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OPTIMIZATION

Weak sharp minima for interval-valued functions and its primal-dual characterizations using generalized Hukuhara subdifferentiability

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Abstract
This article introduces the concept of weak sharp minima for convex interval-valued functions. To solve constrained and unconstrained convex IOPs by WSM, we provide primal and dual characterizations of the set of WSM. The primal characterization is given in terms of gH-directional derivatives. On the other hand, to derive dual characterizations, we propose the notions of the support function of a subset of $I(\mathbb{R})^n$ and gH-subdifferentiability for convex IVFs. Further, we develop the required gH-subdifferential calculus for convex IVFs. Thereafter, by using the proposed gH-subdifferential calculus, we provide dual characterizations for the set of WSM of objective IVFs of convex constrained and unconstrained IOPs. Two applications of the proposed theory are presented. The first one determines the set of WSM of a minimum risk portfolio interval optimization problem. In the second application, we propose a way to find weak efficient solutions of linear and nonlinear IOPs using WSM.

Keywords Interval-valued function · gH-directional derivative · gH-subgradient · Interval optimization · Weak sharp minima

1 Introduction

Due to the presence of uncertainty, deterministic optimization fails to represent many real-life optimization problems. In such cases, we need to proceed with the tools of uncertain optimization. If the uncertainty is given by a random variable, then these optimization problems come under the umbrella of stochastic optimization. On the other, if the uncertainty is given by a membership function, then these optimization problems are solved with the techniques of fuzzy optimization. Also, it is seen that the uncertainty of many practical problems is expressed using closed and bounded intervals. Thus, interval optimization is an indispensable way to deal with the uncertainty present in many real-life problems.

In 1966, Moore (1966) introduced interval analysis to investigate interval-valued functions (IVFs). In Moore (1966), Moore extensively gave arithmetic of intervals. Subsequently, there was a need to improve this arithmetic (Hukuhara 1967), especially the subtraction. Due to this, Hukuhara (1967) presented a notion of the difference between intervals, which is known as Hukuhara difference (H-difference). Stefanini and Bede (2009) proposed an extended version of H-difference, known as generalized Hukuhara difference (gH-difference), which has been comprehensively adopted in interval analysis.

We know that the solution concepts of optimization problems depend widely on ordering the range set of the objective function. Unlike real numbers, the set of intervals is not linearly ordered. Thus, to introduce a solution concept for optimization problems under interval uncertainties, many partial ordering relations of intervals were proposed in the literature (see Ishibuchi and Tanaka 1990; Sengupta et al. 2001; Wu 2008a, b; Jiang et al. 2008; Bhurjee and Panda 2012, and the references therein). With the help of the existing ordering concepts of a pair of intervals, many theories and methods have been developed regarding solutions of optimization problems with IVFs or of interval optimization problems (IOPs) (Shaocheng 1994; Inuiguchi and Sakawa 1995; Mráz 1998; Wu 2009, 2010; Jana and Panda 2014). Inuiguchi and

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Sakawa (1995) proposed treatment of optima for IVFs by minimax regret criteria. Chanas and Kuchta (1996) gave a solution concept based on a preference relation of intervals. A robust efficient solution for interval linear programming was given in Ida (2003). Chen et al. (2004) reported a solution concept by midpoint deterministic approach. Wu (2007) defined type I and type II LU-optimal solution concepts similar to Pareto optimality. A survey on the different ordering of intervals and related optimality concepts can be found in Jiang et al. (2012), Ghosh et al. (2020) and from their references.

The major developments on IOPs started after a rich calculus of IVFs was ready to be used. Hukuhara (1967) laid the foundation to develop the calculus of IVFs by introducing the concept of H-differentiability of IVFs. However, this definition of H-differentiability is restrictive (Chalco-Cano et al. 2013). To overcome the deficiencies of H-differentiability, Stefanini and Bede (2009) proposed the notion of gH-differentiability for IVFs. Later, by using gH-differentiability, many concepts on calculus have been developed, for instance, see Chalco-Cano et al. (2013), Lupulescu (2013), Ghosh et al. (2020). In 2007, Wu (2007) proposed two solution concepts by considering two partial ordering concepts on the set of all closed intervals and derived KKT optimality conditions for IOPs using H-derivative. Subsequently, Wu investigated KKT optimality conditions for multi-objective IOPs (Wu 2009). In 2012, Bhurjee and Panda (2012) developed a methodology to study the existence of the solution of general IOPs by expressing IVFs in the parametric form. Chalco-Cano et al. (2013) derived KKT optimality conditions for IOPs using gH-derivative and explained the advantages of using gH-derivative instead of H-derivative. In 2016, Singh et al. (2016) proposed the concept of Pareto optimal solution for the interval-valued multi-objective programming problems. Many other researchers have also proposed optimality conditions and solution concepts for IOPs, see for instance (Ahmad et al. 2019; Ghosh 2017; Ghosh et al. 2018, 2019; Treanţă 2021) and the references therein.

1.1 Motivation and work done

Consider an IOP

$$\min_{x \in S} \mathbf{F}(x), \tag{1}$$

where $\mathbf{F} : \mathbb{R}^n \rightarrow I(\mathbb{R})$ is gH-lsc, convex IVF and S is a convex subset of $\mathbb{R}^n$. In our best knowledge of literature of IVFs, to solve the IOP (1) having nondifferentiable objective IVF $\mathbf{F}$, a proper method or concept has not been given yet. This is our primary motivation for this article. Due to nonsmoothness of IVF $\mathbf{F}$ in IOP (1) at some points, the gH-gradient of $\mathbf{F}$ at these points cannot be calculated. Hence, we introduce the concepts of gH-subgradient and gH-subdifferentiability of $\mathbf{F}$. Using this gH-subdifferentiability, we propose the notion of WSM to solve the IOP (1).

Also, it is known that WSM plays an important role in the sensitivity analysis and convergence analysis of conventional optimization problems (Burke and Ferris 1993; Burke and Deng 2002, 2005). In Burke and Ferris (1993), Burke and Ferris introduced the notion of WSM and explained the convergence theory of algorithms with the help of WSM in conventional optimization. After that, Burke and Deng (2002) generalized the concept of WSM to the normed linear space setting and dissected the normal cone inclusion characterization for WSM. Further, in Burke and Deng (2005), Burke et al. gave the study in which they provided a link between the WSM, linear regularity and error bounds in convex programming. It is also seen that many algorithms exhibit finite termination at WSM (Burke and Ferris 1993; Ferris 1990; Zhou and Wang 2012; Matsushita and Xu 2012; Wang et al. 2015). In Zhou and Wang (2012), Zhou and Wang presented the concept of WSM by using conjugate functions and established the finite termination property for convex programming and variational inequality problem, respectively. Subsequently, Matsushita and Xu (2012) solved convex optimization problem by the proximal point algorithm in a finite number of steps under the assumption that the solution set is a set of WSM. Wang et al. (2015) studied the finite termination of sequences generated by inexact proximal point algorithms for finding zeroes of a maximal monotone (set-valued) operator T on a Hilbert space. Motivated by these properties and wide applications of WSM in conventional optimization, in this article, we attempt to propose and mathematically characterize the notion of WSM for convex IVFs. To give characterizations of WSM for convex IVFs, we defined gH-subdifferentiability for convex IVFs and support function of a subset of $I(\mathbb{R})^n$. Some required fundamental characteristics of gH-subdifferential set are proposed. A few related results on the support function of a nonempty subset of $I(\mathbb{R})^n$ are also derived. As an application of the proposed study, first we find the set of WSM of a minimum risk portfolio interval optimization problem and secondly we find the set of weakly efficient solutions of the interval linear and nonlinear programming problems using WSM.

1.2 Delineation

The article is presented in the following manner. In Sect. 2, basic terminologies and definitions on intervals and IVFs are provided. In Sect. 3, we propose the concept of the support function of a subset of $I(\mathbb{R})^n$; alongside, a few necessary results on extended support function are also given. Next, we derive the idea of gH-subdifferentiability for convex IVFs, in Sect. 4 that are required in the subsequent sections. The concept of WSM for convex IVFs is presented in Sect. 5; further, we give primal and dual characterizations of WSM. Applications of the proposed study are given in Sect. 6. Lastly, the conclusion and future scopes are given in Sect. 7.

2 Preliminaries and Terminologies

In this article, the following notations are used throughout.

- $\mathbb{R}$ denotes the set of real numbers
- $\mathbb{R}^+$ denotes the set of nonnegative real numbers
- $\|\cdot\|$ denotes the Euclidean norm and $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^n$
- $I(\mathbb{R})$ represents the set of all closed and bounded intervals
- Bold capital letters refer to the elements of $I(\mathbb{R})$
- $\overline{I(\mathbb{R})} = I(\mathbb{R}) \cup \{-\infty, +\infty\}$
- $\mathbb{B} = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ denotes the closed unit ball in $\mathbb{R}^n$
- $\psi_S^*(x)$ is a support function of a subset S of $\mathbb{R}^n$ at $x \in \mathbb{R}^n$
- $\text{cl}(S)$ denotes the closure of the set S.

2.1 Fundamental Operations on Intervals

Arithmetic operations of two intervals $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$ are defined by $\mathbf{A} \oplus \mathbf{B} = [\underline{a} + \underline{b},\ \overline{a} + \overline{b}]$, $\mathbf{A} \ominus \mathbf{B} = [\underline{a} - \overline{b},\ \overline{a} - \underline{b}]$, $\mathbf{A} \otimes \mathbf{B} = [\min\{\underline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \overline{a}\overline{b}\}]$ and

$$\lambda \odot \mathbf{A} = \mathbf{A} \odot \lambda = \begin{cases} [\lambda\underline{a},\ \lambda\overline{a}], & \text{if } \lambda \geq 0 \\ [\lambda\overline{a},\ \lambda\underline{a}], & \text{if } \lambda < 0, \end{cases}$$

where λ is a real constant.

The norm (Moore 1966) of an interval $\mathbf{A} = [\underline{a}, \overline{a}]$ in $I(\mathbb{R})$ is defined by $\|\mathbf{A}\|_{I(\mathbb{R})} = \max\{|\underline{a}|, |\overline{a}|\}$.

The norm of an interval vector $\widehat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n) \in I(\mathbb{R})^n$ is given by (see Moore 1966)

$$\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n} = \sum_{i=1}^{n} \|\mathbf{A}_i\|_{I(\mathbb{R})}.$$

It is to note that a real number p can be represented by the interval $[p, p]$.

Definition 1 (*gH-difference of intervals* Stefanini and Bede 2009). Let $\mathbf{A}$ and $\mathbf{B}$ be two elements in $I(\mathbb{R})$. The gH-difference between $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \ominus_{gH} \mathbf{B}$, is defined by the interval $\mathbf{C}$ such that

$$\mathbf{A} = \mathbf{B} \oplus \mathbf{C} \quad \text{or} \quad \mathbf{B} = \mathbf{A} \ominus \mathbf{C}.$$

It is to be noted that for $\mathbf{A} = [\underline{a},\ \overline{a}]$ and $\mathbf{B} = [\underline{b},\ \overline{b}]$,

$$\mathbf{A} \ominus_{gH} \mathbf{B} = \left[\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\},\ \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}\right],$$

and $\mathbf{A} \ominus_{gH} \mathbf{A} = \mathbf{0}$.

Definition 2 (*Algebraic operations on* $I(\mathbb{R})^n$). Let $\widehat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n)$ and $\widehat{\mathbf{B}} = (\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n)$ be two elements in $I(\mathbb{R})^n$. An algebraic operation $\star$ between $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$, denoted by $\widehat{\mathbf{A}}\star\widehat{\mathbf{B}}$, is defined by

$$\widehat{\mathbf{A}}\star\widehat{\mathbf{B}} = (\mathbf{A}_1\star\mathbf{B}_1, \mathbf{A}_2\star\mathbf{B}_2, \ldots, \mathbf{A}_n\star\mathbf{B}_n),$$

where $\star \in \{\oplus,\ \ominus,\ \ominus_{gH}\}$.

Definition 3 (*Special product*). For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and a vector of intervals $\widehat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n) \in I(\mathbb{R})^n$ with $\mathbf{A}_i = [\underline{a}_i, \overline{a}_i]$ for each $i = 1, 2, \ldots, n$, the special product between x and $\widehat{\mathbf{A}}$, denoted by $x^\top \odot \widehat{\mathbf{A}}$, is given by

$$x^\top \odot \widehat{\mathbf{A}} = \left[\min\left\{\sum_{i=1}^{n} x_i\underline{a}_i, \sum_{i=1}^{n} x_i\overline{a}_i\right\},\ \max\left\{\sum_{i=1}^{n} x_i\underline{a}_i, \sum_{i=1}^{n} x_i\overline{a}_i\right\}\right].$$

Remark 1 It is to notice that if all the components of $\widehat{\mathbf{A}}$ are degenerate intervals, i.e., $\widehat{\mathbf{A}} \in \mathbb{R}^n$, then the special product $x^\top \odot \widehat{\mathbf{A}}$ reduces to the standard inner product of $x \in \mathbb{R}^n$ with $\widehat{\mathbf{A}}$.

Definition 4 (*Dominance of intervals* Wu 2008b). Let $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$ be two elements in $I(\mathbb{R})$.

(i) $\mathbf{B}$ is said to be dominated by $\mathbf{A}$ if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$, and then we write $\mathbf{A} \preceq \mathbf{B}$;

(ii) $\mathbf{B}$ is said to be strictly dominated by $\mathbf{A}$ if $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$, and then we write $\mathbf{A} \prec \mathbf{B}$. Equivalently, $\mathbf{A} \prec \mathbf{B}$ if and only if any of the following holds: '$\underline{a} < \underline{b}$ and $\overline{a} \leq \overline{b}$' or '$\underline{a} \leq \underline{b}$ and $\overline{a} < \overline{b}$' or '$\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$';

(iii) if neither $\mathbf{A} \preceq \mathbf{B}$ nor $\mathbf{B} \preceq \mathbf{A}$, we say that none of $\mathbf{A}$ and $\mathbf{B}$ dominates the other, or $\mathbf{A}$ and $\mathbf{B}$ are not comparable. Equivalently, $\mathbf{A}$ and $\mathbf{B}$ are not comparable if either '$\underline{a} < \underline{b}$ and $\overline{a} > \overline{b}$' or '$\underline{a} > \underline{b}$ and $\overline{a} < \overline{b}$'.

2.2 Calculus of IVFs

Throughout this subsection, $\mathbf{F}$ is an IVF defined on a nonempty subset X of $\mathbb{R}^n$.

Definition 5 (gH*-continuity* Ghosh 2017). Let $\mathbf{F}$ be an IVF and let $\bar{x}$ be a point of X and $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$. The function $\mathbf{F}$ is said to be gH-continuous at $\bar{x}$ if

$$\lim_{\|h\|\to 0} \left(\mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x})\right) = \mathbf{0}.$$

Definition 6 (*gH-derivative* Stefanini and Bede 2009). The gH-derivative of an IVF $\mathbf{F} : \mathbb{R} \to I(\mathbb{R})$ at $\bar{x} \in \mathbb{R}$ is defined by

$$\mathbf{F}'(\bar{x}) = \lim_{h\to 0} \frac{\mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x})}{d}, \text{ provided the limit exists.}$$

Remark 2 (See Stefanini and Bede 2009). Let $\mathbf{F} = [\underline{F}, \overline{F}]$ be an IVF on X, where $\underline{F}$ and $\overline{F}$ are real-valued functions defined on X. Then, the gH-derivative of $\mathbf{F}$ at $\bar{x} \in X$ exists if the derivatives of $\underline{F}$ and $\overline{F}$ at $\bar{x}$ exist and

$$\mathbf{F}'(\bar{x}) = \left[\min\left\{\underline{F}'(\bar{x}), \overline{F}'(\bar{x})\right\}, \max\left\{\underline{F}'(\bar{x}), \overline{F}'(\bar{x})\right\}\right].$$

Definition 7 (*gH-partial derivative* Ghosh 2017). Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)^\top$ be a point of X. For a given $i \in \{1, 2, \ldots, n\}$, we define a function $\mathbf{G}_i$ by $\mathbf{G}_i(x_i) = \mathbf{F}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n)$. If gH-derivative of $\mathbf{G}_i$ exists at $\bar{x}_i$, then we say that $\mathbf{F}$ has the ith gH-partial derivative at $\bar{x}$. We denote the ith gH-partial derivative of $\mathbf{F}$ at $\bar{x}$ by $D_i\mathbf{F}(\bar{x})$, i.e., $D_i\mathbf{F}(\bar{x}) = \mathbf{G}_i'(\bar{x}_i)^\top$.

