Chapter 7

Legendre Collocation Method to Solve the Standard as well as Fractional Order Linear/Non-Linear Two-Dimensional Partial Differential Equations

7.1 Introduction

Here, Linear/non-linear two-dimensional standard, as well as fractional-order models are considered which significantly describe many physical problems of the real world (Debnath (1998, 2012)). Out of which fractional order form of these models is more important in comparison to standard order to explain the many complex physical problems as discussed in Chapter 1. Most of these fractional differential equations do not have the analytical solutions mainly for nonlinear fractional differential equations. Therefore numerical methods are considered to give the approximate solutions of these models. In literature, several numerical methods are available to solve these time, space and time-space fractional differential equations. In 2013, Sun et al. (2013) have developed a semi-analytical finite element method to solve a class of time-fractional diffusion equations. Hoz and Vadillo (2013) describe a new technique to solve two-dimensional advection-diffusion equations via operational matrices avoiding Kronecker tensor product. A new Legendre collocation method with finite difference scheme for time derivative is given by Bhrawy (2014) to solve two-dimensional fractional diffusion equation. Yokus (2017) solved the space and time fractional order Burger type equation using finite difference and generalized Taylor series method. The simplified reproducing kernel method is used by Jia et al. (2017) to solve variable order fractional

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functional differential equation. Rahimkhani et al. (2017) have defined new functions called generalized fractional-order Bernoulli wavelet functions (GFBWFs) based on the Bernoulli wavelets to solve fractional order pantograph differential equations in a large interval. Ali et al. (2017) used optimal homotopy asymptotic method for the numerical solutions of fractional order Cauchy reaction-diffusion equations. A new numerical method based upon the hybrid function to solve the disturbed fractional differential equations is given by Mashayekhi and Razzaghi (2016). To solve multi-dimensional nonlinear fractional sub-diffusion equations, a Jacobi spectral collocation method is given by Bhrawy (2016). Bhrawy et al. (2016) have solved the two-sided space-time Caputo fractional diffusion-wave equation using a space-time Legendre tau method. Saadatmandi and Dehgan (2011) used a Tau approach with shifted Legendre polynomials to solve space fractional diffusion equation. The same authors (2010) used a new Legendre operational matrix approach to solve fractional-order differential equations. Gupta and Ray (2015) gave a numerical treatment for the solution of fractional fifth-order Sawada-Kotera equation using second kind Chebyshev wavelet method in their manuscript. Liu et al. (2016) proposed an approach to solving the multi-term variable order fractional differential equation based on operational matrix approach of the second-kind of Chebyshev polynomial.

In last decade, a lot of attention was given to the spectral methods during the solution of standard as well as fractional order form of many physical models. Therefore, in this chapter, a new algorithm based on spectral collocation method is derived to solve the linear/non-linear standard as well as fractional order two-dimensional partial differential equations. For that, the solution of the considered problem is approximated by triple shifted Legendre polynomials for space and time variables and then applied to the considered problem with the operational matrix of the shifted Legendre polynomials and then collocate it at Legendre Gauss-Lobatto points which convert the considered problem into a system of algebraic equations. This system of algebraic equations can be solved using any standard numerical technique from where one can get the unknown coefficients of expansion. To demonstrate the efficiency of the proposed method, some examples are considered, and for each case, a comparison of the approximate solution with the existing analytical solution is found. Graphical and tabular presentations of absolute error confirm the validity and efficiency of the proposed approach.

7.2 Preliminaries

7.2.1 Shifted Legendre Polynomials

The classical Legendre polynomials (given in Section 1.19.3 of Chapter 1) can be defined for any arbitrary finite interval $[a,b]$ by making this interval corresponding to the interval $[-1,1]$ with the use of the definition given in Section 1.20 and called shifted Legendre polynomials. The shifted Legendre polynomials $L_p\left(\frac{2x-(a+b)}{(b-a)}\right)$ are denoted by $L_{b-a,p}(x)$.

Since in most of the problems interval of interest is $[0,\ell]$, so substitute $z=\frac{2x}{\ell}-1,$ in the definition given in Section 1.19.3 which leads to the alleged shifted Legendre polynomials denoted by $L_{\ell,p}(x)$ given by

$$L_{\ell,p+1}(x)=\frac{(2p+1)(2x-\ell)}{(p+1)\ell}L_{\ell,p}(x)-\frac{p}{p+1}L_{\ell,p-1}(x),\;\; p=1,2,\ldots\;, \tag{7.1}$$

with $L_{\ell,0}(x)=1$ and $L_{\ell,1}(x)=\frac{2x}{\ell}-1.$

The analytical expression of $L_{\ell,p}(x)$ is given by

$$L_{\ell,p}(x)=\sum_{i=0}^{p}(-1)^{p+i}\frac{(p+i)!}{(p-i)!(i!)^2\,\ell^i}x^i, \tag{7.2}$$

which satisfies the following orthogonality condition

$$\int_0^{\ell} L_{\ell,p}(x)L_{\ell,q}(x)\ dx = \frac{\ell}{2p+1}\delta_{pq}, \tag{7.3}$$

where δ_{pq} is the Kronecker function.

7.2.2 Function Approximation

Let $h(x) \in L^2(0,\ell),$ then it can be expressed as

$$h(x) = \sum_{p=0}^{\infty} a_p\, L_{\ell,p}(x), \tag{7.4}$$

where a_p are the coefficients given by

$$a_p = \frac{2p+1}{\ell}\int_0^{\ell} h(x)\, L_{\ell,p}(x)\, dx, \qquad p = 0,1,...; \tag{7.5}$$

As usual, the first $(M+1)$-terms of $L_{\ell,p}(x)$ are considered during approximation. Thus we have

$$h_M(x) \simeq \sum_{p=0}^{M} a_p\, L_{\ell,p}(x) \simeq A^T \Omega_{\ell,M}(x), \tag{7.6}$$

with

$$A^T \equiv [a_0, a_1, ... , a_M],\ \Omega_{\ell,M}(x) \equiv [L_{\ell,0}(x), L_{\ell,1}(x), ... , L_{\ell,M}(x)]^T. \tag{7.7}$$

Similarly, a function $h(x,t)$ which is defined on the interval $[0,\ell]\times[0,\tau]$ can be expressed as

$$h_{M,N}(x,t) = \sum_{p=0}^{M}\sum_{q=0}^{N} c_{pq}\, L_{\tau,p}(t)L_{\ell,q}(x) = \Omega_{\tau,M}^T(t)\, C\, \Omega_{\ell,N}(x), \tag{7.8}$$

where the matrix C is given by

$$C=\begin{pmatrix} c_{00} & c_{01} & . & . & . & c_{0N} \\ c_{10} & c_{11} & . & . & . & c_{1N} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ c_{M0} & c_{M1} & . & . & . & c_{MN} \end{pmatrix}. \tag{7.9}$$

The elements of C are obtained from

$$c_{pq}=\frac{2p+1}{\tau}\frac{2q+1}{\ell}\int_0^{\tau}\int_0^{\ell} h(x,t)L_{\tau,p}(t)L_{\ell,q}(x)dx\,dt,\ p=0,1,...,M,\quad q=0,1,...,N. \tag{7.10}$$

In this way $h(x,y,t)$, defined on the interval $[0,\ell_1]\times[0,\ell_2]\times[0,\tau]$ may be expressed as

$$h_{M_1,M_2,N}(x,y,t)=\sum_{p=0}^{M_1}\sum_{q=0}^{M_2}\sum_{r=0}^{N}c_{pqr}\,L_{\ell_1,p}(x)L_{\ell_2,q}(y)L_{\tau,r}(t)=\Omega_{\tau,N}^T(t)\overline{C}(\Omega_{\ell_1,M_1}(x)\otimes\Omega_{\ell_2,M_2}(y)), \tag{7.11}$$

where the symbol $\otimes$ stands for Kronecker tensor product and the matrix $\overline{C}$ is given as

$$\begin{aligned}\overline{C}&=[C_0,C_1,...,C_{M_1}],\\ C_p&=[C_{p0},C_{p1},...,C_{pM_2}],\\ C_{pq}&=[c_{pq0},c_{pq1},...,c_{pqN}]^T.\end{aligned} \tag{7.12}$$

