, and $\mathcal{M}'_{lc}(n,L)$ is a dense open subset of $\mathcal{M}_{lc}(n,L)$. □

Therefore, we have a natural question.

Question 6.8 Is the moduli space $\mathcal{M}_{lc}(n,L)$ rational ?

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