Definition 8 (*gH-gradient* Ghosh 2017). The gH-gradient of $\mathbf{F}$ at a point $\bar{x} \in X$, denoted by $\nabla\mathbf{F}(\bar{x}) \in I(\mathbb{R})^n$, is defined by

$$\nabla\mathbf{F}(\bar{x}) = (D_1\mathbf{F}_1(\bar{x}), D_2\mathbf{F}_2(\bar{x}), \ldots, D_n\mathbf{F}_n(\bar{x}))^\top.$$

Lemma 1 *Let $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{C}$ are in $I(\mathbb{R})$. Then, for any real number r,*

(i) $[r, r] \preceq \boldsymbol{A}$ *and* $\boldsymbol{A} \preceq \boldsymbol{B} \ominus_{gH} \boldsymbol{C} \implies \boldsymbol{C} \oplus [r, r] \preceq \boldsymbol{B}$ *and*
(i) $((1-\lambda) \odot \boldsymbol{A} \oplus \lambda \odot \boldsymbol{B}) \ominus_{gH} \boldsymbol{A} = \lambda \odot (\boldsymbol{B} \ominus_{gH} \boldsymbol{A})$ *for any* $\lambda \in [0, 1]$.

Proof See Appendix A. □

Definition 9 (*Convex IVF* Wu 2007). Let X be a nonempty convex subset of $\mathbb{R}^n$. An IVF $\mathbf{F} : X \to I(\mathbb{R})$ is said to be convex on X if for any x_1 and x_2 in X,

$$\mathbf{F}(\lambda_1 x_1 + \lambda_2 x_2) \preceq \lambda_1 \odot \mathbf{F}(x_1) \oplus \lambda_2 \odot \mathbf{F}(x_2)$$
$$\text{for all } \lambda_1, \lambda_2 \in [0, 1] \text{ with } \lambda_1 + \lambda_2 = 1.$$

Lemma 2 (See Wu 2007). *Let X be a nonempty convex subset of $\mathbb{R}^n$, and $\boldsymbol{F} = [\underline{F}, \overline{F}]$ be an IVF on X, where $\underline{F}$ and $\overline{F}$ are real-valued functions defined on X. Then, $\boldsymbol{F}$ is convex on X if and only if $\underline{F}$ and $\overline{F}$ are convex on X.*

Definition 10 (*gH-directional derivative* Ghosh et al. 2020). Let $\mathbf{F}$ be an IVF on X. Let $\bar{x} \in X$ and $d \in \mathbb{R}^n$. If the limit

$$\lim_{\lambda\to 0+} \frac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x}+\lambda d) \ominus_{gH} \mathbf{F}(\bar{x})\right)$$

exists, then the limit is said to be gH-directional derivative of $\mathbf{F}$ at $\bar{x}$ in the direction d, and it is denoted by $\mathbf{F}_{\mathscr{D}}(\bar{x})(d)$.

Definition 11 (*gH-differentiability* Ghosh 2017). An IVF $\mathbf{F}$ is said to be gH-differentiable at $\bar{x} \in X$ if there exist two IVFs $\mathbf{E}(\mathbf{F}(\bar{x}); h)$ and $\mathbf{L}_{\bar{x}} : \mathbb{R}^n \to I(\mathbb{R})$ such that

$$\mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x}) = \mathbf{L}_{\bar{x}}(h) \oplus \|h\| \odot \mathbf{E}(\mathbf{F}(\bar{x}); h)$$

for $\|h\| < \delta$ for some $\delta > 0$, where $\lim_{\|h\|\to 0} \mathbf{E}(\mathbf{F}(\bar{x}); h) = \mathbf{0}$ and $\mathbf{L}_{\bar{x}}$ is such a function that satisfies

(i) $\mathbf{L}_{\bar{x}}(x+y) = \mathbf{L}_{\bar{x}}(x) \oplus \mathbf{L}_{\bar{x}}(y)$ for all $x, y \in X$, and
(ii) $\mathbf{L}_{\bar{x}}(cx) = c \odot \mathbf{L}_{\bar{x}}(x)$ for all $c \in \mathbb{R}$ and $x \in X$.

Theorem 1 (See Ghosh 2017). *Let $\boldsymbol{F} : X \to I(\mathbb{R})$ be gH-differential at $\bar{x}$. Then, $\boldsymbol{L}_{\bar{x}}$ exists for every $h \in \mathbb{R}^n$ and*

$$\boldsymbol{L}_{\bar{x}} = \sum_{i=1}^{n} h_i \odot D_i\boldsymbol{F}(\bar{x}),$$

where $\sum_{i=1}^{n} h_i \odot D_i\boldsymbol{F}(\bar{x}) = h_1 \odot D_1\boldsymbol{F}(\bar{x}) \oplus h_2 \odot D_2\boldsymbol{F}(\bar{x}) \oplus \cdots \oplus h_n \odot D_n\boldsymbol{F}(\bar{x})$.

Remark 3 (See Ghosh 2017). Let $\mathbf{F} : X \to I(\mathbb{R})$ be gH-differentiable at $\bar{x} \in X$. Then, there exist a nonzero λ and $\delta > 0$ such that

$$\lim_{\lambda\to 0} \frac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x})\right) = \mathbf{L}_{\bar{x}}(h)$$

for all $h \in \mathbb{R}^n$ with $|\lambda|\|h\| < \delta$, where $\mathbf{L}_{\bar{x}}$ is an IVF, defined in Definition 11 of gH-differentiability.

Lemma 3 *Let $\boldsymbol{F} : X \to I(\mathbb{R})$ be a gH-differentiable at $\bar{x} \in X$. Then, $\boldsymbol{F}$ has gH-directional derivative at $\bar{x}$ for every direction $h \in \mathbb{R}^n$ and*

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h) = \boldsymbol{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n,$$

where $\boldsymbol{L}_{\bar{x}}$ is as defined in Definition 11.

Proof Since $\mathbf{F}$ is gH-differentiable at $\bar{x}$, by Remark 3, we have

$$\lim_{\lambda\to 0} \frac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x})\right) = \mathbf{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n$$
$$\implies \lim_{\lambda\to 0+} \frac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x})\right)$$
$$= \mathbf{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n.$$

Hence, by Definition 10, we conclude that $\mathbf{F}$ has gH-directional derivative at $\bar{x}$ and

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(h) = \mathbf{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n.$$ □

Definition 12 (*Proper IVF*). An extended IVF $\mathbf{F} : X \to \overline{I(\mathbb{R})}$ is called a proper IVF if there exists $\bar{x} \in X$ such that $\mathbf{F}(\bar{x}) \prec [+\infty, +\infty]$ and $[-\infty, -\infty] \prec \mathbf{F}(x)$ for all $x \in X$.

Definition 13 (*Effective domain of IVF*). The effective domain of an extended IVF $\mathbf{F} : X \to \overline{I(\mathbb{R})}$ is the collection of all such points at which $\mathbf{F}$ is finite. It is denoted by dom($\mathbf{F}$), i.e.,

$$\text{dom}(\mathbf{F}) = \left\{x \in X : \mathbf{F}(x) \prec [+\infty, +\infty]\right\}.$$

Definition 14 (*Linear IVF*). An IVF $\mathbf{F} : X \to I(\mathbb{R})$ is said to be linear if the following two conditions hold:

(i) $\mathbf{F}(c\, x) = c \odot \mathbf{F}(x)$ for all $x \in X$ and for all $c \in \mathbb{R}$, and
(ii) for all $x,\ y \in X$,

$$\mathbf{F}(x) \oplus \mathbf{F}(y) = \mathbf{F}(x + y).$$

Theorem 2 *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\boldsymbol{F} : X \to I(\mathbb{R})$ be a convex IVF with $\boldsymbol{F}(x) = [\underline{F}(x), \overline{F}(x)]$, where $\underline{F}$ and $\overline{F}$ are real-valued functions defined on X. Then, at any $\bar{x} \in X$, gH-directional derivative $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(d)$ exists and*

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(d) = \Big[\min\left\{\underline{F}_{\mathscr{D}}(\bar{x})(d), \overline{F}_{\mathscr{D}}(\bar{x})(d)\right\}, \max\left\{\underline{F}_{\mathscr{D}}(\bar{x})(d), \overline{F}_{\mathscr{D}}(\bar{x})(d)\right\}\Big].$$

Proof Similar to the proof of Theorem 3.1 in Ghosh et al. (2020). □

Definition 15 (*gH-Lipschitz continuous IVF* Ghosh et al. 2020). An IVF $\mathbf{F}$ is said to be gH-Lipschitz continuous on X if there exists $M > 0$ such that

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq M\|x - y\| \text{ for all } x, y \in X.$$

The constant M is called a Lipschitz constant.

Definition 16 (*gH-locally Lipschitz continuous IVF*). An IVF $\mathbf{F}$ is said to be gH-locally Lipschitz continuous on X if there exists $M' > 0$ and $\delta > 0$ such that

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq M'\|x - y\| \text{ and } \|x - y\| \leq \delta \text{ for all } x, y \in X.$$

Definition 17 (*Weak efficient solution* Ghosh et al. 2021). A point $\bar{x} \in S$ is said to be a weak efficient solution of IOP (27), if $\mathbf{F}(\bar{x}) \preceq \mathbf{F}(x)$ for all $x \in S$.

Definition 18 (*Supremum of a subset of $\overline{I(\mathbb{R})}$* Kumar and Ghosh 2021). Let $\mathbf{S}$ be a nonempty subset of $\overline{I(\mathbb{R})}$. An interval $\bar{\mathbf{A}} \in \overline{I(\mathbb{R})}$ is said to be an upper bound of $\mathbf{S}$ if $\mathbf{B} \preceq \bar{\mathbf{A}}$ for all $\mathbf{B}$ in $\mathbf{S}$. An upper bound $\bar{\mathbf{A}}$ of $\mathbf{S}$ is called a supremum of $\mathbf{S}$, denoted by sup $\mathbf{S}$, if for all upper bounds $\mathbf{C}$ of $\mathbf{S}$ in $\overline{I(\mathbb{R})}$, $\bar{\mathbf{A}} \preceq \mathbf{C}$. Moreover, if the supremum of the set $\mathbf{S}$ belongs to the set itself, then it is called maximum of $\mathbf{S}$, denoted by max $\mathbf{S}$.

Remark 4 (See Kumar and Ghosh 2021). Let Λ be an index set, and $\lambda \in \Lambda$. For any subset $\mathbf{S} = [a_\lambda, b_\lambda]$ of $\overline{I(\mathbb{R})}$, we have $\sup \mathbf{S} = \left[\sup_{\lambda\in\Lambda} a_\lambda,\ \sup_{\lambda\in\Lambda} b_\lambda\right]$.

Lemma 4 (See Kumar and Ghosh 2021). *Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be two proper extended IVFs defined on $\mathcal{S}$, which is a nonempty subset of X. Then,*

(i) $\inf_{x\in\mathcal{S}} \mathbf{F}_1(x) \oplus \inf_{x\in\mathcal{S}} \mathbf{F}_2(x) \preceq \inf_{x\in\mathcal{S}} (\mathbf{F}_1(x) \oplus \mathbf{F}_2(x))$ *and*
(ii) $\sup_{x\in\mathcal{S}} (\mathbf{F}_1(x) \oplus \mathbf{F}_2(x)) \preceq \sup_{x\in\mathcal{S}} \mathbf{F}_1(x) \oplus \sup_{x\in\mathcal{S}} \mathbf{F}_2(x)$.

Definition 19 (*Lower limit and gH-lower semicontinuity of an extended IVF* Kumar and Ghosh 2021). The lower limit of an extended IVF $\mathbf{F}$ at $\bar{x} \in X$, denoted by $\liminf_{x\to\bar{x}} \mathbf{F}(x)$, is defined by

$$\liminf_{x\to\bar{x}} \mathbf{F}(x) = \lim_{\delta\downarrow 0}(\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}) = \sup_{\delta>0}(\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}),$$

where $B_\delta(\bar{x})$ is an open ball with radius δ centered at $\bar{x}$. $\mathbf{F}$ is called gH-lower semicontinuous (gH-lsc) at a point $\bar{x}$ if $\mathbf{F}(\bar{x}) \preceq \liminf_{x\to\bar{x}} \mathbf{F}(x)$. Further, $\mathbf{F}$ is called gH-lsc on X if $\mathbf{F}$ is gH-lsc at every $\bar{x} \in X$.

Remark 5 By Note 5 of Kumar and Ghosh (2021), we see that $\mathbf{F}$ is gH-lsc at $\bar{x} \in X$ if and only if $\underline{F}$ and $\overline{F}$ both are lsc at $\bar{x}$.

Lemma 5 *Let $\boldsymbol{F}$: $\mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper convex IVF. Then, for all $x, y \in$ dom($\boldsymbol{F}$), we have*

$$\boldsymbol{F}_{\mathscr{D}}(x)(y - x) \preceq \boldsymbol{F}(y) \ominus_{gH} \boldsymbol{F}(x).$$

Proof By Definition 10 of gH-directional derivative, we have

$$\mathbf{F}_{\mathscr{D}}(x)(d) = \lim_{\lambda\to 0+} \frac{1}{\lambda} \odot \left(\mathbf{F}(x + \lambda d) \ominus_{gH} \mathbf{F}(x)\right). \tag{2}$$

By taking $d = y - x$ in (2), we get

$$\begin{aligned}\mathbf{F}_{\mathscr{D}}(x)(d) &= \lim_{\lambda\to 0+} \frac{1}{\lambda} \odot \left(\mathbf{F}(x + \lambda(y - x)) \ominus_{gH} \mathbf{F}(x)\right)\\ &= \lim_{\lambda\to 0+} \frac{1}{\lambda} \odot \left(\mathbf{F}((1 - \lambda)x + \lambda(y)) \ominus_{gH} \mathbf{F}(x)\right)\\ &\preceq \lim_{\lambda\to 0+} \frac{1}{\lambda} \odot \{((1 - \lambda) \odot \mathbf{F}(x) \oplus \lambda\\ &\quad \odot\mathbf{F}(y)) \ominus_{gH} \mathbf{F}(x)\}\end{aligned}$$

$$= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \lambda \left(\mathbf{F}(y) \ominus_{gH} \mathbf{F}(x)\right) \text{ by (ii) of Lemma 1}$$
$$= \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x).$$

Definition 20 (Convergence of a sequence in $I(\mathbb{R})^n$). A sequence $\widehat{\mathbf{G}} : \mathbb{N} \to I(\mathbb{R})^n$ is said to be convergent if there exists a $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$ such that

$$\|\widehat{\mathbf{G}}_k \ominus_{gH} \widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} \to 0 \text{ as } k \to \infty,$$

where $\widehat{\mathbf{G}}(k) = \widehat{\mathbf{G}}_k$, $k \in \mathbb{N}$.