7.2.3 Legendre Operational Matrix of Standard Order Derivative

The first order derivative of the column vector $\Omega_{\ell,M}(x)$ is given as

$$\frac{d}{dx}\Omega_{\ell,M}(x)=D\,\Omega_{\ell,M}(x), \tag{7.13}$$

where D is the $(M+1)\times(M+1)$ order operational matrix of derivative given by

$$D=(d_{pq})=\begin{cases}\dfrac{2(2q+1)}{\ell}, & for\, q=p-r,\ \begin{cases} r=1,3,...M, & if\ M\ odd,\\ r=1,3,...,M-1, & if\ M\ even,\end{cases}\\ 0 & otherwise,\end{cases} \tag{7.14}$$

For even M, we have

$$D = 2\begin{pmatrix} 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & . & .. & . & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & & & . & . & . \\ . & . & . & . & & . & & . & . & . \\ . & . & . & . & & & . & . & . & . \\ 1 & 0 & 5 & 0 & . & . & . & 2M-3 & 0 & 0 \\ 0 & 3 & 0 & 7 & . & . & . & 0 & 2M-1 & 0 \end{pmatrix}. \tag{7.15}$$

It is clear from equation (7.13) that the higher order derivative of the column vector $\Omega_{\ell,M}(x)$ is given by

$$\frac{d^m}{dx^m}\Omega_{\ell,M}(x) = D^m\,\Omega_{\ell,M}(x), \tag{7.16}$$

where m is a natural number and D^m denotes the mth-order derivative of $\Omega_{\ell,M}(x)$.

7.2.4 Legendre Operational Matrix of Fractional Order Derivative

The Caputo fractional derivative of order μ of the column vector $\Omega_{\ell,M}(x)$ is given by

$$D^{\mu}\Omega_{\ell,M}(x) \simeq {}_{\ell}D_{\mu}\Omega_{\ell,M}(x), \tag{7.17}$$

where ${}_{\ell}D_{\mu}$ denotes the $(M+1)\times(M+1)$ Legendre operational matrix of the fractional derivative of order μ in the interval $[0,\ell]$, and it is given as

$${}_{\ell}D_{\mu} = \begin{pmatrix} 0 & 0 & . \; . \; . & 0 \\ \vdots & \vdots & \vdots \; \vdots \; \vdots & \vdots \\ 0 & 0 & . \; . \; . & 0 \\ \sum_{r=\lceil\mu\rceil}^{\lceil\mu\rceil}\Phi_{\lceil\mu\rceil,0,r} & \sum_{r=\lceil\mu\rceil}^{\lceil\mu\rceil}\Phi_{\lceil\mu\rceil,1,r} & . \; . \; . & \sum_{r=\lceil\mu\rceil}^{\lceil\mu\rceil}\Phi_{\lceil\mu\rceil,M,r} \\ \vdots & \vdots & \vdots \; \vdots \; \vdots & \vdots \\ \sum_{r=\lceil\mu\rceil}^{p}\Phi_{p,0,r} & \sum_{r=\lceil\mu\rceil}^{p}\Phi_{p,1,r} & . \; . \; . & \sum_{r=\lceil\mu\rceil}^{p}\Phi_{p,M,r} \\ \vdots & \vdots & \vdots \; \vdots \; \vdots & \vdots \\ \sum_{r=\lceil\mu\rceil}^{M}\Phi_{M,0,r} & \sum_{r=\lceil\mu\rceil}^{M}\Phi_{M,1,r} & . \; . \; . & \sum_{r=\lceil\mu\rceil}^{M}\Phi_{M,M,r} \end{pmatrix}, \tag{7.18}$$

where

$$\Phi_{p,q,r} = \frac{2q+1}{\ell^{r+1}} \sum_{i=0}^{q} \frac{(-1)^{p+q+r+i}(p+r)!(i+q)!}{(p-r)!r!\Gamma(r-\mu+1)(q-i)!(i!)^2(r+i-\mu+1)}. \tag{7.19}$$

It is to be noted that in ${}_{\ell}D_{\mu}$, the first $\lceil \mu \rceil$ rows are all zeros.

7.3 Application of the Proposed Algorithm

In this section, a new algorithm is proposed to solve the standard order as well as fractional order linear/nonlinear two-dimensional PDEs using spectral collocation method based on shifted Legendre operational matrix. A two-dimensional fractional order PDE is given by

$$\begin{aligned} \frac{\partial}{\partial t}u(x,y,t) &= d_x(x,y,t)\frac{\partial^{\gamma_1}}{\partial x^{\gamma_1}}u(x,y,t) + d_y(x,y,t)\frac{\partial^{\gamma_2}}{\partial y^{\gamma_2}}u(x,y,t) \\ &\quad - v_x(x,y,t)\frac{\partial^{\eta_1}}{\partial x^{\eta_1}}u(x,y,t) - v_y(x,y,t)\frac{\partial^{\eta_2}}{\partial x^{\eta_2}}u(x,y,t) + \Psi\big(u(x,y,t),x,y,t\big), \\ &\qquad (x,y,t) \in [0,\ell_1]\times[0,\ell_2]\times[0,\tau], \end{aligned} \tag{7.20}$$

with the initial condition

$$u(x,y,0) = f(x,y), \quad (x,y) \in [0,\ell_1]\times[0,\ell_2], \tag{7.21}$$

and the boundary conditions

$$u(0,y,t) = g_0(y,t), \qquad u(\ell_1,y,t) = g_{\ell_1}(y,t), \quad (y,t) \in [0,\ell_2]\times[0,\tau], \tag{7.22}$$

$$u(x,0,t) = h_0(x,t), \qquad u(x,\ell_2,t) = h_{\ell_2}(x,t), \quad (x,t) \in [0,\ell_1]\times[0,\tau], \tag{7.23}$$

where $1 \le \gamma_1, \gamma_2 \le 2$ and $0 \le \eta_1, \eta_2 \le 1$. If we take $\gamma_1 = \gamma_2 = 2$ and $\eta_1 = \eta_2 = 1$, then it represents the two-dimensional classical reaction-convection-diffusion equation subject to initial and boundary conditions.

Now to start proposed algorithm, approximate the solution $u(x,y,t)$ by triple shifted Legendre polynomials as

$$u_{M_1,M_2,N}(x,y,t) = \sum_{p=0}^{M_1}\sum_{q=0}^{M_2}\sum_{r=0}^{N} c_{pqr} L_{\ell_1,p}(x) L_{\ell_2,q}(y) L_{\tau,r}(t) = \Omega_{\tau,N}^{T}(t)\,\overline{C}(\Omega_{\ell_1,M_1}(x) \otimes \Omega_{\ell_2,M_2}(y)), \tag{7.24}$$

where $\Omega^{T}_{\tau,N}(t), \Omega_{\ell_1,M_1}(x)$ and $\Omega_{\ell_2,M_2}(y)$ are defined in equation (7.7). The matrix $\overline{C}$ is defined in equation (7.12).

Now with the help of equations (7.17) and (7.24), the terms of equation (7.20) can be re-written as

$$\frac{\partial}{\partial t}u(x,y,t) \simeq \Omega^{T}_{\tau,N}(t)\, {}_{\tau}D_1^{T}\, \tilde{C}(\Omega_{\ell_1,M_1}(x)\otimes\Omega_{\ell_2,M_2}(y)), \tag{7.25}$$

$$\frac{\partial^{\gamma_1}}{\partial x^{\gamma_1}}u(x,y,t) \simeq \Omega^{T}_{\tau,N}(t)\tilde{C}({}_{\ell_1}D_{\gamma_1}\Omega_{\ell_1,M_1}(x)\otimes\Omega_{\ell_2,M_2}(y)), \tag{7.26}$$

$$\frac{\partial^{\gamma_2}}{\partial y^{\gamma_2}}u(x,y,t) \simeq \Omega^{T}_{\tau,N}(t)\tilde{C}(\Omega_{\ell_1,M_1}(x)\otimes{}_{\ell_2}D_{\gamma_2}\Omega_{\ell_2,M_2}(y)), \tag{7.27}$$

$$\frac{\partial^{\eta_1}}{\partial x^{\eta_1}}u(x,y,t) \simeq \Omega^{T}_{\tau,N}(t)\tilde{C}({}_{\ell_1}D_{\eta_1}\Omega_{\ell_1,M_1}(x)\otimes\Omega_{\ell_2,M_2}(y)), \tag{7.28}$$

$$\frac{\partial^{\eta_2}}{\partial y^{\eta_2}}u(x,y,t) \simeq \Omega^{T}_{\tau,N}(t)\tilde{C}(\Omega_{\ell_1,M_1}(x)\otimes{}_{\ell_2}D_{\eta_2}\Omega_{\ell_2,M_2}(y)), \tag{7.29}$$