Remark 6 It is noteworthy that if a sequence $\{\widehat{\mathbf{G}}_k\}$ in $I(\mathbb{R})^n$, where $\widehat{\mathbf{G}}_k = (\mathbf{G}_{k1}, \mathbf{G}_{k2}, \ldots, \mathbf{G}_{kn}) \in I(\mathbb{R})^n$ with $\mathbf{G}_{ki} = [\underline{g}_{ki}, \overline{g}_{ki}]$, converges to $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n) \in I(\mathbb{R})^n$ with $\mathbf{G}_i = [\underline{g}_i, \overline{g}_i]$, then according to Definition 2 and norm on $I(\mathbb{R})^n$, the corresponding sequence $\{\mathbf{G}_{ki}\}$ in $I(\mathbb{R})$ converges to $\mathbf{G}_i \in I(\mathbb{R})$ for each $i = 1, 2, \ldots, n$. Also, by Definition 20, the sequences $\underline{g}_{ki}$ and $\overline{g}_{ki}$ in $\mathbb{R}$ converge to $\underline{g}_i$ and $\overline{g}_i$ in $\mathbb{R}$, respectively, for each $i = 1, 2, \ldots, n$.

2.3 Results from convex analysis

Apart from the results of interval analysis, we use the following results from classical convex analysis throughout the article.

Definition 21 (*Projection* Rockafellar and Wets 2009). Let A be a nonempty closed set in $\mathbb{R}^n$. Then, the projection of a point $x \in \mathbb{R}^n$ onto the set A is denoted by $P(x \mid A)$, and is defined by

$$P(x \mid A) = \{y \in A : \|x - y\| = \inf\{\|x - u\| : u \in A\}\}.$$

Definition 22 (*Polar cone* Rockafellar and Wets 2009). Let A be a nonempty set in $\mathbb{R}^n$. Then, the polar cone of the set A is

$$A^o = \{x^* \in \mathbb{R}^n : \langle x^*, x\rangle \le 0 \text{ for all } x \in A\}.$$

Definition 23 (*Tangent cone* Rockafellar and Wets 2009). Let A be a nonempty closed convex set in $\mathbb{R}^n$. Then, the tangent cone to the set A at $x \in A$ is defined by

$$T_A(x) = \text{cl}\left(\bigcup_{t>0} \frac{A - x}{t}\right).$$

Definition 24 (*Normal cone* Rockafellar and Wets 2009). The normal cone to a nonempty set A in $\mathbb{R}^n$ at x is polar of the tangent cone at x to the A, i.e., $N_A(x) = T_A(x)^o$. Therefore,

$$N_A(x) = \left\{x^* \in \mathbb{R}^n : \langle x^*, y - x\rangle \le 0, \text{ for any } y \in A\right\}.$$

Lemma 6 (Hiriart-Urruty and Lemaréchal 2004). *Consider a convex set $S \subseteq \mathbb{R}^n$. Then, $\bar{x}$ is an element in the closure of S if and only if $\langle x, \bar{x}\rangle \le \psi_S^*(x)$ for all $x \in \mathbb{R}^n$, where ψ_S^* is the support function of S, i.e., $\psi_S^*(x) = \sup_{s\in S}\langle x, s\rangle$.*

Lemma 7 (Burke and Deng 2002). *Let C be a nonempty closed convex subset of $\mathbb{R}^n$.*

(i) *For all $y \in \mathbb{R}^n$, $\text{dist}(y, C) = \sup_{x\in C} \text{dist}(y, x + T_C(x))$, where the distance function is given by $\text{dist}(y, C) = \inf_{\overline{x}\in C} \|y - \overline{x}\|$.*

(ii) *Define $\rho(x) = \text{dist}(x, C)$. Then, for all $x \in C$ and $d \in \mathbb{R}^n$,*

$$\rho_{\mathscr{D}}(x)(d) = \text{dist}(d, T_C(x)) = \psi^*_{\mathbb{B}\cap N_C(x)}(d).$$

Moreover, if $d \in N_C(x)$, then $\rho_{\mathscr{D}}(x)(d) = \text{dist}(d, T_C(x)) = \psi^*_{\mathbb{B}\cap N_C(x)}(d) = \|d\|$.

Theorem 3 (Burke and Ferris 1993). *Suppose we have a linear programming problem (LPP)*

$$\min_{x\in S'} F(x), \tag{3}$$

where F is a linear real-valued function on $\mathbb{R}^n$ and S' is polyhedral subset of $\mathbb{R}^n$. Then, the solution set of LPP (3) is equal to the set of WSM of F over S'.

3 Support function in $I(\mathbb{R})^n$

In this section, we attempt to extend the conventional notion of support functions for subsets of $I(\mathbb{R})^n$. The derived concepts of support function are used later in Sect. 5 to derive dual characterizations of WSM for convex IVFs.

Definition 25 (*Support function of a subset of $I(\mathbb{R})^n$*). Let $\mathbf{S}$ be a nonempty subset of $I(\mathbb{R})^n$. Then, the support function of $\mathbf{S}$ at $x \in \mathbb{R}^n$, denoted by $\boldsymbol{\psi}_{\mathbf{S}}^*(x)$, is defined by

$$\boldsymbol{\psi}_{\mathbf{S}}^*(x) = \sup_{\widehat{\mathbf{A}}\in\mathbf{S}} x^\top \odot \widehat{\mathbf{A}}.$$

Lemma 8 *Let $\mathbf{S}_1, \mathbf{S}_2$ be two nonempty subsets of $I(\mathbb{R})^n$ such that $\mathbf{S}_1 \subseteq \mathbf{S}_2$. Then, for any $x \in X \subseteq \mathbb{R}^n$,*

$$\boldsymbol{\psi}_{\mathbf{S}_1}^*(x) \preceq \boldsymbol{\psi}_{\mathbf{S}_2}^*(x).$$

Proof For any $\widehat{\mathbf{B}} = \left([\underline{b}_1, \overline{b}_1], [\underline{b}_2, \overline{b}_2], \ldots, [\underline{b}_n, \overline{b}_n]\right) \in \mathbf{S}_2$ and $x \in X$, we have $x^\top \odot \widehat{\mathbf{B}} \preceq \boldsymbol{\psi}_{\mathbf{S}_2}^*(x)$. Given $\mathbf{S}_1 \subseteq \mathbf{S}_2$, i.e., for any $\widehat{\mathbf{D}} = \left([\underline{d}_1, \overline{d}_1], [\underline{d}_2, \overline{d}_2], \ldots, [\underline{d}_n, \overline{d}_n]\right) \in$

$\mathbf{S}_1$, we have $\widehat{\mathbf{D}} \in \mathbf{S}_2$. Therefore,

$$x^\top \odot \widehat{\mathbf{D}} \preceq \boldsymbol{\psi}^*_{\mathbf{S}_2}(x).$$

Since $\widehat{\mathbf{D}}$ is arbitrary, we get

$$\boldsymbol{\psi}^*_{\mathbf{S}_1}(x) = \sup_{\widehat{\mathbf{E}} \in \mathbf{S}_1} x^\top \odot \widehat{\mathbf{E}} \preceq \boldsymbol{\psi}^*_{\mathbf{S}_2}(x).$$

□

Theorem 4 *Let K be a nonempty closed convex cone in $X \subseteq \mathbb{R}^n$. Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be two nonempty subsets of $I(\mathbb{R})^n$. Then,*

$$\boldsymbol{\psi}^*_{\boldsymbol{P}}(x) \preceq \boldsymbol{\psi}^*_{\boldsymbol{Q}}(x) \text{ for all } x \in K \quad \text{if and only if} \quad \boldsymbol{\psi}^*_{\boldsymbol{P}}(x) \preceq \boldsymbol{\psi}^*_{\boldsymbol{Q} \oplus K^o}(x) \text{ for all } x \in X,$$

where K^o is the polar cone of K.

Proof Let $\boldsymbol{\psi}^*_{\mathbf{P}}(x) \preceq \boldsymbol{\psi}^*_{\mathbf{Q}}(x)$ for all $x \in K$. Consider $x \in K$. Clearly $\mathbf{Q} \subseteq \mathbf{Q} \oplus K^o$. Then, by Lemma 8, we have

$$\begin{aligned}\boldsymbol{\psi}^*_{\mathbf{Q}}(x) &\preceq \boldsymbol{\psi}^*_{\mathbf{Q}\oplus K^o}(x)\\ &\preceq \boldsymbol{\psi}^*_{\mathbf{Q}}(x) \oplus \psi^*_{K^o}(x) \text{ by (ii) of Lemma 4 and Definition 25}\\ &\preceq \boldsymbol{\psi}^*_{\mathbf{Q}}(x) \text{ because } \psi^*_{K^o}(x) = 0.\end{aligned}$$

Therefore,

$$\boldsymbol{\psi}^*_{\mathbf{Q}}(x) = \boldsymbol{\psi}^*_{\mathbf{Q}\oplus K^o}(x) \text{ for all } x \in K. \tag{4}$$

Also, by hypothesis, we have $\boldsymbol{\psi}^*_{\mathbf{P}}(x) \preceq \boldsymbol{\psi}^*_{\mathbf{Q}}(x)$ for all $x \in K$, and hence

$$\boldsymbol{\psi}^*_{\mathbf{P}}(x) \preceq \boldsymbol{\psi}^*_{\mathbf{Q}\oplus K^o}(x) \text{ for all } x \in K. \tag{5}$$

Suppose now if $x \notin K$, then there exists $z \in K^o$ such that $\langle z, x\rangle > 0$. Thus, for any $\widehat{\mathbf{A}} \in \mathbf{Q}$ and $\lambda \geq 0$, $\widehat{\mathbf{A}} \oplus \lambda z \in \mathbf{Q} \oplus K^o$. Also, $x^\top \odot (\widehat{\mathbf{A}} \oplus \lambda z)$

$$\begin{aligned}&= \left[\min\left\{\sum_{i=1}^n x_i(\underline{a}_i + \lambda z_i), \sum_{i=1}^n x_i(\overline{a}_i + \lambda z_i)\right\},\right.\\ &\qquad \left.\max\left\{\sum_{i=1}^n x_i(\underline{a}_i + \lambda z_i), \sum_{i=1}^n x_i(\overline{a}_i + \lambda z_i)\right\}\right]\\ &= \left[\min\left\{\sum_{i=1}^n x_i\underline{a}_i + \lambda x^\top z, \sum_{i=1}^n x_i\overline{a}_i + \lambda x^\top z\right\},\right.\\ &\qquad \left.\max\left\{\sum_{i=1}^n x_i\underline{a}_i + \lambda x^\top z, \sum_{i=1}^n x_i\overline{a}_i + \lambda x^\top z\right\}\right].\end{aligned}$$

Note that as $\lambda \to +\infty$, $\lambda x^\top z \to +\infty$, and therefore $x^\top \odot (\widehat{\mathbf{A}} \oplus \lambda z) \to +\infty$, which implies

$$\boldsymbol{\psi}^*_{\mathbf{Q}\oplus K^o}(x) = [+\infty, +\infty].$$

Thus,

$$\boldsymbol{\psi}^*_{\mathbf{P}}(x) \preceq \boldsymbol{\psi}^*_{\mathbf{Q}\oplus K^o}(x) \text{ for all } x \in X\backslash K. \tag{6}$$

Therefore, from (5) and (6), we have

$$\boldsymbol{\psi}^*_{\mathbf{P}}(x) \preceq \boldsymbol{\psi}^*_{\mathbf{Q}\oplus K^o}(x) \text{ for all } x \in X.$$

Proof of the converse part follows from (4). This completes the proof. □

Lemma 9 *Let P be a nonempty subset of $\mathbb{R}^n$ and $\boldsymbol{Q}$ be a nonempty closed convex subset of $I(\mathbb{R})^n$. Then, for any $x \in \mathbb{R}^n$,*

$$\psi^*_P(x) \preceq \boldsymbol{\psi}^*_{\boldsymbol{Q}}(x) \text{ if and only if } P \subseteq \boldsymbol{Q}.$$

Proof Let $\psi^*_P(x) \preceq \boldsymbol{\psi}^*_{\mathbf{Q}}(x)$ for $x \in \mathbb{R}^n$. Therefore, for any $p \in P$ and $x \in \mathbb{R}^n$, we have

$$\begin{aligned}&\langle x, p\rangle \preceq \sup_{\widehat{\mathbf{Q}}_i \in \mathbf{Q}} x^\top \odot \widehat{\mathbf{Q}}_i, \text{ where } \widehat{\mathbf{Q}}_i\\ &= \left(\left[\underline{q}_{i1}, \overline{q}_{i1}\right], \left[\underline{q}_{i2}, \overline{q}_{i2}\right], \ldots, \left[\underline{q}_{in}, \overline{q}_{in}\right]\right)\\ &\implies \langle x, p\rangle \preceq \sup_{\widehat{\mathbf{Q}}_i \in \mathbf{Q}}\\ &\left[\min\left\{\sum_{j=1}^n x_j\underline{q}_{ij}, \sum_{j=1}^n x_j\overline{q}_{ij}\right\},\right.\\ &\left.\max\left\{\sum_{j=1}^n x_j\underline{q}_{ij}, \sum_{j=1}^n x_j\overline{q}_{ij}\right\}\right].\end{aligned}$$

We now consider the following two possible cases.

- Case 1. Let $\sum_{j=1}^n x_j\underline{q}_{ij} \leq \sum_{j=1}^n x_j\overline{q}_{ij}$. In this case, we have

$$\langle x, p\rangle \preceq \sup_{\widehat{\mathbf{Q}}_i \in \mathbf{Q}} \left[\sum_{j=1}^n x_j\underline{q}_{ij}, \sum_{j=1}^n x_j\overline{q}_{ij}\right]. \tag{7}$$

  Next, define two sets S_1 and S_2 such that $S_1 = \left\{\underline{Q}_1, \underline{Q}_2, \ldots, \underline{Q}_n, \ldots\right\}$ and $S_2 = \{\overline{Q}_1, \overline{Q}_2, \ldots, \overline{Q}_n, \ldots\}$, where $\underline{Q}_i = \left(\underline{q}_{i1}, \underline{q}_{i2}, \ldots, \underline{q}_{in}\right) \in \mathbb{R}^n$ and $\overline{Q}_i = (\overline{q}_{i1}, \overline{q}_{i2}, \ldots, \overline{q}_{in}) \in \mathbb{R}^n$.
  Therefore, (7) along with Remark 4 gives,

$$\langle x, p\rangle \leq \sup_{\underline{Q}_i \in S_1} \left\langle x, \underline{Q}_i\right\rangle \tag{8}$$

$$\text{and } \langle x, p\rangle \leq \sup_{\overline{Q}_i \in S_2} \left\langle x, \overline{Q}_i\right\rangle. \tag{9}$$

Thus, from (8) and Lemma 6, we have $p \in S_1$, i.e., $p = \underline{Q}_m$ for some m.
To show that $p \in \mathbf{Q}$, we have to show that $p = \overline{Q}_m$ as well.
Note that

$$\langle x, p\rangle = \left\langle x, \underline{Q}_m \right\rangle \leq \langle x, \overline{Q}_m\rangle \text{ for all } x \in \mathbb{R}^n \text{ because}$$

$$\sum_{j=1}^{n} x_j \underline{q}_{ij} \leq \sum_{j=1}^{n} x_j \overline{q}_{ij}$$

$$\implies \langle x, p\rangle \leq \sup\langle x, \overline{Q}_m\rangle \text{ for all } x \in \mathbb{R}^n$$

$$\implies \langle x, p\rangle \leq \psi^*_{S'}(x) \text{ for all } x \in \mathbb{R}^n,$$

$$\text{where } S' \text{ is the singleton set } \left\{\overline{Q}_m\right\}. \tag{10}$$

Thus, from equation (10) and Lemma 6, we have $p \in S'$, i.e., $p = \overline{Q}_m$.
Hence, $p \in \mathbf{Q}$. Since p is arbitrary, $P \subseteq \mathbf{Q}$.