Substituting equations (7.25)-(7.29) in equations (7.20)-(7.23), we get

$$\begin{aligned}
&\Omega^{T}_{\tau,N}(t)\, {}_{\tau}D_1^{T}\, \tilde{C}(\Omega_{\ell_1,M_1}(x)\otimes\Omega_{\ell_2,M_2}(y))\\
&\qquad = d_x(x,y,t)\Omega^{T}_{\tau,N}(t)\tilde{C}({}_{\ell_1}D_{\gamma_1}\Omega_{\ell_1,M_1}(x)\otimes\Omega_{\ell_2,M_2}(y))\\
&\qquad +d_y(x,y,t)\Omega^{T}_{\tau,N}(t)\tilde{C}(\Omega_{\ell_1,M_1}(x)\otimes{}_{\ell_2}D_{\gamma_2}\Omega_{\ell_2,M_2}(y))\\
&\qquad -v_x(x,y,t)\Omega^{T}_{\tau,N}(t)\tilde{C}({}_{\ell_1}D_{\eta_1}\Omega_{\ell_1,M_1}(x)\otimes\Omega_{\ell_2,M_2}(y))\\
&\qquad -v_y(x,y,t)\Omega^{T}_{\tau,N}(t)\tilde{C}(\Omega_{\ell_1,M_1}(x)\otimes{}_{\ell_2}D_{\eta_2}\Omega_{\ell_2,M_2}(y))\\
&\qquad +\Psi\left(\Omega^{T}_{\tau,N}(t)\tilde{C}(\Omega_{\ell_1,M_1}(x)\otimes\Omega_{\ell_2,M_2}(y)),x,y,t\right)
\end{aligned} \tag{7.30}$$

$$\Omega^{T}_{\tau,N}(0)\tilde{C}(\Omega_{\ell_1,M_1}(x)\otimes\Omega_{\ell_2,M_2}(y)) = f(x,y), \tag{7.31}$$

$$\begin{aligned}
&\Omega^{T}_{\tau,N}(t)\tilde{C}(\Omega_{\ell_1,M_1}(0)\otimes\Omega_{\ell_2,M_2}(y)) = g_0(y,t),\\
&\Omega^{T}_{\tau,N}(t)\tilde{C}(\Omega_{\ell_1,M_1}(\ell_1)\otimes\Omega_{\ell_2,M_2}(y)) = g_{\ell_1}(y,t),
\end{aligned} \tag{7.32}$$

$$\begin{aligned} &\Omega_{\tau,N}^{T}(t)\tilde{C}(\Omega_{\ell_1,M_1}(x)\otimes\Omega_{\ell_2,M_2}(0))=h_0(x,t),\\ &\Omega_{\tau,N}^{T}(t)\tilde{C}(\Omega_{\ell_1,M_1}(x)\otimes\Omega_{\ell_2,M_2}(\ell_2))=h_{\ell_2}(x,t), \end{aligned}\tag{7.33}$$

Now collocating equation (7.30) at $(M_1-1)\times(M_2-1)\times N,$ equation (7.31) at $(M_1+1)\times(M_2+1),$ equation (7.32) at $(M_2+1)\times N$ and equation (7.33) at $(M_1-1)\times N,$ we get

$$\begin{aligned} &\Omega_{\tau,N}^{T}(t_p)\,{}_{\tau}D_1^{T}\,\tilde{C}(\Omega_{\ell_1,M_1}(x_q)\otimes\Omega_{\ell_2,M_2}(y_r))\\ &\quad=d_x(x_q,y_r,t_p)\Omega_{\tau,N}^{T}(t_p)\tilde{C}({}_{\ell_1}D_{\gamma_1}\Omega_{\ell_1,M_1}(x_q)\otimes\Omega_{\ell_2,M_2}(y_r))\\ &\quad+d_y(x_q,y_r,t_p)\Omega_{\tau,N}^{T}(t_p)\tilde{C}(\Omega_{\ell_1,M_1}(x_q)\otimes{}_{\ell_2}D_{\gamma_2}\Omega_{\ell_2,M_2}(y_r))\\ &\quad-v_x(x_q,y_r,t_p)\Omega_{\tau,N}^{T}(t_p)\tilde{C}({}_{\ell_1}D_{\eta_1}\Omega_{\ell_1,M_1}(x_q)\otimes\Omega_{\ell_2,M_2}(y_r))\\ &\quad-v_y(x_q,y_r,t_p)\Omega_{\tau,N}^{T}(t_p)\tilde{C}(\Omega_{\ell_1,M_1}(x_q)\otimes{}_{\ell_2}D_{\eta_2}\Omega_{\ell_2,M_2}(y_r))\\ &\quad+\Psi\left(\Omega_{\tau,N}^{T}(t_p)\tilde{C}(\Omega_{\ell_1,M_1}(x_q)\otimes\Omega_{\ell_2,M_2}(y_r)),x_q,y_r,t_p\right)\\ &\qquad 1\le p\le N,\ 1\le q\le(M_1-1),\ 1\le r\le(M_2-1), \end{aligned}\tag{7.34}$$

$$\Omega_{\tau,N}^{T}(0)\tilde{C}(\Omega_{\ell_1,M_1}(x_q)\otimes\Omega_{\ell_2,M_2}(y_r))=f(x_q,y_r),\ 0\le q\le M_1,\ 0\le r\le M_2,\tag{7.35}$$

$$\begin{aligned} &\Omega_{\tau,N}^{T}(t_p)\tilde{C}(\Omega_{\ell_1,M_1}(0)\otimes\Omega_{\ell_2,M_2}(y_r))=g_0(y_r,t_p),\\ &\Omega_{\tau,N}^{T}(t_p)\tilde{C}(\Omega_{\ell_1,M_1}(\ell_1)\otimes\Omega_{\ell_2,M_2}(y_r))=g_{\ell_1}(y_r,t_p), \end{aligned}\quad 1\le p\le N,\ 1\le r\le(M_2+1),\tag{7.36}$$

$$\begin{aligned} &\Omega_{\tau,N}^{T}(t_p)\tilde{C}(\Omega_{\ell_1,M_1}(x_q)\otimes\Omega_{\ell_2,M_2}(0))=h_0(x_q,t_p),\\ &\Omega_{\tau,N}^{T}(t_p)\tilde{C}(\Omega_{\ell_1,M_1}(x_q)\otimes\Omega_{\ell_2,M_2}(\ell_2))=h_{\ell_2}(x_q,t_p), \end{aligned}\quad 1\le p\le N,\ 1\le q\le(M_1-1),\tag{7.37}$$

where $x_q\,(0\le q\le M_1)$ and $y_r\,(0\le r\le M_2)$ are the shifted Legendre-Gauss-Lobatto quadratures of $L_{\ell_1,M_1}(x)$ and $L_{\ell_2,M_2}(y),$ respectively while t_p's $(1\le p\le N)$ are the roots of $L_{\tau,N}(t).$ Equations (7.34)-(7.37) generate the system of $(M_1+1)\times(M_2+1)\times(N+1)$ linear/nonlinear equations in unknowns c_{pqr} which are easier to solve. Consequently $u_{M_1,M_2,N}(x,y,t)$ given in equation (7.24) can be calculated.

7.4 Numerical Results and Discussions

To demonstrate the efficiency and validity of the proposed algorithm, some illustrative examples are carried out in this section. The proposed algorithm is applied to solve the considered two-dimensional standard order as well as fractional order PDEs. In addition, the results obtained using the proposed algorithm are compared with the exact solutions of considered problems which show that the proposed algorithm is providing accurate results.