- Case 2. Let $\sum_{j=1}^{n} x_j \overline{q}_{ij} \leq \sum_{j=1}^{n} x_j \underline{q}_{ij}$. By following similar steps as in Case 1, in this case also, we get $P \subseteq \mathbf{Q}$.

Proof of the converse part follows from Lemma 8. □

Lemma 10 *For $x \in \mathbb{R}^n$ and $\widehat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n) \in \mathbf{S} \subseteq \mathbb{R}^n$, we have*

$$x^\top \odot \widehat{\mathbf{A}} \preceq \|x\| \odot \left[\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}\right].$$

Proof Note that $x^\top \odot \widehat{\mathbf{A}}$

$$= \left[\min\left\{\sum_{i=1}^{n} x_i \underline{a}_i, \sum_{i=1}^{n} x_i \overline{a}_i\right\}, \max\left\{\sum_{i=1}^{n} x_i \underline{a}_i, \sum_{i=1}^{n} x_i \overline{a}_i\right\}\right]$$

$$\preceq \left[\min\left\{\sum_{i=1}^{n} |x_i| \|\mathbf{A}_i\|_{I(\mathbb{R})}, \sum_{i=1}^{n} |x_i| \|\mathbf{A}_i\|_{I(\mathbb{R})}\right\}, \max\left\{\sum_{i=1}^{n} |x_i| \|\mathbf{A}_i\|_{I(\mathbb{R})}, \sum_{i=1}^{n} |x_i| \|\mathbf{A}_i\|_{I(\mathbb{R})}\right\}\right]$$

$$= \left[\min\left\{\|x\| \|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|x\| \|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}\right\}, \max\left\{\|x\| \|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|x\| \|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}\right\}\right]$$

$$\preceq \|x\| \odot \left[\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}\right].$$

□

Lemma 11 *The support function of a nonempty set $\mathbf{S} \subseteq I(\mathbb{R})^n$ is finite everywhere if and only if $\mathbf{S}$ is bounded.*

Proof Suppose that $\mathbf{S}$ is bounded, i.e., we have $M > 0$ such that $\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n} \leq M$ for all $\widehat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n) \in \mathbf{S}$ with $\mathbf{A}_i = [\underline{a}_i, \overline{a}_i]$ for each $i = 1, 2, \ldots, n$. By Lemma 10 and $\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n} \leq M$, for any $x \in \mathbb{R}^n$, we have

$$x^\top \odot \widehat{\mathbf{A}} \preceq \|x\| \odot \left[\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}\right]$$

$$\preceq \|x\| \odot [M, M] \preceq \|x\| M.$$

Since $\widehat{\mathbf{A}} \in \mathbf{S}$ is arbitrary chosen, therefore

$$\boldsymbol{\psi}^*_{\mathbf{S}}(x) = \sup_{\widehat{\mathbf{A}} \in \mathbf{S}} x^\top \odot \widehat{\mathbf{A}} \preceq \|x\| M.$$

Hence, $\boldsymbol{\psi}^*_{\mathbf{S}}(x)$ is finite everywhere.

Conversely, let $\boldsymbol{\psi}^*_{\mathbf{S}}(x)$ is finite for every $x \in \mathbb{R}^n$. Therefore, there exists an $M > 0$ such that $\boldsymbol{\psi}^*_{\mathbf{S}}(x) \preceq M$, which implies that for any $x \in \mathbb{R}^n$ and $\widehat{\mathbf{A}} \in \mathbf{S}$, we have

$$x^\top \odot \widehat{\mathbf{A}} = \left[\min\left\{\sum_{i=1}^{n} x_i \underline{a}_i, \sum_{i=1}^{n} x_i \overline{a}_i\right\}, \max\left\{\sum_{i=1}^{n} x_i \underline{a}_i, \sum_{i=1}^{n} x_i \overline{a}_i\right\}\right] \preceq M$$

$$\implies \sum_{i=1}^{n} x_i \underline{a}_i \leq M \text{ and } \sum_{i=1}^{n} x_i \overline{a}_i \leq M.$$

Take $\sum_{i=1}^{n} x_i \underline{a}_i \leq M$, then by Remark 1, we have

$$\langle x, \underline{a}\rangle \leq M, \text{ where } \underline{a} = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n) \in \mathbb{R}^n. \tag{11}$$

If $\underline{a} \neq 0$, choose $x = \frac{\underline{a}}{\|\underline{a}\|}$, then (11) gives

$$\left\langle \frac{\underline{a}}{\|\underline{a}\|}, \underline{a} \right\rangle \leq M$$

$$\implies \|\underline{a}\| \leq M, \text{ where } \underline{a} = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n) \in \mathbb{R}^n$$

$$\implies |\underline{a}_i| \leq M \text{ for each } i = 1, 2, \ldots, n.$$

Similarly, when we take $\sum_{i=1}^{n} x_i \overline{a}_i \leq M$, we get $|\overline{a}_i| \leq M$ for each $i = 1, 2, \ldots, n$. Therefore, we have

$$\mathbf{A}_i = [\underline{a}_i, \overline{a}_i] \preceq M \text{ for each } i = 1, 2, \ldots, n$$

$$\implies \widehat{\mathbf{A}} \preceq M.$$

Since $\widehat{\mathbf{A}} \in \mathbf{S}$ was arbitrary chosen, therefore we have $\widehat{\mathbf{A}} \preceq M$ for all $\widehat{\mathbf{A}} \in \mathbf{S}$. Hence, $\mathbf{S}$ is bounded. □

4 *gH*-subdifferentiability of convex IVFs

In this section, we develop gH-subdifferential calculus for convex IVFs that are used later to find dual characterization of WSM for convex IVFs.

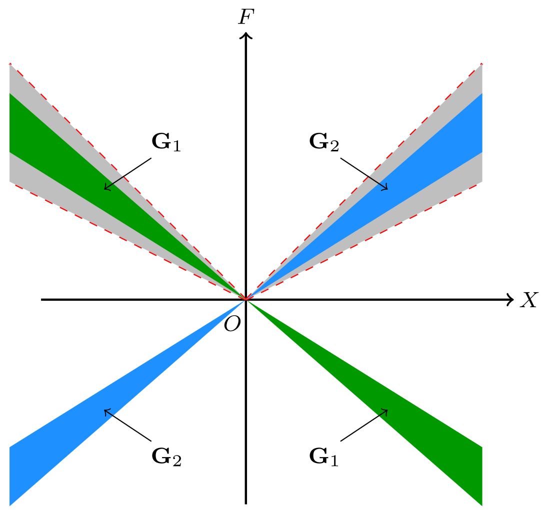

Fig. 1 The IVF $\mathbf{F}$ of Example 1

Definition 26 (gH-*subdifferentiability*). Let $\mathbf{F} : X \subseteq \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper convex IVF and $\bar{x} \in \text{dom}(\mathbf{F})$. Then, gH-subdifferential of $\mathbf{F}$ at $\bar{x}$, denoted by $\boldsymbol{\partial}\mathbf{F}(\bar{x})$ is defined by

$$\boldsymbol{\partial}\mathbf{F}(\bar{x}) = \Big\{\widehat{\mathbf{G}} \in I(\mathbb{R})^n : (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } x \in X\Big\}. \quad (12)$$

The elements of (12) are known as gH-subgradients of $\mathbf{F}$ at $\bar{x}$. Further, if $\boldsymbol{\partial}\mathbf{F}(\bar{x}) \neq \emptyset$, we say that $\mathbf{F}$ is gH-subdifferentiable at $\bar{x}$.

Example 1 Consider $\mathbf{F} : \mathbb{R} \to I(\mathbb{R})$ be a convex IVF such that $\mathbf{F}(x) = |x| \odot \mathbf{A}$, where $\mathbf{0} \preceq \mathbf{A}$. Let us check gH-subdifferentiability of $\mathbf{F}$ at 0.

$$\begin{aligned}\boldsymbol{\partial}\mathbf{F}(0) &= \{\mathbf{G} \in I(\mathbb{R}) : (x - 0) \odot \mathbf{G} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(0) \text{ for all } x \in \mathbb{R}\} \\ &= \{\mathbf{G} \in I(\mathbb{R}) : x \odot \mathbf{G} \preceq |x| \odot \mathbf{A} \text{ for all } x \in \mathbb{R}\}. \end{aligned} \quad (13)$$

- Case 1. $x \leq 0$. In this case, for all $x \in \mathbb{R}$, (13) gives,

$$x \odot \mathbf{G} \preceq (-x) \odot \mathbf{A} \implies (-1) \odot \mathbf{A} \preceq \mathbf{G}.$$

- Case 2. $x > 0$. In this case, for all $x \in \mathbb{R}$, (13) gives,

$$x \odot \mathbf{G} \preceq x \odot \mathbf{A} \implies \mathbf{G} \preceq \mathbf{A}.$$

Hence, from Case 1 and Case 2, we have $\boldsymbol{\partial}\mathbf{F}(0) = \{\mathbf{G} \in I(\mathbb{R}) : (-1) \odot \mathbf{A} \preceq \mathbf{G} \preceq \mathbf{A}\}$.

In Fig. 1, the IVF $\mathbf{F}$, with $\mathbf{A} = \left[\frac{1}{4}, 1\right]$, is drawn by the gray shaded region between two red dashed lines, and its possible two gH-subgradients $\mathbf{G}_1$ and $\mathbf{G}_2$ at 0 are shown by blue and green shaded regions, respectively.

Lemma 12 *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\boldsymbol{F} : X \to \overline{I(\mathbb{R})}$ be a proper convex IVF. Then, for any $\bar{x} \in \text{dom}(\boldsymbol{F})$ and $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$, the gH-subdifferential set of $\boldsymbol{F}$ at $\bar{x}$ is*

$$\boldsymbol{\partial}\boldsymbol{F}(\bar{x}) = \Big\{\widehat{\mathbf{G}} \in I(\mathbb{R})^n : h^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{F}_{\mathscr{D}}(\bar{x})(h)\Big\},$$

where $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h)$ is gH-directional derivative of $\boldsymbol{F}$ at $\bar{x}$ in the direction of h.

Proof Suppose $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$. Then, by Definition 26, we have

$$(x - \bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } x \in X. \quad (14)$$

By taking $x = \bar{x} + \lambda h$ with $\lambda > 0$ and $h \in \mathbb{R}^n$ in (14), we get

$$\begin{aligned} & h^\top \odot \widehat{\mathbf{G}} \preceq \frac{\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})}{\lambda} \\ \implies & h^\top \odot \widehat{\mathbf{G}} \preceq \lim_{\lambda \to 0} \frac{\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})}{\lambda} \\ \implies & h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}_{\mathscr{D}}(\bar{x})(h). \end{aligned}$$

Next, if we take any $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$ such that $h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}_{\mathscr{D}}(\bar{x})(h)$ for all $h \in \mathbb{R}^n$. Then, by a similar reasoning as above it can be seen that $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$. □

Theorem 5 *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\boldsymbol{F} : X \to \overline{I(\mathbb{R})}$ be a proper convex IVF with $\boldsymbol{F}(x) = [\underline{F}(x), \overline{F}(x)]$, where $\underline{F}, \overline{F} : X \to \overline{\mathbb{R}}$ are extended real-valued functions. Then, for any $\bar{x} \in \text{dom}(\boldsymbol{F})$, $\boldsymbol{\partial}\boldsymbol{F}(\bar{x})$ is closed and convex.*

Proof We first prove the closedness of $\boldsymbol{\partial}\mathbf{F}(\bar{x})$. Let $\{\widehat{\mathbf{G}}_k\}$ be a sequence in $\boldsymbol{\partial}\mathbf{F}(\bar{x})$, which converges to $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$, where $\widehat{\mathbf{G}}_k = (\mathbf{G}_{k1}, \mathbf{G}_{k2}, \ldots, \mathbf{G}_{kn})$ and $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n)$. Since $\widehat{\mathbf{G}}_k \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$, for all $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$, we have

$$\begin{aligned} & h^\top \odot \widehat{\mathbf{G}}_k \preceq \mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}), \\ & \implies \min\left\{\sum_{i=1}^n h_i \underline{g}_{ki}, \sum_{i=1}^n h_i \overline{g}_{ki}\right\} \\ & \leq \min\left\{\underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x})\right\} \\ & \text{and } \max\left\{\sum_{i=1}^n h_i \underline{g}_{ki}, \sum_{i=1}^n h_i \overline{g}_{ki}\right\} \\ & \leq \max\left\{\underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x})\right\}. \end{aligned} \quad (15)$$

Since the sequence $\{\widehat{\mathbf{G}}_k\}$ converges to $\widehat{\mathbf{G}}$, in view of Remark 6, the sequences $\{\underline{g}_{ki}\}$ and $\{\overline{g}_{ki}\}$ converge to $\underline{g}_i$ and

$\overline{g}_i$, respectively, for each $i = 1, 2, \ldots, n$. Thus,

$$\sum_{i=1}^{n} h_i \underline{g}_{ki} \to \sum_{i=1}^{n} h_i \underline{g}_i \text{ and } \sum_{i=1}^{n} h_i \overline{g}_{ki} \to \sum_{i=1}^{n} h_i \overline{g}_i \text{ as } k \to \infty. \tag{16}$$

Therefore, in view of (15) and (16), we have

$$\left(\min \left\{\sum_{i=1}^{n} h_i \underline{g}_{ki}, \sum_{i=1}^{n} h_i \overline{g}_{ki}\right\}\right) \to \left(\min \left\{\sum_{i=1}^{n} h_i \underline{g}_i, \sum_{i=1}^{n} h_i \overline{g}_i\right\}\right) \leq \min \left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\right\}$$

and

$$\left(\max \left\{\sum_{i=1}^{n} h_i \underline{g}_{ki}, \sum_{i=1}^{n} h_i \overline{g}_{ki}\right\}\right) \to \left(\max \left\{\sum_{i=1}^{n} h_i \underline{g}_i, \sum_{i=1}^{n} h_i \overline{g}_i\right\}\right) \leq \max \left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\right\}.$$

Thus,

$$\left[\min \left\{\sum_{i=1}^{n} h_i \underline{g}_i, \sum_{i=1}^{n} h_i \overline{g}_i\right\}, \max \left\{\sum_{i=1}^{n} h_i \underline{g}_i, \sum_{i=1}^{n} h_i \overline{g}_i\right\}\right] \preceq \mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x})$$
$$\implies h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } h \in X.$$

Therefore, $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$, and hence $\boldsymbol{\partial}\mathbf{F}(\bar{x})$ is closed.

To prove the convexity of $\boldsymbol{\partial}\mathbf{F}(\bar{x})$, let $\widehat{\mathbf{H}} = (\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_n)$ and $\widehat{\mathbf{K}} = (\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_n)$ be any two elements of $\boldsymbol{\partial}\mathbf{F}(\bar{x})$ with $\mathbf{H}_i = [\underline{h}_i, \overline{h}_i]$ and $\mathbf{K}_i = [\underline{k}_i, \overline{k}_i]$ for each $i = 1, 2, \ldots, n$. Then, for all $\lambda_1, \lambda_2 \geq 0$, with $\lambda_1 + \lambda_2 = 1$ and for any $d \in \mathbb{R}^n$, we have

$$d^\top \odot (\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}}) = \left[\min \left\{\sum_{i=1}^{n} d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^{n} d_i(\lambda_1 \overline{h}_i + \lambda_2 \overline{k}_i)\right\}, \max \left\{\sum_{i=1}^{n} d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^{n} d_i(\lambda_1 \overline{h}_i + \lambda_2 \overline{k}_i)\right\}\right].$$

- Case 1. Let $\min \left\{\sum_{i=1}^{n} d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^{n} d_i(\lambda_1 \overline{h}_i + \lambda_2 \overline{k}_i)\right\} = \sum_{i=1}^{n} d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i)$. Then,
$$\begin{aligned} d^\top \odot (\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}}) &= \left[\sum_{i=1}^{n} d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^{n} d_i(\lambda_1 \overline{h}_i + \lambda_2 \overline{k}_i)\right] \\ &= \left[\sum_{i=1}^{n} \lambda_1 d_i \underline{h}_i, \sum_{i=1}^{n} \lambda_1 d_i \overline{h}_i\right] \oplus \left[\sum_{i=1}^{n} \lambda_2 d_i \underline{k}_i, \sum_{i=1}^{n} \lambda_2 d_i \overline{k}_i\right] \\ &= \lambda_1 \odot d^\top \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot d^\top \odot \widehat{\mathbf{K}} \\ &\preceq \lambda_1 \odot \mathbf{F}_{\mathscr{D}}(\bar{x})(d) \oplus \lambda_2 \odot \mathbf{F}_{\mathscr{D}}(\bar{x})(d) \text{ by Lemma 12} \\ &= \mathbf{F}_{\mathscr{D}}(\bar{x})(d) \text{ for any } d \in \mathbb{R}^n. \end{aligned}$$
Hence, $d^\top \odot (\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}}) \preceq \mathbf{F}_{\mathscr{D}}(\bar{x})(d)$ for any $d \in \mathbb{R}^n$. Therefore, by Lemma 12, $\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$.
- Case 2. Let $\min \left\{\sum_{i=1}^{n} d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^{n} d_i(\lambda_1 \overline{h}_i + \lambda_2 \overline{k}_i)\right\} = \sum_{i=1}^{n} d_i(\lambda_1 \overline{h}_i + \lambda_2 \overline{k}_i)$. Proof contains similar steps as in Case 1.

Thus, for any $\bar{x} \in \text{dom}(\mathbf{F})$, $\boldsymbol{\partial}\mathbf{F}(\bar{x})$ is convex. □

Theorem 6 *Let X be a nonempty convex subset of $\mathbb{R}^n$ and let $\boldsymbol{F} : X \to I(\mathbb{R})$ be a gH-differentiable convex IVF at $\bar{x} \in X$. Then,*

$$\boldsymbol{\partial F}(\bar{x}) = \{\nabla \boldsymbol{F}(\bar{x})\}.$$

Proof Let $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$. Since $\mathbf{F}$ is gH-differentiable at $\bar{x}$, with the help of Lemma 3 and Lemma 12, we get

$$h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n$$
$$\implies h^\top \odot \widehat{\mathbf{G}} \preceq \sum_{i=1}^{n} h_i \odot D_i\mathbf{F}(\bar{x}) \text{ by Theorem 1.} \tag{17}$$

Replacing h by $-h$ in (17), we obtain

$$(-h)^\top \odot \widehat{\mathbf{G}} \preceq \sum_{i=1}^{n} (-h_i) \odot D_i\mathbf{F}(\bar{x})$$
$$\implies \sum_{i=1}^{n} h_i \odot D_i\mathbf{F}(\bar{x}) \preceq h^\top \odot \widehat{\mathbf{G}} \text{ for all } h \in \mathbb{R}^n. \tag{18}$$

Thus, (17) and (18), simultaneously give

$$\sum_{i=1}^{n} h_i \odot D_i\mathbf{F}(\bar{x}) = h^\top \odot \widehat{\mathbf{G}} \text{ for all } h \in \mathbb{R}^n. \tag{19}$$

Therefore, for each $i \in \{1, 2, \ldots, n\}$, by choosing $h = e_i$ in (19), we have $D_i\mathbf{F}(\bar{x}) = \mathbf{G}_i$.

Hence, $\nabla \mathbf{F}(\bar{x}) = \widehat{\mathbf{G}}$. Since $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$ is arbitrary, $\boldsymbol{\partial}\mathbf{F}(\bar{x}) = \{\nabla \mathbf{F}(\bar{x})\}$. □

Lemma 13 *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\boldsymbol{F} : X \to \overline{I(\mathbb{R})}$ be a proper convex IVF with $\boldsymbol{F}(x) = [\underline{F}(x), \overline{F}(x)]$, where $\underline{F}$, $\overline{F} : X \to \overline{\mathbb{R}}$ are extended real-valued functions. Then, the subdifferential set of $\boldsymbol{F}$ at $\bar{x} \in$ int(dom($\boldsymbol{F}$)) can be obtained by the subdifferential sets of $\underline{F}$ and $\overline{F}$ at $\bar{x}$ and vice-versa.*

Proof Since $\mathbf{F}$ is proper convex, with the help of Lemma 2, we note that $\underline{F}$ and $\overline{F}$ are also convex. Therefore, by the property of real-valued proper convex functions, the subdifferential sets of $\underline{F}$ and $\overline{F}$ at $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$ are nonempty (see Beck 2017). Let $\underline{g} = (\underline{g}_1, \underline{g}_2, \ldots, \underline{g}_n) \in \partial \underline{F}(\bar{x})$ and $\overline{g} = (\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n) \in \partial \overline{F}(\bar{x})$. Then, by Definition 26 of gH-subdifferentiability, for any $h \in \mathbb{R}^n$ such that $\bar{x}+h \in X$, we have

$$h^\top \odot \underline{g} \leq \underline{F}(\bar{x}+h) - \underline{F}(\bar{x}) \text{ and } h^\top \odot \overline{g} \leq \overline{F}(\bar{x}+h) - \overline{F}(\bar{x}). \tag{20}$$

Note that $\mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x})$

$$\begin{aligned}
&= \Big[\min\left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\right\}, \\
&\qquad \max\left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\right\}\Big] \\
&\implies \left[\min\left\{h^\top \odot \underline{g}, h^\top \odot \overline{g}\right\}, \max\left\{h^\top \odot \underline{g}, h^\top \odot \overline{g}\right\}\right] \\
&\preceq \mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ by (20)} \\
&\implies h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x}), \\
&\quad \text{where } \widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n) \text{ with } \mathbf{G}_i = [\underline{g}_i, \overline{g}_i] \\
&\implies \widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x}).
\end{aligned}$$

Thus, for any $\underline{g} \in \partial \underline{F}(\bar{x})$ and $\overline{g} \in \partial \overline{F}(\bar{x})$, we have the corresponding $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$.

To prove the converse part, for any $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$, take $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n) \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$ with $\mathbf{G}_i = [\underline{g}_i, \overline{g}_i]$, $i = 1, 2, \ldots, n$. Then, by Definition 26 of gH-subdifferentiability, we have

$$\begin{aligned}
&h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x}) \\
&\quad \text{for all } h \in \mathbb{R}^n \text{ such that } \bar{x}+h \in X \\
&\implies \left[\min\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\}, \right. \\
&\quad \left. \max\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\}\right] \preceq \mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\min\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\} \\
&\leq \min\left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\right\}
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\max\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\} \\
&\leq \max\left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\right\}.
\end{aligned} \tag{22}$$

We now consider the following two possible cases.

- Case 1. Let $\min\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\} = \sum_{i=1}^n h_i \underline{g}_i$ and $\min\left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\right\} = \underline{F}(\bar{x}+h) - \underline{F}(\bar{x})$. In this case, by (21) and (22), we have

$$\begin{aligned}
&\sum_{i=1}^n h_i \underline{g}_i \leq \underline{F}(\bar{x}+h) - \underline{F}(\bar{x}) \\
&\quad \text{and } \sum_{i=1}^n h_i \overline{g}_i \leq \overline{F}(\bar{x}+h) - \overline{F}(\bar{x}) \\
&\implies h^\top \odot \underline{g} \leq \underline{F}(\bar{x}+h) - \underline{F}(\bar{x}) \\
&\quad \text{and } h^\top \odot \overline{g} \leq \overline{F}(\bar{x}+h) - \overline{F}(\bar{x}), \\
&\quad \text{where } \underline{g} = (\underline{g}_1, \underline{g}_2, \ldots, \underline{g}_n) \in \mathbb{R}^n \\
&\quad \text{and } \overline{g} = (\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n) \in \mathbb{R}^n.
\end{aligned}$$

  Thus, we get $\underline{g} \in \partial \underline{F}(\bar{x})$ and $\overline{g} \in \partial \overline{F}(\bar{x})$, which are required.
- Case 2. Let $\min\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\} = \sum_{i=1}^n h_i \overline{g}_i$ and $\min\left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\right\} = \underline{F}(\bar{x}+h) - \underline{F}(\bar{x})$. Proof contains similar steps as in Case 1.

From Case 1 and Case 2, it is clear that for any $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$, we can obtain the subgradients of $\overline{F}$ and $\underline{F}$ at $\bar{x}$. This completes the proof for the converse part. □

Remark 7 By Lemma 13, it is easy to note that for any proper convex IVF $\mathbf{F}(x) = [\underline{F}(x), \overline{F}(x)]$ and $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$, $\boldsymbol{\partial}\mathbf{F}(\bar{x})$ is nonempty.

Theorem 7 *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\boldsymbol{F} : X \to \overline{I(\mathbb{R})}$ be a proper convex IVF with $\boldsymbol{F}(x) = [\underline{F}(x), \overline{F}(x)]$, where $\underline{F}$, $\overline{F} : X \to \overline{\mathbb{R}}$ are extended real-valued functions. Then, at any $\bar{x} \in$ int(dom($\boldsymbol{F}$)),*

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h) = \boldsymbol{\psi}^*_{\boldsymbol{\partial}\boldsymbol{F}(\bar{x})}(h) \text{ for all } h \in \mathbb{R}^n \text{ such that } \bar{x}+h \in X,$$

where $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h)$ is gH-directional derivative of $\boldsymbol{F}$ at $\bar{x}$ in the direction of h.

Proof Note that $\underline{F}(x)$ and $\overline{F}(x)$ are proper convex, and therefore $\partial \underline{F}(\bar{x})$ and $\partial \overline{F}(\bar{x})$ are nonempty. Let $\underline{g} = (\underline{g}_1, \underline{g}_2, \ldots, \underline{g}_n) \in \partial \underline{F}(\bar{x})$ and $\overline{g} = (\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n) \in \partial \overline{F}(\bar{x})$ for $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$. By the property of real-valued convex functions (see Dhara and Dutta 2011), we have

$$\underline{F}_{\mathscr{D}}(\bar{x})(h) = \psi^*_{\partial \underline{F}(\bar{x})}(h) \text{ and } \overline{F}_{\mathscr{D}}(\bar{x})(h) = \psi^*_{\partial \overline{F}(\bar{x})}(h) \quad \text{for all } h \in \mathbb{R}^n \text{ such that } \bar{x} + h \in X.$$

Due to Theorem 2, we get

$$\begin{aligned} \mathbf{F}_{\mathscr{D}}(\bar{x})(h) &= \Big[\min \Big\{ \underline{F}_{\mathscr{D}}(\bar{x})(h), \overline{F}_{\mathscr{D}}(\bar{x})(h) \Big\}, \max \Big\{ \underline{F}_{\mathscr{D}}(\bar{x})(h), \overline{F}_{\mathscr{D}}(\bar{x})(h) \Big\} \Big] \\ &= \Big[\min \Big\{ \psi^*_{\partial \underline{F}(\bar{x})}(h), \psi^*_{\partial \overline{F}(\bar{x})}(h) \Big\}, \max \Big\{ \psi^*_{\partial \underline{F}(\bar{x})}(h), \psi^*_{\partial \overline{F}(\bar{x})}(h) \Big\} \Big]. \end{aligned} \tag{23}$$

We now consider the following two possible cases.

- Case 1. Let $\psi^*_{\partial \underline{F}(\bar{x})}(h) \leq \psi^*_{\partial \overline{F}(\bar{x})}(h)$. In this case, by (23), we get

$$\begin{aligned} \mathbf{F}_{\mathscr{D}}(\bar{x})(h) &= \Big[\psi^*_{\partial \underline{F}(\bar{x})}(h), \psi^*_{\partial \overline{F}(\bar{x})}(h) \Big] \\ &= \Big[\sup_{\underline{g} \in \partial \underline{F}(\bar{x})} h^\top \odot \underline{g}, \sup_{\overline{g} \in \partial \overline{F}(\bar{x})} h^\top \odot \overline{g} \Big] \\ &= \sup_{\underline{g} \in \partial \underline{F}(\bar{x}),\ \overline{g} \in \partial \overline{F}(\bar{x})} \Big[h^\top \odot \underline{g}, h^\top \odot \overline{g} \Big] \\ &= \sup_{\underline{g} \in \partial \underline{F}(\bar{x}),\ \overline{g} \in \partial \overline{F}(\bar{x})} \Big[\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i \Big]. \end{aligned} \tag{24}$$

  We have seen in Lemma 13 that corresponding to every $\underline{g} \in \partial \underline{F}(\bar{x})$ and $\overline{g} \in \partial \overline{F}(\bar{x})$, we get $\widehat{\mathbf{G}} \in \boldsymbol{\partial} \mathbf{F}(\bar{x})$ and vice-versa. Thus, for $\underline{g} = (\underline{g}_1, \underline{g}_2, \ldots, \underline{g}_n) \in \partial \underline{F}(\bar{x})$ and $\overline{g} = (\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n) \in \partial \overline{F}(\bar{x})$, by (24), we obtain

$$\begin{aligned} \mathbf{F}_{\mathscr{D}}(\bar{x})(h) &= \sup_{\widehat{\mathbf{G}} \in \boldsymbol{\partial} \mathbf{F}(\bar{x})} h^\top \odot \widehat{\mathbf{G}}, \text{ where } \widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n) \text{ with } \mathbf{G}_i = [\underline{g}_i, \overline{g}_i] \\ &= \boldsymbol{\psi}^*_{\boldsymbol{\partial} \mathbf{F}(\bar{x})}(h). \end{aligned}$$

- Case 2. Let $\psi^*_{\partial \overline{F}(\bar{x})}(h) \leq \psi^*_{\partial \underline{F}(\bar{x})}(h)$. Proof contains similar steps as in Case 1.

Hence, from Case 1 and Case 2, we have

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(h) = \boldsymbol{\psi}^*_{\boldsymbol{\partial} \mathbf{F}(\bar{x})}(h) \text{ for all } h \in \mathbb{R}^n \text{ such that } \bar{x} + h \in X.$$

□

Theorem 8 *Let* $\boldsymbol{F} : X \to \overline{I(\mathbb{R})}$ *be a proper convex IVF and* $\bar{x} \in \text{int}(\text{dom}(\boldsymbol{F}))$. *Then, the gH-subdifferential set of* $\boldsymbol{F}$ *at* $\bar{x}$ *is bounded.*

Proof Note that by Theorem 2, for $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$, the directional derivative of $\mathbf{F}$ at $\bar{x}$ exists everywhere. Thus, for all $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$, we have

$$\begin{aligned} &\mathbf{F}_{\mathscr{D}}(\bar{x})(h) \text{ is finite} \\ &\implies \boldsymbol{\psi}^*_{\boldsymbol{\partial} \mathbf{F}(\bar{x})}(h) \text{ is finite by Theorem 7} \\ &\implies \boldsymbol{\partial} \mathbf{F}(\bar{x}) \text{ is bounded by Lemma 11.} \end{aligned}$$

Hence, the gH-subdifferential set of $\mathbf{F}$ at $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$ is bounded, i.e., for every $\widehat{\mathbf{G}} \in \boldsymbol{\partial} \mathbf{F}(\bar{x})$, there exists an $M > 0$ such that $\|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} \leq M$. □

Lemma 14 *Let* $\boldsymbol{F}$ *be an IVF on a nonempty set* $X \subseteq \mathbb{R}^n$ *such that*

$$\boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(y) \preceq c\|x - y\| \text{ for all } x, y \in X,$$

where $c \in \mathbb{R}$. *Then,*

$$\|\boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(y)\|_{I(\mathbb{R})} \leq c\|x - y\| \text{ for all } x, y \in X.$$

Proof We have $\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y) \preceq c\|x - y\|$ for all $x, y \in X$, which implies that

$$\underline{F}(x) - \underline{F}(y) \leq c\|x - y\| \text{ and } \overline{F}(x) - \overline{F}(y) \leq c\|x - y\|. \tag{25}$$

Interchanging x and y in (25), we obtain

$$\underline{F}(y) - \underline{F}(x) \leq c\|x - y\| \text{ and } \overline{F}(y) - \overline{F}(x) \leq c\|x - y\|. \tag{26}$$

With the help of (25) and (26), we get

$$|\underline{F}(x) - \underline{F}(y)| \leq c\|x - y\| \text{ and } |\overline{F}(x) - \overline{F}(y)| \leq c\|x - y\| \text{ for all } x, y \in X,$$

which implies

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq c\|x - y\| \text{ for all } x, y \in X.$$

□

Theorem 9 *Let* X *be a nonempty convex subset of* $\mathbb{R}^n$ *and* $\boldsymbol{F}$ *be a convex IVF on* X *such that* $\boldsymbol{F}$ *has gH-subgradient at every* $x \in X$. *Then,* $\boldsymbol{F}$ *is gH-Lipschitz continuous on* X.