Example 1. Consider the time-fractional heat problem

$$\frac{\partial^{\nu}}{\partial t^{\nu}} u(x,y,t) = \frac{1}{\pi^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x,y,t), \qquad (x,y) \in [0,1]\times[0,1], t > 0,$$

with the initial condition

$$u(x,y,0) = \sin(\pi x)\sin(\pi y), \qquad (x,y) \in [0,1]\times[0,1],$$

and the boundary conditions

$$u(0,y,t) = u(1,y,t) = 0, \qquad y \in [0,1], t > 0,$$

$$u(x,0,t) = u(x,1,t) = 0, \qquad x \in [0,1], t > 0,$$

which has the exact solution $u(x,y,t) = \sin(\pi x)\sin(\pi y)E_{\nu}(-2t^{\nu})$ (Sun et al. (2013)) where $E_{\nu}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\nu i + 1)}$ is the Mittang Leffler function.

To demonstrate the accuracy of the proposed algorithm, the absolute errors are tabulated in Tables 7.1 and 7.2. In Table 7.1, absolute errors are tabulated for various choices of t with fixed values of $x = y$ and $N = M_1 = M_2 = 3$ for Example 1. In Table 7.2, absolute errors are tabulated at various choices of $x = y$ with fixed values of t and

$N = M_1 = M_2 = 3.$ In Fig. 7.1(a) – (e), the absolute errors vs. x and y are plotted for various choices of time t with $N = M_1 = M_2 = 3.$

Table 7.1 The absolute errors at different t with fixed values of $x = y$ and $N = M_1 = M_2 = 3$ for Example 1

t	x	$e(x,y,t)$	t	x	$e(x,y,t)$	t	x	$e(x,y,t)$
0.1	0.1	2.86e-04	0.1	0.5	5.53e-05	0.1	0.9	2.86e-04
0.2		3.26e-04	0.2		7.52e-04	0.2		3.26e-04
0.3		3.49e-04	0.3		1.25e-03	0.3		3.49e-04
0.4		3.59e-04	0.4		1.59e-03	0.4		3.59e-04
0.5		3.59e-04	0.5		1.80e-03	0.5		3.59e-04
0.6		3.53e-04	0.6		1.93e-03	0.6		3.53e-04
0.7		3.42e-04	0.7		1.99e-03	0.7		3.42e-04
0.8		3.29e-04	0.8		2.00e-03	0.8		3.29e-04
0.9		3.12e-04	0.9		1.97e-03	0.9		3.12e-04

Table 7.2 The absolute errors at different choices of $x = y$ with fixed values of t and $N = M_1 = M_2 = 3$ for Example 1

x	t	$e(x,y,t)$	x	t	$e(x,y,t)$	x	t	$e(x,y,t)$
0.1	0.1	2.86e-04	0.1	0.5	3.59e-04	0.1	0.9	3.12e-04
0.2		5.48e-04	0.2		9.75e-04	0.2		9.15e-04
0.3		4.57e-04	0.3		1.46e-03	0.3		1.48e-03
0.4		1.90e-04	0.4		1.73e-03	0.4		1.85e-03
0.5		5.53e-05	0.5		1.80e-03	0.5		1.97e-03
0.6		1.90e-04	0.6		1.73e-03	0.6		1.85e-03
0.7		4.57e-04	0.7		1.46e-03	0.7		1.48e-03
0.8		5.48e-04	0.8		9.75e-04	0.8		9.15e-04
0.9		2.86e-04	0.9		3.59e-04	0.9		3.12e-04

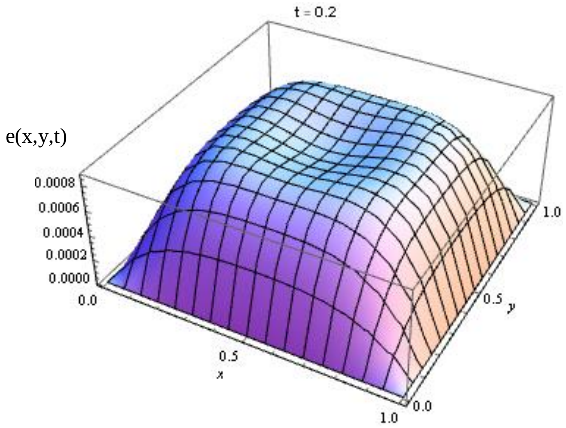

Fig. 7.1(a)

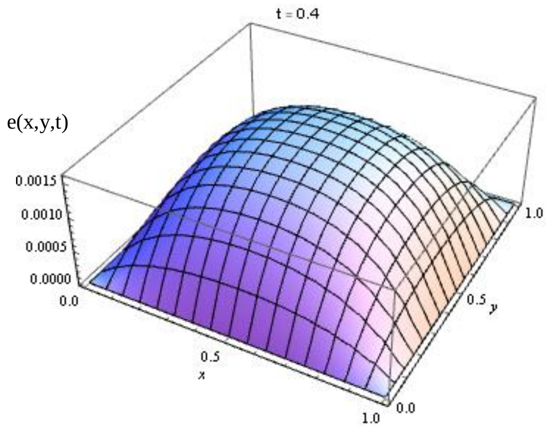

Fig.7.1(b)

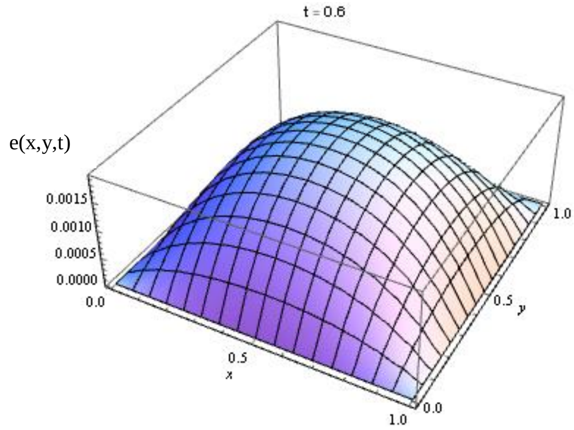

Fig. 7.1(c)

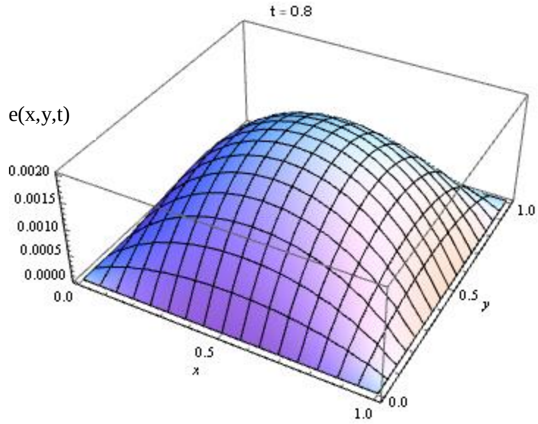

Fig.7.1(d)

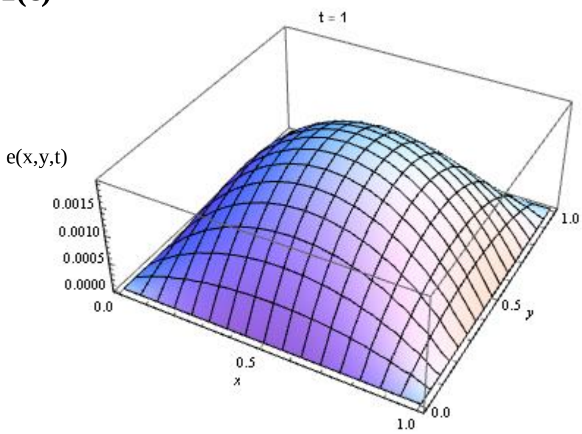

Fig. 7.1(e)

Fig. 7.1 The absolute errors of Example 1 vs. x and y with $N = M_1 = M_2 = 3$ at (a) $t = 0.2$, (b) $t = 0.4$, (c) $t = 0.6$, (d) $t = 0.8$, (e) $t = 1$

Example 2. Consider the space-fractional reaction-diffusion problem

$$\frac{\partial}{\partial t}u(x,y,t) = \frac{\Gamma(2.2)}{6}x^{2.8}y\frac{\partial^{1.8}}{\partial x^{1.8}}u(x,y,t) + \frac{2xy^{2.6}}{\Gamma(4.6)}\frac{\partial^{1.6}}{\partial y^{1.6}}u(x,y,t) - (1+2xy)e^{-t}x^3y^{3.6},$$
$$(x,y) \in [0,1]\times[0,1], t > 0,$$

with the initial condition

$$u(x,y,0) = x^3y^{3.6}, \qquad (x,y) \in [0,1]\times[0,1],$$

and the boundary conditions

$$u(0,y,t) = 0, u(1,y,t) = e^{-t}y^{3.6}, \qquad y \in [0,1], t > 0,$$

$$u(x,0,t) = 0, u(x,1,t) = e^{-t}x^3, \qquad x \in [0,1], t > 0,$$

which has the exact solution $u(x,y,t) = e^{-t}x^3y^{3.6}$ (Bhrawy (2014)).