Proof Since $\mathbf{F}$ has gH-subgradient at every $x \in X$, then there exists a $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$ such that

$$(y - x)^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x) \text{ for all } y \in X$$

$$\implies (-1) \odot \left((x-y)^\top \odot \widehat{\mathbf{G}}\right) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x)$$
$$\implies \mathbf{F}(x) \ominus_{gH} \mathbf{F}(y) \preceq (x-y)^\top \odot \widehat{\mathbf{G}}$$
$$\implies \mathbf{F}(x) \ominus_{gH} \mathbf{F}(y) \preceq \|x-y\| \odot \left[\|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n}, \|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n}\right] \text{ by Lemma 10}$$
$$\implies \|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq \|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} \|x-y\| \text{ by Lemma 14}$$
$$\implies \|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq M\|x-y\|, \text{ where } \|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} \leq M \text{ by Theorem 8.}$$

Thus, $\mathbf{F}$ is gH-Lipschitz continuous on X. □

5 Weak sharp minima and its characterizations

In this section, we present the main results—primal and dual characterizations of WSM for a gH-lsc and convex IVF.

Definition 27 (*WSM for an IVF*). Let $\mathbf{F}\colon \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a gH-lsc and convex IVF. Let $\bar{S}$ and S be two nonempty closed convex sets such that $\bar{S} \subseteq S \subseteq \mathbb{R}^n$. Further, let $\text{dom}(\mathbf{F}) \cap S \neq \emptyset$. Then, the set $\bar{S}$ is said to be a set of WSM of $\mathbf{F}$ over the set S with modulus $\alpha > 0$ if

$$\mathbf{F}(\bar{x}) \oplus \alpha \,\text{dist}(x, \bar{S}) \preceq \mathbf{F}(x) \text{ for all } \bar{x} \in \bar{S} \text{ and } x \in S.$$

Remark 8 Let $\mathbf{F}\colon \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a gH-lsc and convex IVF with $\mathbf{F}(x) = [\underline{F}(x), \overline{F}(x)]$ for all $x \in \mathbb{R}^n$, where $\underline{F}, \overline{F} : \mathbb{R}^n \to \overline{\mathbb{R}}$ be two extended real-valued functions. Then, $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$ if and only if $\bar{S}$ is a set of WSM of $\underline{F}$ and $\overline{F}$ over S with modulus $\alpha > 0$. The reason is as follows. By Remark 5 and Lemma 2, it is easy to see that the functions $\underline{F}$ and $\overline{F}$ are lsc and convex. Let $\bar{S}$ be a set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$. Then,

$$\begin{aligned}
&\mathbf{F}(\bar{x}) \oplus \alpha \,\text{dist}(x, \bar{S}) \preceq \mathbf{F}(x) \text{ for all } \bar{x} \in \bar{S} \text{ and } x \in S\\
&\iff [\underline{F}(\bar{x}) + \alpha \,\text{dist}(x, \bar{S}), \overline{F}(\bar{x}) + \alpha \,\text{dist}(x, \bar{S})] \preceq [\underline{F}(x), \overline{F}(x)] \text{ for all } \bar{x} \in \bar{S} \text{ and } x \in S\\
&\iff \underline{F}(\bar{x}) + \alpha \,\text{dist}(x, \bar{S}) \leq \underline{F}(x) \text{ and } \overline{F}(\bar{x}) + \alpha \,\text{dist}(x, \bar{S}) \leq \overline{F}(x) \text{ for all } \bar{x} \in \bar{S} \text{ and } x \in S\\
&\iff \bar{S} \text{ is a set of WSM of both } \underline{F} \text{ and } \overline{F} \text{ over } S \text{ with modulus } \alpha > 0.
\end{aligned}$$

Example 2 Let $\mathbf{F} : \mathbb{R}^2 \to \overline{I(\mathbb{R})}$ be an IVF defined by

$$\mathbf{F}(x) = [5 - x_1x_2 - x_1, 10 - x_1^2x_2 - x_2^2x_1].$$

Let $S = [-a, 0] \times [-a, 0] \subseteq \mathbb{R}^2$ and $\bar{S} = \{0\} \times [-a, 0]$, where $a > 0$. Thus, $\bar{S} \subseteq S \subseteq \mathbb{R}^n$. Clearly, the functions $\underline{F}$ and $\overline{F}$ are $5 - x_1x_2 - x_1$ and $10 - x_1^2x_2 - x_2^2x_1$, respectively. Note that for any $\alpha > 0$,

$$\underline{F}(\bar{x}) + \alpha \,\text{dist}(x, \bar{S}) \leq \underline{F}(x) \text{ and } \overline{F}(\bar{x}) + \alpha \,\text{dist}(x, \bar{S}) \leq \overline{F}(x) \text{ for all } \bar{x} \in \bar{S} \text{ and } x \in S.$$

Thus, $\bar{S} = \{0\} \times [-a, 0]$ is a set of WSM of both $\underline{F}$ and $\overline{F}$ over S with modulus α, for any $\alpha > 0$. Therefore, by Remark 8, $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$.

Let us consider an IOP

$$\min_{x \in S} \mathbf{F}(x), \tag{27}$$

where $\mathbf{F}$ and S are same as in Definition 27. Note that constrained IOP (27) can be converted into unconstrained IOP (28)

$$\min_{x \in \mathbb{R}^n} \mathbf{F}_o(x), \tag{28}$$

where $\mathbf{F}_o(x) = \begin{cases} \mathbf{F}(x), & \text{if } x \in S,\\ +\infty, & \text{otherwise.}\end{cases}$

Thus, we can solve the constrained and unconstrained IOP both using the concepts of WSM. Regarding this, we give two characterizations primal and dual below.

Theorem 10 (Primal characterization). *Let $\mathbf{F}$, and S be as in IOP (27). Further, define an IVF $\mathbf{F}_o : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ as in IOP (28). Let $\bar{S}$ be the set of WSM of $\mathbf{F}$. Then, the set $\bar{S}$ is a set of WSM of $\mathbf{F}$ over the set S with modulus $\alpha > 0$ if and only if*

$$\alpha \,\text{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ for all } x \in \bar{S} \text{ and } d \in \mathbb{R}^n. \tag{29}$$

Proof Suppose $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$. Then, by Definition 27, for any $x \in \bar{S}$, $d \in \mathbb{R}^n$, and $t > 0$, we have

$$\begin{aligned}
&\mathbf{F}_o(x) \oplus \alpha \,\text{dist}(x + td, \bar{S}) \preceq \mathbf{F}_o(x + td)\\
&\implies \alpha \,\text{dist}(x + td, \bar{S}) \preceq \mathbf{F}_o(x + td) \ominus_{gH} \mathbf{F}_o(x)\\
&\implies \frac{\alpha}{t}\left(\text{dist}(x + td, \bar{S}) - \text{dist}(x, \bar{S})\right) \preceq \frac{1}{t} \odot \left(\mathbf{F}_o(x + td) \ominus_{gH} \mathbf{F}_o(x)\right)\\
&\implies \lim_{t\to 0} \frac{\alpha}{t}\left(\text{dist}(x + td, \bar{S}) - \text{dist}(x, \bar{S})\right) \preceq \lim_{t\to 0} \frac{1}{t} \odot \left(\mathbf{F}_o(x + td) \ominus_{gH} \mathbf{F}_o(x)\right)\\
&\implies \alpha \lim_{t\to 0} \frac{1}{t}\left(\text{dist}(x + td, \bar{S}) - \text{dist}(x, \bar{S})\right) \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ by Definition 10}
\end{aligned}$$

$$\implies \alpha\,\mathrm{dist}(d, T_{\bar{S}}(x))$$
$$\preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ by part (ii) of Lemma 7.}$$

Thus,

$$\alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ for all } x \in \bar{S} \text{ and } d \in \mathbb{R}^n.$$

For the converse part, let $y \in S$ and $x \in \bar{S}$. Therefore, from Lemma 5, we get

$$\begin{aligned}
&\mathbf{F}_{o\mathscr{D}}(x)(y-x) \preceq \mathbf{F}_o(y) \ominus_{gH} \mathbf{F}_o(x)\\
&\implies \mathbf{F}_o(x) \oplus \mathbf{F}_{o\mathscr{D}}(x)(y-x) \preceq \mathbf{F}_o(y)\\
&\implies \mathbf{F}_o(x) \oplus \alpha\,\mathrm{dist}(y-x, T_{\bar{S}}(x)) \preceq \mathbf{F}_o(y) \text{ by (29)}\\
&\implies \mathbf{F}_o(x) \oplus \alpha\,\mathrm{dist}(y, x + T_{\bar{S}}(x)) \preceq \mathbf{F}_o(y).
\end{aligned}$$

Since $x \in \bar{S}$ is arbitrary, we have

$$\begin{aligned}
&\mathbf{F}_o(x) \oplus \alpha \sup_{x \in \bar{S}} \mathrm{dist}(y, x + T_{\bar{S}}(x)) \preceq \mathbf{F}_o(y)\\
&\implies \mathbf{F}_o(x) \oplus \alpha\,\mathrm{dist}(y, \bar{S}) \preceq \mathbf{F}_o(y)\\
&\quad \text{for all } x \in \bar{S} \text{ and } \alpha > 0 \text{ by part (i) of Lemma 7.}
\end{aligned}$$

Hence, $\bar{S}$ is the set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$, and the proof is complete. □

Theorem 11 (Dual characterizations). *Let $\boldsymbol{F}$, and S be as in IOP (27). Further, define an IVF $\boldsymbol{F}_o : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ as in IOP (28). Let $\bar{S}$ be the set of WSM of $\boldsymbol{F}$. Then, for any $\alpha > 0$, the following statements are equivalent.*

(a) *The set $\bar{S}$ is a set of WSM of $\boldsymbol{F}$ over the set S with modulus α.*
(b) *The normal cone inclusion holds. That is,*

$$\alpha\mathbb{B} \cap N_{\bar{S}}(x) \subseteq \boldsymbol{\partial F}_o(x) \text{ for all } x \in \bar{S}.$$

(c) *For all $x \in \bar{S}$ and $d \in T_S(x)$,*

$$\alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \boldsymbol{F}_{\mathscr{D}}(x)(d).$$

(d) *The following inclusion holds,*

$$\alpha\mathbb{B} \bigcap \left(\bigcup_{x \in \bar{S}} N_{\bar{S}}(x) \right) \subseteq \bigcup_{x \in \bar{S}} \boldsymbol{\partial F}_o(x).$$

(e) *For all $x \in \bar{S}$ and $d \in T_S(x) \cap N_{\bar{S}}(x)$,*

$$\alpha \|d\| \preceq \boldsymbol{F}_{\mathscr{D}}(x)(d).$$

(f) *For all $y \in S$,*

$$\alpha\,\mathrm{dist}(y, \bar{S}) \preceq \boldsymbol{F}_{\mathscr{D}}(p)(y-p),$$

where $p \in P(y \mid \bar{S})$.

Proof (a) $\iff$ (b). Let $x \in \bar{S}$. By hypothesis, $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S. Therefore, by Theorem 10, we get

$$\alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ for all } d \in \mathbb{R}^n,$$

which along with Theorem 7 imply

$$\alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d) \text{ for all } d \in \mathbb{R}^n. \tag{30}$$

Notice that for all $x \in \bar{S}$ and $d \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\alpha\,\mathrm{dist}(d, T_{\bar{S}}(x))\\
&= \alpha \psi^*_{\mathbb{B} \cap N_{\bar{S}}(x)}(d) \text{ by (ii) of Lemma 7}\\
&= \alpha \sup \langle z, d \rangle, \text{ where } z \in \mathbb{B} \cap N_{\bar{S}}(x)\\
&= \sup \langle \alpha z, d \rangle, \text{ where } z \in \mathbb{B} \cap N_{\bar{S}}(x) \text{ and } \alpha > 0\\
&= \sup \langle z, d \rangle, \text{ where } z \in \alpha\mathbb{B} \cap N_{\bar{S}}(x)\\
&= \psi^*_{\alpha\mathbb{B} \cap N_{\bar{S}}(x)}(d) \text{ for all } d \in \mathbb{R}^n.
\end{aligned}$$

That is,

$$\alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) = \psi^*_{\alpha\mathbb{B} \cap N_{\bar{S}}(x)}(d) \text{ for all } d \in \mathbb{R}^n. \tag{31}$$

Thus, by (30) and (31), we get

$$\psi^*_{\alpha\mathbb{B} \cap N_{\bar{S}}(x)}(d) \preceq \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d). \tag{32}$$

Next, with the help of Lemma 9, we get the desired result

$$\alpha\mathbb{B} \cap N_{\bar{S}}(x) \subseteq \boldsymbol{\partial}\mathbf{F}_o(x) \text{ for all } d \in \mathbb{R}^n. \tag{33}$$

Conversely, we have

$$\alpha\mathbb{B} \cap N_{\bar{S}}(x) \subseteq \boldsymbol{\partial}\mathbf{F}_o(x) \text{ for all } x \in \bar{S} \tag{34}$$
$$\begin{aligned}
&\implies \psi^*_{\alpha\mathbb{B} \cap N_{\bar{S}}(x)}(d)\\
&\preceq \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d) \text{ for all } d \in \mathbb{R}^n \text{ by Lemma 8}\\
&\implies \alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d) \text{ for all } d \in \mathbb{R}^n \text{ by (31).}
\end{aligned} \tag{35}$$

Also, by Theorem 7, we have

$$\boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d) = \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ for all } d \in \mathbb{R}^n.$$

Thus, by (34), we get

$$\alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ for all } d \in \mathbb{R}^n.$$

Therefore, by Theorem 10, $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S with modulus α.

(a) $\iff$ (c). Let the statement (a) holds. Let $x \in \bar{S}$. Therefore, by Theorem 10, we have

$$\alpha\, \text{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ for all } d \in T_S(x).$$

Note that for $x \in \bar{S}$, $\mathbf{F}_o(x) = \mathbf{F}(x)$. Thus,

$$\mathbf{F}_{o\mathscr{D}}(x)(d) = \mathbf{F}_{\mathscr{D}}(x)(d) \text{ for } x \in \bar{S} \text{ and } d \in T_S(x). \tag{36}$$

By (36) and Theorem 10, we get

$$\alpha\, \text{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{\mathscr{D}}(x)(d) \text{ for all } d \in T_S(x) \text{ and } x \in \bar{S}.$$

Conversely, we are given that

$$\begin{aligned} &\alpha\, \text{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{\mathscr{D}}(x)(d) \text{ for all } x \in \bar{S} \text{ and } d \in T_S(x) \\ &\quad \implies \alpha\, \text{dist}(d, T_{\bar{S}}(x)) \\ &\quad \preceq \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}(x)}(d) \text{ for all } d \in T_S(x) \text{ by Theorem 7.} \end{aligned} \tag{37}$$

Note that for $x \in \bar{S}$, we have

$$\boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}(x)}(d) = \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d) \text{ for all } d \in \mathbb{R}^n. \tag{38}$$

In view of (37) and (38), we have

$$\begin{aligned} &\alpha\, \text{dist}(d, T_{\bar{S}}(x)) \preceq \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d) \text{ for all } d \in T_S(x) \\ &\quad \implies \alpha\, \text{dist}(d, T_{\bar{S}}(x)) \\ &\quad \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ for all } d \in T_S(x) \text{ by Theorem 7.} \end{aligned}$$

Hence, by Theorem 10, $\bar{S}$ is the set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$.

(b) $\iff$ (d). If the statement (b) holds, then obviously the statement (d) also holds.

Conversely, let the statement (d) holds. Let $x \in \bar{S}$ and $\widehat{G} \in \alpha\mathbb{B} \cap N_{\bar{S}}(x)$. Therefore, there exists a $\bar{y} \in \bar{S}$ such that $\widehat{G} \in \boldsymbol{\partial}\mathbf{F}_o(\bar{y})$. Thus, by Definition 26, we get

$$(z - \bar{y})^\top \odot \widehat{G} \preceq \mathbf{F}_o(z) \ominus_{gH} \mathbf{F}_o(\bar{y}) \text{ for all } z \in \mathbb{R}^n. \tag{39}$$

In particular, for any $z \in \bar{S}$, $\mathbf{F}_o(z) = \mathbf{F}_o(\bar{y})$. Thus, (39) reduces to

$$(z - \bar{y})^\top \odot \widehat{G} \preceq \mathbf{0} \text{ for all } z \in \bar{S}.$$

Since $\widehat{G} \in \mathbb{R}^n$, by using Remark 1, $(z - \bar{y})^\top \odot \widehat{G} = \langle \widehat{G}, z - \bar{y} \rangle \leq 0$ for all $z \in \bar{S}$. Therefore,

$$\begin{aligned} &\langle \widehat{G}, z \rangle \leq \langle \widehat{G}, \bar{y} \rangle \text{ for all } z \in \bar{S} \\ &\quad \implies \sup_{z \in \bar{S}} \langle \widehat{G}, z \rangle \leq \langle \widehat{G}, \bar{y} \rangle \\ &\quad \implies \psi^*_{\bar{S}}(\widehat{G}) = \langle \widehat{G}, \bar{y} \rangle \text{ because } \bar{y} \in \bar{S}. \end{aligned} \tag{40}$$

Since $\widehat{G} \in N_{\bar{S}}(x)$, by Definition 24, we have

$$\begin{aligned} &\langle \widehat{G}, z - x \rangle \leq 0 \text{ for all } z \in \bar{S} \\ &\quad \implies \psi^*_{\bar{S}}(\widehat{G}) = \langle \widehat{G}, x \rangle. \end{aligned} \tag{41}$$

Combining (40) and (41), we get

$$\langle \widehat{G}, x \rangle = \langle \widehat{G}, \bar{y} \rangle. \tag{42}$$

Note that

$$\begin{aligned} &(z - x)^\top \odot \widehat{G} \\ &\quad = \langle \widehat{G}, z - x \rangle \text{ for all } z \in \mathbb{R}^n \\ &\quad = \langle \widehat{G}, z - \bar{y} \rangle \text{ for all } z \in \mathbb{R}^n \text{ by (42)} \\ &\quad = (z - \bar{y})^\top \odot \widehat{G} \text{ for all } z \in \mathbb{R}^n \text{ by Remark 1} \\ &\quad \preceq \mathbf{F}_o(z) \ominus_{gH} \mathbf{F}_o(\bar{y}) \text{ for all } z \in \mathbb{R}^n \text{ by (39)} \\ &\quad = \mathbf{F}_o(z) \ominus_{gH} \mathbf{F}_o(x) \text{ for all } z \in \mathbb{R}^n \text{ because } \mathbf{F}(x) = \mathbf{F}(\bar{y}). \end{aligned}$$

Hence, $\widehat{G} \in \boldsymbol{\partial}\mathbf{F}_o(x)$. Since $x \in \bar{S}$ is arbitrary, the statement (b) holds.

(c) $\implies$ (e). From the statement (c), we have

$$\begin{aligned} &\alpha\, \text{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{\mathscr{D}}(x)(d) \text{ for all } d \in T_S(x) \text{ and } x \in \bar{S} \\ &\quad \implies \alpha\|d\| \\ &\quad \preceq \mathbf{F}_{\mathscr{D}}(x)(d) \text{ for all } d \in T_S(x) \cap N_{\bar{S}}(x) \text{ by (ii) of Lemma 7.} \end{aligned}$$

Hence, the statement (e) holds.

(e) $\implies$ (a). Let $y \in S$. Set $x = P(y \mid \bar{S})$, then $(y - x) \in T_S(x) \cap N_{\bar{S}}(x)$. Therefore, according to the hypothesis, we obtain

$$\begin{aligned} &\alpha\|y - x\| \preceq \mathbf{F}_{\mathscr{D}}(x)(y - x) \\ &\quad \implies \alpha\, \text{dist}(y, \bar{S}) \preceq \mathbf{F}_{\mathscr{D}}(x)(y - x) \text{ by Definition 21} \\ &\quad \implies \alpha\, \text{dist}(y, \bar{S}) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x) \text{ by Lemma 5} \\ &\quad \implies \mathbf{F}(x) \oplus \alpha\, \text{dist}(y, \bar{S}) \preceq \mathbf{F}(y) \text{ by (i) of Lemma 1,} \end{aligned}$$

which shows that $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S.

(a) $\iff$ (f). Let the statement (a) holds. Let $y \in S$ and $p = P(y \mid \bar{S})$. Thus, the statement (a) gives

$$\mathbf{F}(p) \oplus \alpha\, \text{dist}(y, \bar{S}) \preceq \mathbf{F}(y), \text{ i.e., } \mathbf{F}(p) \oplus \alpha\|y - p\| \preceq \mathbf{F}(y). \tag{43}$$

Define $z_\lambda = \lambda y + (1 - \lambda)p$ for $\lambda \in [0, 1]$. Then, $p = P(z_\lambda \mid \bar{S})$ for all $\lambda \in [0, 1]$. From (43), we have

$$\begin{aligned} &\mathbf{F}(p) \oplus \alpha\|z_\lambda - p\| \preceq \mathbf{F}(z_\lambda) \\ &\quad \implies \mathbf{F}(p) \oplus \alpha\lambda\|y - p\| \preceq \mathbf{F}(z_\lambda) \\ &\quad \implies \alpha\|x - p\| \preceq \frac{1}{\lambda} \odot \Big(\mathbf{F}(p + \lambda(y - p)) \ominus_{gH} \mathbf{F}(p)\Big). \end{aligned} \tag{44}$$

By taking limit as $\lambda \downarrow 0$ in (44), we get

$$\alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}_{\mathscr{D}}(p)(y-p), \text{ where } p \in P(y \mid \bar{S}).$$

Conversely, let $y \in S$ and set $x = P(y \mid \bar{S})$. Then, from the statement (f), we get

$$\begin{aligned} &\alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}_{\mathscr{D}}(x)(y-x) \\ &\implies \alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x) \text{ by Lemma 5} \\ &\implies \mathbf{F}(x) \oplus \alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}(y) \text{ by (i) of Lemma 1,} \end{aligned}$$

which is the required result. □

6 Applications

In this section, we present two applications of the proposed study.

6.1 Application 1

As a first application, we find the set of WSM of a Minimum Risk Portfolio Interval Optimization Problem (MRPIOP), where the returns and the components of risk covariance matrix of returns are intervals.

The conventional minimum risk portfolio optimization problem of two assets is given by (see Bartholomew-Biggs 2006)

$$\begin{aligned} \min \quad & y^\top Q y \\ \text{subject to} \quad & y_1 + y_2 = 1 \\ & 0 \le y_i, i = 1, 2, \end{aligned}$$

where $y = (y_1, y_2)^\top$, y_i is the proportion of investment corresponding to the i-th asset, and Q is the risk covariance matrix of returns. Conventionally, the entries in Q are real numbers. However, in practice, for a portfolio optimization problems, realistic data involves uncertainty. We, thus, aim to formulate and solve an MRPIOP with interval-valued data.

We define an MRPIOP as

$$\min_{x \in [0,1]} \mathbf{F}(x), \tag{45}$$

where $\mathbf{F}(x) = x^2 \odot \mathbf{Q}_{11} \oplus (1-x)^2 \odot \mathbf{Q}_{21} \oplus x^2 \odot \mathbf{Q}_{12} \oplus (1-x)^2 \odot \mathbf{Q}_{22}$, and $\mathbf{Q}_{ik}$ for $i, k = 1, 2$ are (interval-valued) entries of the risk covariance matrix $\mathbf{Q}$ of interval-valued returns.

For instance, we consider two assets namely 'AAL' and 'DAL' of the companies American Airlines Group and Delta Airlines Inc., respectively, for the period January 2021 to December 2021. We calculate the interval-valued returns $\mathbf{R}_{ij}$ of every month using lowest and highest indices of the i-th asset in the j-th month, for $i = 1, 2$ and $j = 1, 2, \ldots, 12$. These interval-valued returns are displayed in Table 1.

Table 1 Sample interval-valued returns of the given assets

Periods	Returns of AAL ($\mathbf{R}_{1j}$)	Returns of DAL ($\mathbf{R}_{2j}$)
Jan-21	[1.1400,5.9200]	[2.4300,2.4700]
Feb-21	[0.9200,5.1900]	[0.9300,11.8500]
Mar-21	[2.8100,4.3400]	[2.8100,6.4600]
Apr-21	[1.2500,2.9500]	[1.0100,4.2500]
May-21	[1.3600,3.7400]	[0.4100,5.5500]
Jun-21	[0.6900,3.1900]	[1.0400,5.9000]
Jul-21	[1.0000,1.8500]	[1.7600,2.7300]
Aug-21	[1.3700,2.1800]	[1.7700,4.4900]
Sep-21	[1.0800,2.1500]	[2.0500,5.1400]
Oct-21	[2.1900,3.2600]	[4.5500,6.3700]
Nov-21	[2.1000,3.4700]	[3.1900,4.2400]
Dec-21	[0.9300,1.7900]	[3.0200,3.4700]

Table 2 The risk covariance interval-valued matrix $\mathbf{Q}$ of the returns of the given assets

Assets	AAL	DAL
AAL	[0.0091,0.1486]	[0.0092,0.1821]
DAL	[0.0092,0.1821]	[0.0475,0.4738]

We calculate the interval-valued mean return $\mathbf{R}_{i_M}$ for the i-th asset by the formula

$$\mathbf{R}_{i_M} = \frac{1}{12} \odot (\mathbf{R}_{i1} \oplus \mathbf{R}_{i2} \oplus \cdots \oplus \mathbf{R}_{i12}),$$

for $i = 1, 2$. The interval-valued mean returns are

$$\mathbf{R}_{1_M} = [1.4033, 3.3358] \text{ and } \mathbf{R}_{2_M} = [2.1183, 5.2433].$$

The interval components $\mathbf{Q}_{ik}$ of risk covariance matrix $\mathbf{Q}$ are computed by the formula

$$\begin{aligned} \mathbf{Q}_{ik} = \frac{1}{144} \odot [&\{(\mathbf{R}_{i1} \ominus_{gH} \mathbf{R}_{i_M}) \otimes (\mathbf{R}_{k1} \ominus_{gH} \mathbf{R}_{k_M})\} \\ &\oplus \{(\mathbf{R}_{i2} \ominus_{gH} \mathbf{R}_{i_M}) \otimes (\mathbf{R}_{k2} \ominus_{gH} \mathbf{R}_{k_M})\} \\ &\oplus \cdots \oplus \{(\mathbf{R}_{i12} \ominus_{gH} \mathbf{R}_{i_M}) \otimes (\mathbf{R}_{k12} \ominus_{gH} \mathbf{R}_{k_M})\}], \end{aligned}$$

for $i, k = 1, 2$, and are shown in Table 2. With the help of $\mathbf{Q}$, MRPIOP (45) can be written as

$$\begin{aligned} &\min_{x \in [0,1]} \{x^2 \odot [0.0091, 0.1486] \oplus (1-x)^2 \\ &\quad \odot [0.0092, 0.1821] \oplus x^2 \odot [0.0092, 0.1821] \\ &\quad \oplus (1-x)^2 \odot [0.0475, 0.4738]\} \\ &= \min_{x \in [0,1]} \{x^2 \odot [0.0183, 0.3307] \end{aligned}$$

$$\oplus(1-x)^2 \odot [0.0567, 0.6559]\}$$
$$= \min_{x\in[0,1]} [0.0750x^2 - 0.1134x + 0.0567, 0.9876x^2 - 1.3118x + 0.6559].$$

Thus,

$$\mathbf{F}(x) = [0.0750x^2 - 0.1134x + 0.0567, 0.9876x^2 - 1.3118x + 0.6559],\ x \in [0, 1].$$

Consider $\alpha = 0.075$, $\bar{S}_1 = [0, 0.6641]$ and $\bar{S}_2 = [0.7560, 1]$.

Then, with respect to the usual Euclidean distance, $\text{dist}(x, \bar{S}_1) = (x - 0.6641)^2$, and $\text{dist}(x, \bar{S}_2) = (x - 0.7560)^2$, where $x \in [0, 1]$.

In Fig. 2, we have depicted the graph of $\mathbf{F}(x)$ by red-shaded region. The black-shaded portions show the graphs of $\mathbf{F}(\bar{x}) \oplus \alpha\, \text{dist}(x, \bar{S}_1)$ for $\bar{x} \in \bar{S}_1$ and of $\mathbf{F}(\bar{x}) \oplus \alpha\, \text{dist}(x, \bar{S}_2)$ for $\bar{x} \in \bar{S}_2$. From the graphs, notice that for any $x \in [0, 1]$,

$$\mathbf{F}(\bar{x}) \oplus \alpha\, \text{dist}(x, \bar{S}_1) \preceq \mathbf{F}(x) \text{ for all } \bar{x} \in \bar{S}_1$$
$$\text{and } \mathbf{F}(\bar{x}) \oplus \alpha\, \text{dist}(x, \bar{S}_2) \preceq \mathbf{F}(x) \text{ for all } \bar{x} \in \bar{S}_2.$$

Hence, by Definition 27 of WSM, $\bar{S}_1 = [0, 0.6641]$ and $\bar{S}_2 = [0.7560, 1]$ are the sets of WSM for (45) with the data in Table 2. Thus, the points belonging to $\bar{S}_1$ and $\bar{S}_2$ are preferable points for the investment.

6.2 Application 2

As a second application, we use the concept of WSM to find the weak efficient solutions of the following linear programming problem with interval-valued objective function:

$$\min_{x\in S'} \mathbf{F}(x), \tag{46}$$

where $\mathbf{F}$ is a linear IVF on $\mathbb{R}^n$ and S' is a polyhedral subset of $\mathbb{R}^n$. The above problem is extensively studied by several authors, for instance, see (Ishibuchi and Tanaka 1990; Inuiguchi and Kume 1991; Ida 2003), and the references therein. The real-world applications of (46) are shown by several authors. For instance, Steuer (1981) used (46) to study feedmix and blending problems. Wu et al. (2006) proposed a method for the planning of waste management system for the region of Hamilton, Ontario, Canada with the help of the tools of (46).