In Table 7.3, absolute errors are tabulated for various choices of t with fixed $x = y$ and $N = M_1 = M_2 = 3$ for Example 2. In Table 7.4, absolute errors are tabulated for different choices of $x = y$ with fixed t and $N = M_1 = M_2 = 3$. In Fig. 7.2(a) – (e), the absolute errors vs. *x* and *y* are displayed for various t with $N = M_1 = M_2 = 3$.

Table 7.3 The absolute errors at various t with fixed values of $x = y$ and $N = M_1 = M_2 = 3$ for Example 2

t	x	$e(x,y,t)$	t	x	$e(x,y,t)$	t	x	$e(x,y,t)$
0.1	0.1	4.06e-05	0.1	0.5	1.12e-04	0.1	0.9	1.88e-04
0.2		6.61e-05	0.2		1.69e-04	0.2		3.28e-04
0.3		7.84e-05	0.3		1.88e-04	0.3		3.94e-04
0.4		7.96e-05	0.4		1.74e-04	0.4		3.97e-04
0.5		7.18e-05	0.5		1.34e-04	0.5		3.48e-04
0.6		5.70e-05	0.6		7.57e-05	0.6		2.59e-04
0.7		3.72e-05	0.7		4.76e-06	0.7		1.41e-04
0.8		1.45e-05	0.8		7.17e-05	0.8		5.23e-06
0.9		9.08e-06	0.9		1.47e-04	0.9		1.36e-04

Table 7.4 The absolute errors at various values of $x = y$ with fixed values of t and $N = M_1 = M_2 = 3$ for Example 2

x	t	$e(x,y,t)$	x	t	$e(x,y,t)$	x	t	$e(x,y,t)$
0.1	0.1	4.06e-05	0.1	0.5	7.18e-05	0.1	0.9	9.08e-06
0.2		3.70e-05	0.2		6.53e-05	0.2		8.26e-06
0.3		1.24e-05	0.3		2.97e-05	0.3		1.16e-05
0.4		9.60e-06	0.4		4.64e-05	0.4		5.40e-05
0.5		1.12e-04	0.5		1.34e-04	0.5		1.47e-04
0.6		3.19e-04	0.6		4.71e-04	0.6		2.60e-04
0.7		4.83e-04	0.7		7.59e-04	0.7		3.30e-04
0.8		4.51e-04	0.8		7.43e-04	0.8		2.91e-04
0.9		1.88e-04	0.9		3.48e-04	0.9		1.36e-04

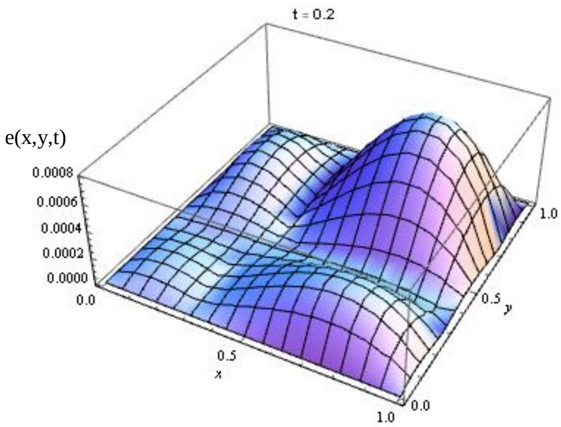

Fig. 7.2(a)

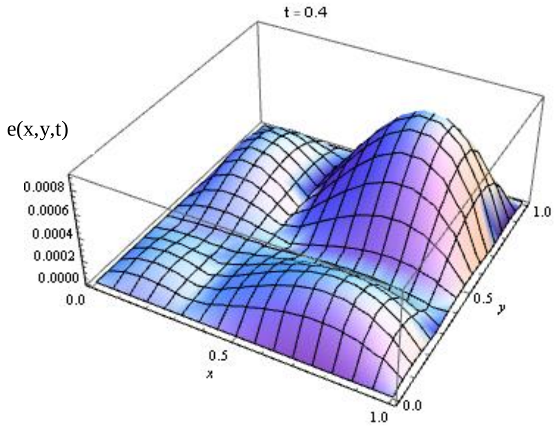

Fig.7.2(b)

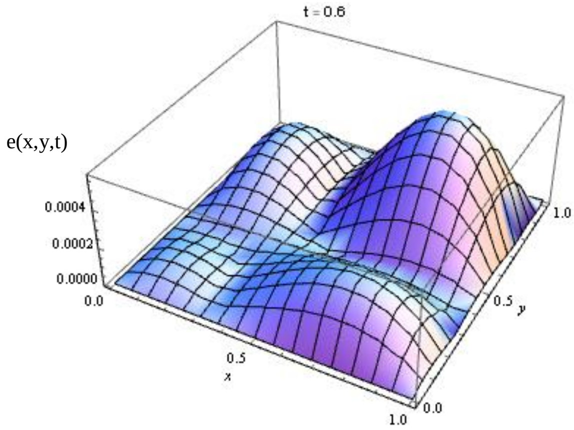

Fig. 7.2(c)

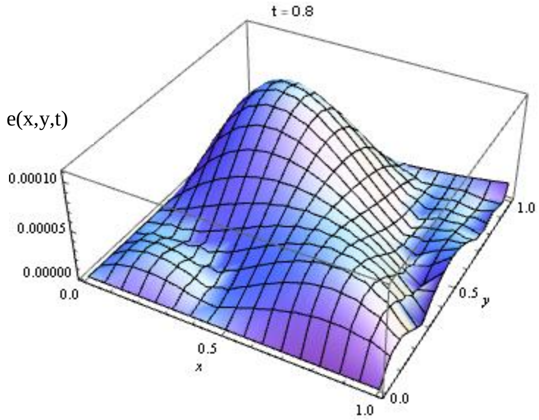

Fig.7.2(d)

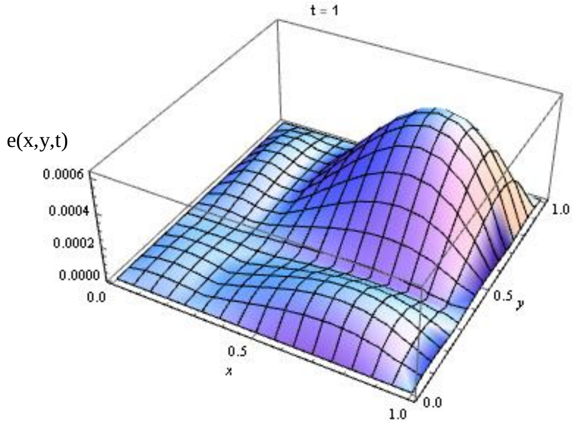

Fig. 7.2(e)

Fig. 7.2 The absolute errors of Example 2 vs. x and y with $N = M_1 = M_2 = 3$ at (a) $t = 0.2$, (b) $t = 0.4$, (c) $t = 0.6$, (d) $t = 0.8$, (e) $t = 1$

Example 3. Consider the space-fractional reaction-diffusion problem

$$\frac{\partial}{\partial t}u(x,y,t) = (3-2x)\frac{\Gamma(3-\gamma_1)}{2}\frac{\partial^{\gamma_1}}{\partial x^{\gamma_1}}u(x,y,t) + (4-y)\frac{\Gamma(4-\gamma_2)}{6}\frac{\partial^{\gamma_2}}{\partial y^{\gamma_2}}u(x,y,t)$$
$$+e^{-t}(x^2(-y^{3/2}+y-4)y^{3/2}+\sqrt{x}(2x-3)y^3),$$
$$(x,y)\in[0,1]\times[0,1], t>0,$$

with the initial condition

$$u(x,y,0) = x^2y^3, \qquad (x,y)\in[0,1]\times[0,1],$$

and the boundary conditions

$$u(0,y,t) = 0, u(1,y,t) = e^{-t}y^3, \qquad y\in[0,1], t>0,$$

$$u(x,0,t) = 0, u(x,1,t) = e^{-t}x^2, \qquad x\in[0,1], t>0,$$

The exact solution of which is given by $u(x,y,t) = e^{-t}x^2y^3$ (Bhrawy (2014)).

In Tables 7.5 and 7.6, the absolute errors are tabulated for different values of t with fixed $x = y$ and for various $x = y$ at t, respectively with $N = M_1 = M_2 = 3$ for Example 3. In Fig. 7.3(a) – (e), the absolute errors vs. x and y are plotted at various time t with $N = M_1 = M_2 = 3$.

Table 7.5 The absolute errors for different t with fixed values of $x = y$ and $N = M_1 = M_2 = 3$ for Example 3

t	x	$e(x,y,t)$	t	x	$e(x,y,t)$	t	x	$e(x,y,t)$
1	0.1	7.04e-07	0.1	0.5	1.04e-04	0.1	0.9	2.39e-05
0.2		1.43e-06	0.2		1.81e-04	0.2		3.81e-05
0.3		2.16e-06	0.3		2.35e-04	0.3		4.51e-05
0.4		2.88e-06	0.4		2.67e-04	0.4		4.73e-05
0.5		3.56e-06	0.5		2.82e-04	0.5		4.65e-05
0.6		4.18e-06	0.6		2.81e-04	0.6		4.41e-05
0.7		4.73e-06	0.7		2.67e-04	0.7		4.11e-05
0.8		5.18e-06	0.8		2.43e-04	0.8		3.84e-05
0.9		5.51e-06	0.9		2.12e-04	0.9		3.62e-05

Table 7.6 The absolute errors at different choices of $x = y$ with fixed values of t and $N = M_1 = M_2 = 3$ for Example 3

x	t	$e(x,y,t)$	x	t	$e(x,y,t)$	x	t	$e(x,y,t)$
0.1	0.1	7.04e-07	0.1	0.5	3.56e-06	0.1	0.9	5.52e-06
0.2		1.37e-05	0.2		4.09e-05	0.2		3.69e-05
0.3		4.18e-05	0.3		1.17e-04	0.3		9.48e-05
0.4		7.61e-05	0.4		2.09e-04	0.4		1.61e-04
0.5		1.04e-04	0.5		2.82e-04	0.5		2.12e-04
0.6		1.13e-04	0.6		3.04e-04	0.6		2.26e-04
0.7		9.87e-05	0.7		2.61e-04	0.7		1.93e-04
0.8		6.40e-05	0.8		1.63e-04	0.8		1.21e-04
0.9		2.39e-05	0.9		4.65e-05	0.9		3.62e-04

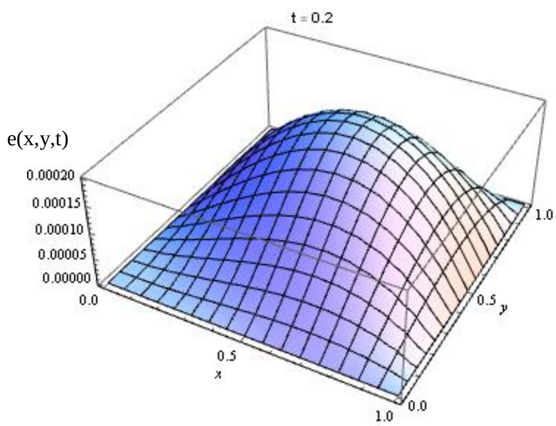

Fig. 7.3(a)

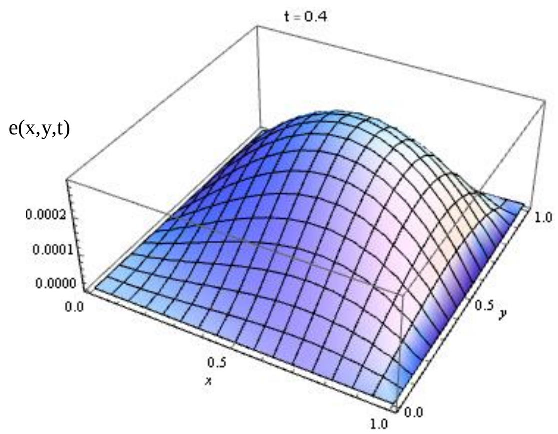

Fig.7.3(b)

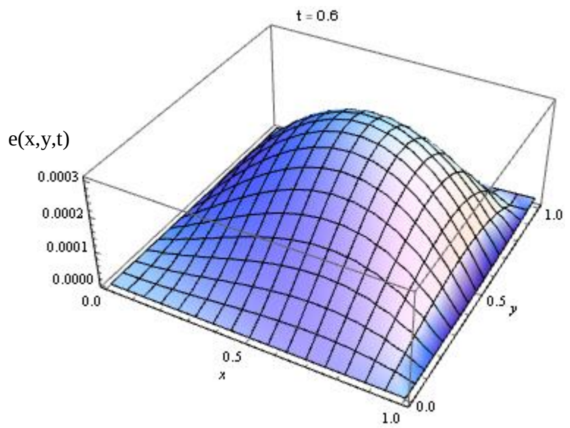

Fig. 7.3(c)

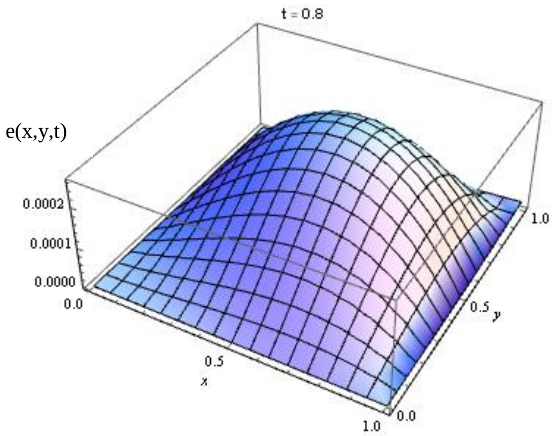

Fig.7.3(d)

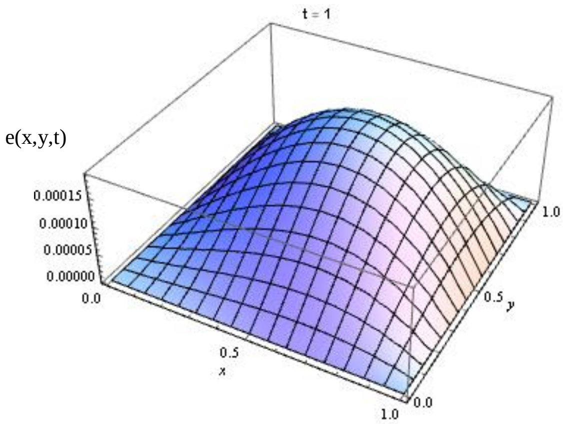

Fig. 7.3(e)

Fig. 7.3 The absolute errors of Example 3 vs. x and y with $N = M_1 = M_2 = 3$ at (a) $t = 0.2$, (b) $t = 0.4$, (c) $t = 0.6$, (d) $t = 0.8$, (e) $t = 1$

Example 4. Consider the reaction-convection-diffusion problem

$$\frac{\partial}{\partial t}u(x,y,t) = \left(\frac{\partial^2}{\partial x^{2_1}} + \frac{\partial^2}{\partial y^2}\right)u(x,y,t) - \tanh(x)\frac{\partial}{\partial x}u(x,y,t) - \tanh(y)\frac{\partial}{\partial y}u(x,y,t)$$
$$+(-3 + 3\sec h^2(x) + 3\sec h^2(y))u(x,y,t) - \sec h(x)\sec h(y),$$
$$(x,y) \in [0,1]\times[0,1], t > 0,$$

with the initial condition

$$u(x,y,0) = 2\sec h(x)\sec h(y), \qquad (x,y) \in [0,1]\times[0,1],$$

and the boundary conditions

$$u(0,y,t) = (1+e^t)\sec h(y), u(1,y,t) = (1+e^t)\sec h(1)\sec h(y),\ y \in [0,1], t > 0,$$

$$u(x,0,t) = (1+e^t)\sec h(x), u(x,1,t) = (1+e^t)\sec h(x)\sec h(1),\ x \in [0,1], t > 0,$$

which has the exact solution $u(x,y,t)=(1+e^{t})\sec h(x)\sec h(y)$ (Hoz and Vadillo (2013)).