Inspired by all these real-life applications of (46), we find out the weak efficient solutions of (46) with the help of the studied concept of WSM. Since interval-linear programming problem (ILPP) (46) has been solved by many researchers, we are providing a theory, to find the set of weak sharp minima of ILPP.

Theorem 12 *Let $\boldsymbol{F} = [\underline{F}(x), \overline{F}(x)]$ be a linear IVF on $\mathbb{R}^n$. Then, the set of weak efficient solutions of (46), where S' is a polyhedral set in $\mathbb{R}^n$, is identical to the set of WSM of $\boldsymbol{F}$ over S'.*

Proof Let $\bar{S}$ be the set of weak efficient solutions of (46). Then, every $\bar{x} \in \bar{S}$ minimizes $\mathbf{F}(x)$ over S'. Therefore, every $\bar{x} \in \bar{S}$ minimizes $\underline{F}(x)$ and $\overline{F}(x)$ as well. Hence, $\bar{S}$ is the solution set of both the linear programming problems

$$\min_{x\in S'} \underline{F} \text{ and } \min_{x\in S'} \overline{F}. \tag{47}$$

Therefore, by Theorem 3, we have

$$\bar{S} \text{ is a set of WSM of both } \underline{F} \text{ and } \overline{F} \text{ over } S'$$
$$\implies \bar{S} \text{ is a set of WSM of } \mathbf{F} \text{ over } S' \text{ by Remark 8.}$$

Next, if $\bar{x}$ belongs to the set of WSM of $\mathbf{F}$, then by Definition 27 of WSM, $\bar{x}$ is also a weak efficient solution of (47), which completes the proof. □

In the next theorem (Theorem 13), we use the concept of WSM to solve the following IOP, need not to be an ILPP:

$$\min_{x\in S} \mathbf{F}_1(x), \tag{48}$$

where S is a closed convex subset of $\mathbb{R}^n$ and $\mathbf{F}_1$ is proper, extended, convex and gH-lsc IVF on $\mathbb{R}^n$. Note that it is not always an easy task to solve IOP (48) by finding WSM of $\mathbf{F}_1$ or by some other methods. In this case, we use perturbation to investigate the weak efficient solution of IOP (48). In the perturbation, we consider a different IOP (49), whose weak efficient solution is known or easy to find:

$$\min_{x\in S} \mathbf{F}_2(x), \tag{49}$$

where $\mathbf{F}_2$ is a proper extended IVF on $\mathbb{R}^n$. Now we consider a perturbed IOP:

$$\min_{x\in S} \{\mathbf{F}_1(x) \oplus \epsilon \odot \mathbf{F}_2(x)\}, \tag{50}$$

where ϵ is nonnegative real number.

Theorem 13 *Let $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and S be as in (50). Let $\bar{S}(\epsilon) \subseteq \bar{S}$ be the set of weak efficient solutions of perturbed IOP (50) and $\bar{S}$ be the set of WSM of $\boldsymbol{F}_1$ over S. Then, $\bar{S}(\epsilon) \subseteq \bar{S}_{\boldsymbol{F}_2}$, where $\bar{S}_{\boldsymbol{F}_2}$ is the set of weak efficient solutions of IOP (49). Moreover, if $\boldsymbol{F}_2$ is gH-locally Lipschitz continuous on $\mathbb{R}^n$, then $\bar{S}_{\boldsymbol{F}_2} = \bar{S}(\epsilon)$.*

Proof Suppose $\bar{x} \in \bar{S}(\epsilon)$. Then, for any $y \in S$, we have

$$\mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}_2(\bar{x}) \preceq \mathbf{F}_1(y) \oplus \epsilon \odot \mathbf{F}_2(y). \tag{51}$$

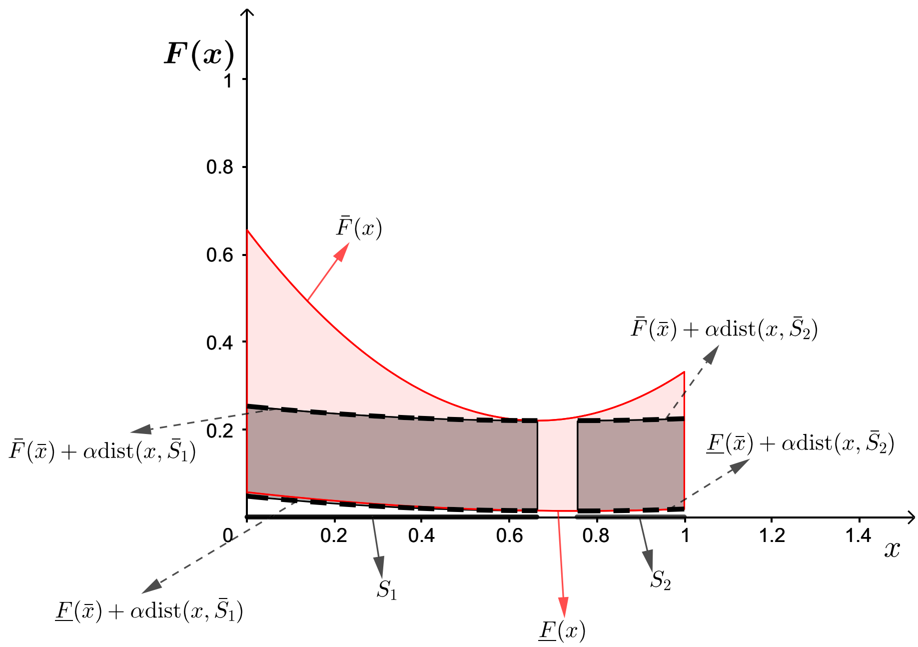

Fig. 2 Objective function of (45) and the locations of the sets of WSM of (45)

Since $\bar{S} \subseteq S$, then (51) holds for any $\bar{y} \in \bar{S}$. Thus,

$$\begin{aligned}
&\mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}_2(\bar{x}) \preceq \mathbf{F}_1(\bar{y}) \oplus \epsilon \odot \mathbf{F}_2(\bar{y}) \\
&\quad\Longrightarrow \epsilon \odot \mathbf{F}_2(\bar{x}) \\
&\qquad \preceq \epsilon \odot \mathbf{F}_2(\bar{y}) \text{ because } \bar{S} \text{ is a set of WSM of } \mathbf{F}_1 \\
&\quad\Longrightarrow \mathbf{F}_2(\bar{x}) \preceq \mathbf{F}_2(\bar{y}) \text{ because } \epsilon > 0.
\end{aligned}$$

Thus, $\bar{x} \in \bar{S}_{\mathbf{F}_2}$. Since $\bar{x} \in \bar{S}(\epsilon)$ is arbitrarily chosen, we get the result.

Conversely, let $\bar{x} \in \bar{S}_{\mathbf{F}_2}$. In order to show $\bar{x} \in \bar{S}(\epsilon)$, we prove that

$$\mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}_2(\bar{x}) \preceq \mathbf{F}_1(\bar{y}) \oplus \epsilon \odot \mathbf{F}_2(\bar{y}) \text{ for } \bar{y} \in \bar{S} \quad (52)$$

and

$$\mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}_2(\bar{x}) \prec \mathbf{F}_1(y) \oplus \epsilon \odot \mathbf{F}_2(y) \text{ for } y \in S \setminus \bar{S}. \quad (53)$$

Note that $\bar{S}$ is a set of WSM of $\mathbf{F}_1$ over S and $\bar{x} \in \bar{S}_{\mathbf{F}_2}$, therefore (52) holds. To establish (53), let $x \in S \setminus \bar{S}$, thus $x \neq \bar{x}$. Then, we have

$$\begin{aligned}
&\epsilon \odot \mathbf{F}_2(\bar{x}) \preceq \epsilon \odot \mathbf{F}_2(\bar{x}) \\
&\quad\Longrightarrow \epsilon \odot \mathbf{F}_2(\bar{x}) \preceq \epsilon \odot \mathbf{F}_2(\bar{x}) \ominus_{gH} \epsilon \odot \mathbf{F}_2(x) \oplus \epsilon \odot \mathbf{F}_2(x) \\
&\quad\Longrightarrow \epsilon \odot \mathbf{F}_2(\bar{x}) \preceq \epsilon K \|\bar{x} - x\| \oplus \epsilon \odot \mathbf{F}_2(x) \\
&\qquad \text{because } \mathbf{F} \text{ is } gH\text{-local Lipschitz continuous} \\
&\quad\Longrightarrow \epsilon \odot \mathbf{F}_2(\bar{x}) \prec \alpha \|\bar{x} - x\| \oplus \epsilon \odot \mathbf{F}_2(x), \text{ where } \epsilon K < \alpha \\
&\quad\Longrightarrow \mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}_2(\bar{x}) \prec \alpha \|\bar{x} - x\| \oplus \mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}(x) \\
&\quad\Longrightarrow \mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}_2(\bar{x}) \prec \mathbf{F}_1(x) \oplus \epsilon \odot \mathbf{F}(x) \\
&\qquad \text{because } \bar{S} \text{ is a set of WSM of } \mathbf{F}_1.
\end{aligned}$$

Therefore, $\bar{x} \in \bar{S}(\epsilon)$. Since $\bar{x}$ is arbitrarily chosen, $\bar{S}_{\mathbf{F}_2} \subseteq \bar{S}(\epsilon)$, and hence we get the desired result. □

7 Conclusion and future scopes

In this article, the conventional concepts of support function and subdifferentiability have been extended for IVFs (Definitions 25 and Definition 26). Also, some important characteristics of the gH-subdifferential set like nonemptiness (Lemma 13), boundedness (Theorem 8), convexity and closedness (Theorem 5) have been presented. Subsequently, we have provided a few necessary results (Lemma 8, Theorem 4 and Lemma 9) based on the support function of a subset of $I(\mathbb{R})^n$. It has been reported that the gH-subdifferential set of a gH-differentiable convex IVF is a singleton set containing the gH-gradient (Theorem 6). The relationship between gH-directional derivative and the support function of gH-subdifferential set of convex IVF has been also established (Theorem 7). Further, we have introduced the notion of WSM for convex IVFs (Definition 27) to solve IOP (1). With the help of the proposed concepts of gH-subdifferentiability and support function, a primal characterization (Theorem 10) and a few dual characterizations (Theorem 11) of WSM have been presented to solve constrained IOP (27) and unconstrained IOP (28). Two applications of the proposed study have been given. In the first application, the sets of WSM of MRPIOP (45) has been given. In the second application, we provide a relationship between the WSM and weak efficient solutions of linear and nonlinear IOPs (Theorems 12 and 13).

In future, we shall apply proposed theory on WSM to derive necessary and sufficient conditions under which a global error bound may exist for a convex inequality system as follows:

$$\mathbf{H}_\lambda(x) \preceq \mathbf{0}, \ \lambda \in \Lambda \text{ and } x \in C, \tag{54}$$

where Λ is an index set, and for each $\lambda \in \Lambda$, $\mathbf{H}_\lambda : X \subseteq \mathbb{R}^n \to I(\mathbb{R})$ is gH-lsc, convex, proper, and the set C is closed convex subset of X. By a global error bound for the inequality system (54), we mean the existence of a constant $\beta > 0$ such that

$$\beta \text{ dist}(x, \Omega) \preceq \text{dist}(x, C) \oplus \mathbf{H}_{\lambda+}(x) \text{ for each } \lambda \in \Lambda \text{ and } x \in X, \tag{55}$$

where $\Omega = \{x : x \in C \text{ and } \mathbf{H}_\lambda(x) \preceq \mathbf{0}\}$ and $\mathbf{H}_{\lambda+}(x) = \max\{\mathbf{0}, \mathbf{H}_\lambda(x)\}$. For the sake of convenience, we define $\mathbf{H}(x) = \sup\{\mathbf{H}_\lambda(x) : \lambda \in \Lambda\}$ for each $x \in X$. Note that if a constant $\bar{\beta} > 0$ exists such that

$$\bar{\beta} \text{ dist}(x, \Omega) \preceq \text{dist}(x, C) \oplus \mathbf{H}_{+}(x) \text{ for all } x \in X, \tag{56}$$

where $\bar{\Omega} = \{x : x \in C \text{ and } \mathbf{H}(x) \preceq \mathbf{0}\}$ and $\mathbf{H}_+(x) = \max\{\mathbf{0}, \mathbf{H}(x)\}$, then (55) holds for $\beta > 0$. The following observation can be useful to solve the problem. If we define an IVF $\mathbf{F} : X \to \overline{I(\mathbb{R})}$ such that

$$\mathbf{F}(x) = \text{dist}(x, C) \oplus \mathbf{H}_+(x),$$

then $\mathbf{F}$ has Ω as a set of WSM with modulus $\bar{\beta} > 0$, which is equivalent to the condition in (56).

Immediately in the next step we shall try to give some numerical algorithms to find WSM of an IVF. It is well known that algorithms in optimization to solve real-world problems have huge reputation (Lee and Geem 2005; Faramarzi et al. 2020; Abd Elaziz et al. 2017; Abualigah et al. 2021; Mohammad Hasani Zade and Mansouri 2021; Abualigah 2020). For instance, the arithmetic optimization algorithm (AOA) and Aquila optimizer (AO) proposed in Abualigah et al. (2021) and Abualigah et al. (2021) have been proved useful to solve some real-life problems. However, some real-life problems may have uncertainties in the given data, for instance see Sahoo et al. (2012), Dey et al. (2020), Das et al. (2016). Hence, to solve these kind of problems, we shall try to extend AOA and AO for IVFs.

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A Proof of Lemma 1

(i). Let $\mathbf{A} = [\underline{a}, \overline{a}]$, $\mathbf{B} = [\underline{b}, \overline{b}]$, and $\mathbf{C} = [\underline{c}, \overline{c}]$. We have

$$r \leq \underline{a} \text{ and } r \leq \overline{a}. \tag{57}$$

Similarly, by $\mathbf{A} \preceq \mathbf{B} \ominus_{gH} \mathbf{C}$, we have

$$\underline{a} \leq \min\left\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\right\} \text{ and } \overline{a} \leq \max\left\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\right\}. \tag{58}$$

- Case 1. Let $\min\left\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\right\} = \overline{b} - \overline{c}$ and $\max\left\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\right\} = \underline{b} - \underline{c}$. Then, from (57) and (58), we get

  $$\overline{c} + r \leq \overline{b} \text{ and } \underline{c} + r \leq \underline{b}.$$

  Hence, $\mathbf{C} \oplus [r, r] \preceq \mathbf{B}$.
- Case 2. When $\min\left\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\right\} = \underline{b} - \underline{c}$ and $\max\left\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\right\} = \overline{b} - \overline{c}$. Proof contains similar steps as in Case 1.

(ii). Let $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$. Then,

$$\begin{aligned}
&\left\{(1-\lambda) \odot \mathbf{A} \oplus \lambda \odot \mathbf{B}\right\} \ominus_{gH} \mathbf{A} \\
&= \left\{(1-\lambda) \odot [\underline{a}, \overline{a}] \oplus \lambda \odot [\underline{b}, \overline{b}]\right\} \ominus_{gH} [\underline{a}, \overline{a}] \\
&= \left[(1-\lambda)\underline{a} + \lambda\underline{b}, (1-\lambda)\overline{a} + \lambda\overline{b}\right] \ominus_{gH} [\underline{a}, \overline{a}] \\
&\quad \text{because } \lambda \in [0, 1] \\
&= \left[\min\left\{\lambda\underline{b} - \lambda\underline{a}, \lambda\overline{b} - \lambda\overline{a}\right\}, \max\left\{\lambda\underline{b} - \lambda\underline{a}, \lambda\overline{b} - \lambda\overline{a}\right\}\right] \\
&= \lambda \odot \left\{\mathbf{A} \ominus_{gH} \mathbf{B}\right\}.
\end{aligned}$$

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