The absolute errors are tabulated in Tables 7.7 and 7.8 for various t with fixed $x=y$ and for various $x=y$ at fixed t, respectively with $N=M_1=M_2=3$ for Example 4. In Fig. 7.4(a) – (e), the absolute errors vs. x and y are displayed for various t with $N=M_1=M_2=3$.

Table 7.7 The absolute errors at various choices of t with fixed values of $x=y$ and $N=M_1=M_2=3$ for Example 4

t	x	$e(x,y,t)$	t	x	$e(x,y,t)$	t	x	$e(x,y,t)$
0.1	0.1	2.38e-06	0.1	0.5	2.66e-04	0.1	0.9	2.96e-05
0.2		4.51e-05	0.2		5.13e-04	0.2		6.29e-05
0.3		9.80e-05	0.3		7.43e-04	0.3		1.00e-04
0.4		1.50e-04	0.4		9.54e-04	0.4		1.37e-04
0.5		2.01e-04	0.5		1.14e-03	0.5		1.70e-04
0.6		2.38e-04	0.6		1.31e-03	0.6		1.97e-04
0.7		2.59e-04	0.7		1.45e-03	0.7		2.16e-04
0.8		2.62e-04	0.8		1.57e-03	0.8		2.26e-04
0.9		2.45e-04	0.9		1.68e-03	0.9		2.26e-04

Table 7.8 The absolute errors at various choices of $x=y$ with fixed values of t and $N=M_1=M_2=3$ for Example 4

x	t	$e(x,y,t)$	x	t	$e(x,y,t)$	x	t	$e(x,y,t)$
0.1	0.1	2.38e-06	0.1	0.5	2.01e-04	0.1	0.9	2.45e-04
0.2		9.38e-05	0.2		5.31e-04	0.2		7.43e-04
0.3		1.90e-04	0.3		8.67e-04	0.3		1.25e-03
0.4		2.51e-04	0.4		1.09e-03	0.4		1.59e-03
0.5		2.66e-04	0.5		1.14e-03	0.5		1.68e-03
0.6		2.35e-04	0.6		1.02e-03	0.6		1.51e-03
0.7		1.73e-04	0.7		7.72e-04	0.7		1.13e-03
0.8		9.72e-05	0.8		4.56e-04	0.8		6.57e-04
0.9		2.96e-05	0.9		1.70e04	0.9		2.26e-04

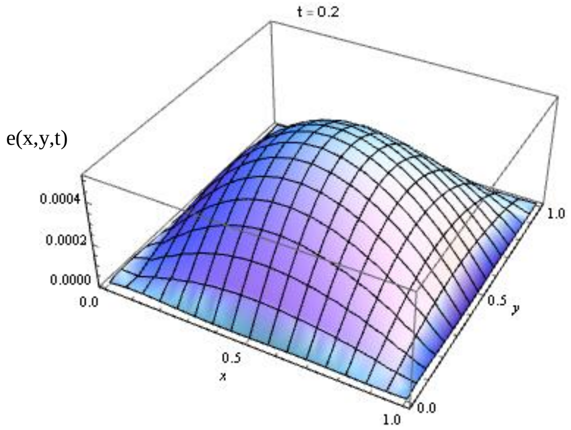

Fig. 7.4(a)

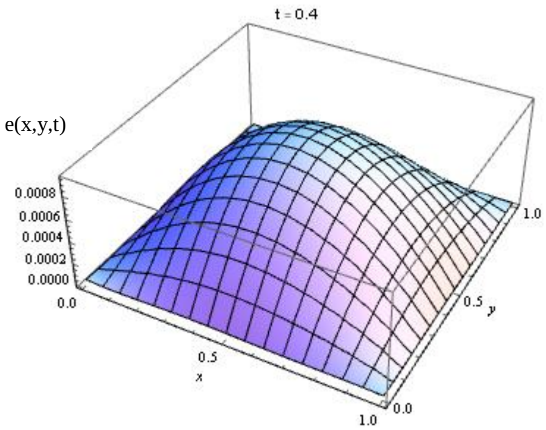

Fig.7.4(b)

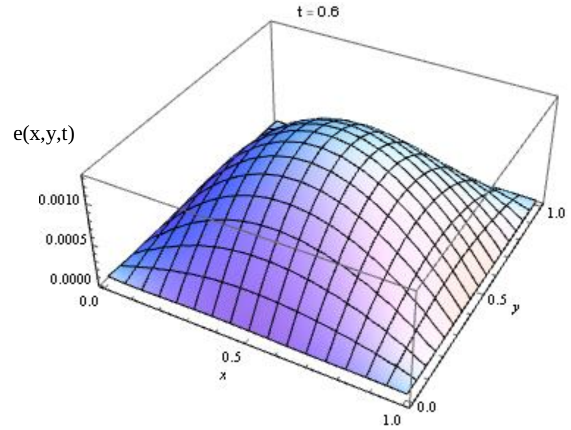

Fig. 7.4(c)

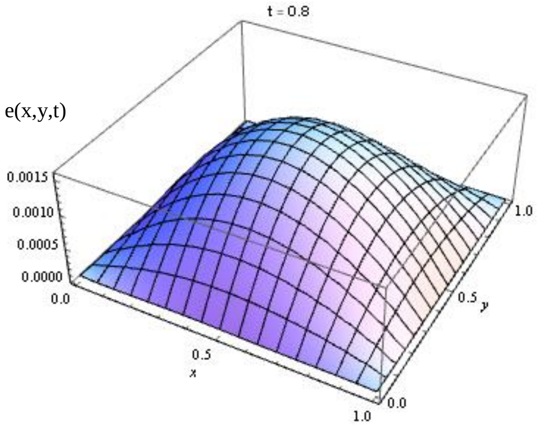

Fig.7.4(d)

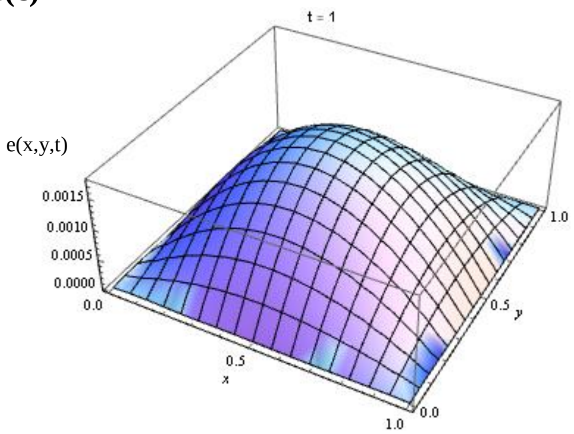

Fig. 7.4(e)

Fig. 7.4 The absolute errors of Example 4 vs. x and y with $N = M_1 = M_2 = 3$ at (a) $t = 0.2$, (b) $t = 0.4$, (c) $t = 0.6$, (d) $t = 0.8$, (e) $t = 1$

Example 5. Consider the reaction-convection-diffusion problem

$$\frac{\partial}{\partial t}u(x,y,t) = \frac{1}{c_1^2 + c_2^2}\left(\frac{\partial^2}{\partial x^{2_1}} + \frac{\partial^2}{\partial y^2}\right)u(x,y,t) - \frac{1}{c_1}\sin(c_1 x)\frac{\partial}{\partial x}u(x,y,t) -$$
$$\frac{1}{c_2}\sin(c_2 y)\frac{\partial}{\partial y}u(x,y,t) + (\cos(c_1 x) + \cos(c_2 y))u(x,y,t) - (\cos(c_1 x) + \cos(c_2 y)),$$
$$(x,y) \in [0,\pi/2]\times[0,\pi/2], t > 0,$$

with the initial condition

$$u(x,y,0)=1+\sin(c_1x)\sin(c_2y), \qquad (x,y)\in[0,\pi/2]\times[0,\pi/2],$$

and the boundary conditions

$$u(0,y,t)=1, u_x(\pi/2,y,t)=0, \qquad y\in[0,\pi/2], t>0,$$

$$u(x,0,t)=1, u(x,\pi/2,t)=0, \qquad x\in[0,\pi/2], t>0,$$

having the exact solution as $u(x,y,t)=1+e^{-t}\sin(c_1x)\sin(c_2y)$ (Hoz and Vadillo (2013)).

The absolute errors are tabulated in Tables 7.9 and 7.10 for various t with fixed $x=y$ and for various $x=y$ at fixed t, respectively with $N=M_1=M_2=3$ for Example 5. Fig. 7.5(a) – (e), depicts the absolute errors vs. x and y at different t with $N=M_1=M_2=3$.

Table 7.9 The absolute errors at different choices of t with fixed values of $x=y$ and $N=M_1=M_2=3$ for Example 5

t	x	$e(x,y,t)$	t	x	$e(x,y,t)$	t	x	$e(x,y,t)$
0.1	0.1	3.49e-06	0.1	0.5	1.16e-04	0.1	0.9	2.78e-04
0.2		8.41e-06	0.2		2.07e-04	0.2		4.87e-04
0.3		1.23e-05	0.3		2.81e-04	0.3		6.59e-04
0.4		1.53e-05	0.4		3.37e-04	0.4		7.96e-04
0.5		1.75e-05	0.5		3.78e-04	0.5		8.99e-04
0.6		1.88e-05	0.6		4.05e-04	0.6		9.70e-04
0.7		1.94e-05	0.7		4.18e-04	0.7		1.01e-03
0.8		1.94e-05	0.8		4.20e-04	0.8		1.03e-03
0.9		1.90e-05	0.9		4.12e-04	0.9		1.02e-03

Table 7.10 The absolute errors at different choices of $x = y$ with fixed values of t and $N = M_1 = M_2 = 3$ for Example 5

x	t	$e(x,y,t)$	x	t	$e(x,y,t)$	x	t	$e(x,y,t)$
0.1	0.1	3.49e-06	0.1	0.5	1.75e-05	0.1	0.9	1.90e-05
0.2		1.62e-05	0.2		6.84e-05	0.2		7.40e-05
0.3		3.99e-05	0.3		1.49e-04	0.3		1.62e-04
0.4		7.39e-05	0.4		2.55e-04	0.4		2.76e-04
0.5		1.16e-04	0.5		3.78e-04	0.5		4.12e-04
0.6		1.61e-04	0.6		5.12e-04	0.6		5.62e-04
0.7		2.07e-04	0.7		6.49e-04	0.7		7.18e-04
0.8		2.47e-04	0.8		7.80e-04	0.8		8.73e-04
0.9		2.78e-04	0.9		8.99e-04	0.9		1.02e-03

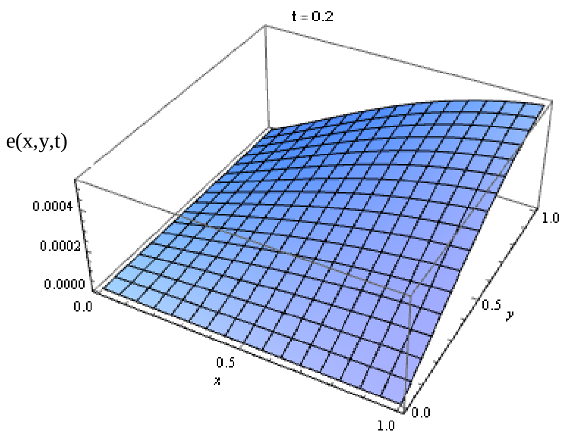

Fig. 7.5(a)

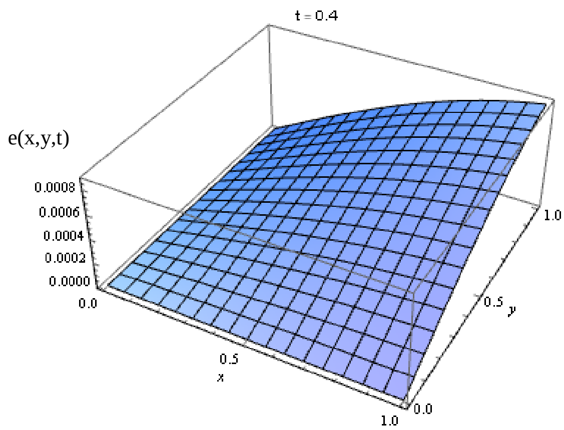

Fig.7.5(b)

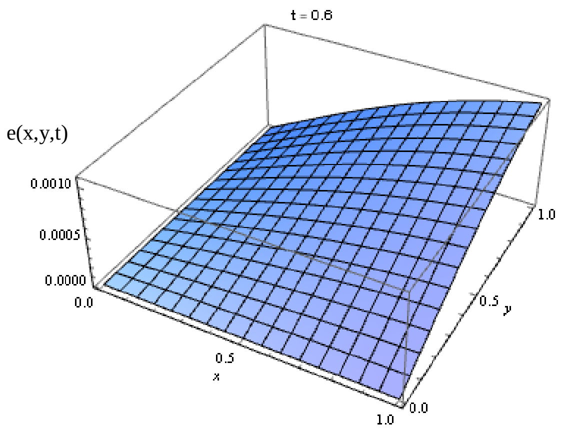

Fig. 7.5(c)

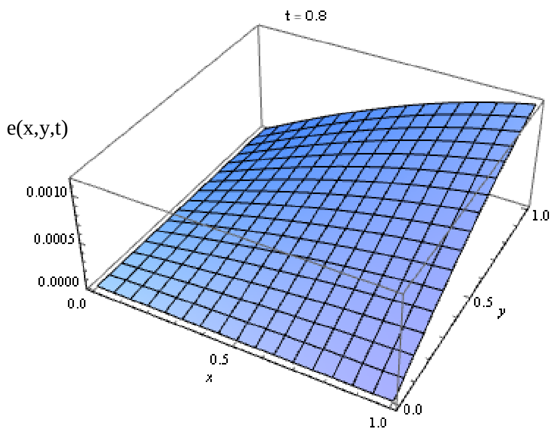

Fig.7.5(d)

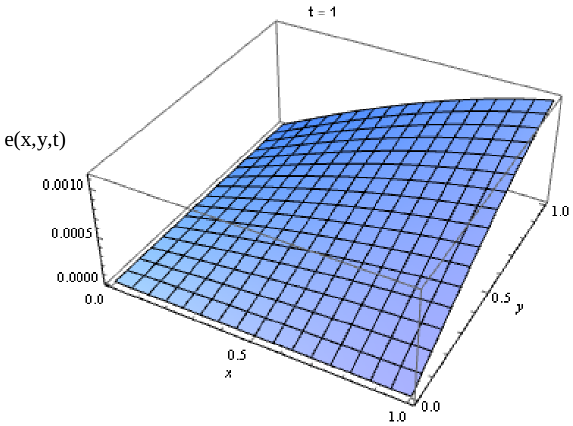

Fig. 7.5(e)

Fig. 7.5 The absolute errors of Example 5 vs. x and y with $N = M_1 = M_2 = 3$ at (a) $t = 0.2$, (b) $t = 0.4$, (c) $t = 0.6$, (d) $t = 0.8$, (e) $t = 1$

7.5 Conclusions

In this chapter, a new algorithm based on spectral collocation method and shifted Legendre polynomials is proposed to get the more accurate approximate solution of the standard as well as fractional order linear/non-linear two-dimensional PDEs subject to initial and boundary conditions. Operational matrices of the shifted Legendre polynomials are used during the application of the proposed algorithm. Some problems are solved using the proposed algorithm, and the results are compared with the existing analytical solutions to show the applicability, efficiency, and validity of the proposed algorithm. Theoretical and graphical studies of the errors of the approximate solution show that the proposed algorithm has the exponential convergence rate in both spatial and temporal discretization.