Chapter 6

Numerical Solution of Porous Media Equations Using Jacobi Operational Matrices

6.1 Introduction

In last chapters, only those models have considered in which the coefficients are constant or the function of space-time and if the model is non-linear then nonlinearity only presents in reaction term. But in general, as discussed in Chapter 1, the coefficients may depend on the unknown due to which nonlinearity may appear in diffusion and advection term. In the present chapter, those models are considered where the diffusion coefficient d depends upon the unknown c. Let us considered the model which can be expressed as

$$c_t = (d(c)c_x)_x + \xi(c), \tag{6.1}$$

The equation (6.1) is a nonlinear reaction-diffusion model in which $d(c)$ can have different functional forms from which various diffusion processes are characterized such as fast diffusion ($d(c) = c^p, p < 0$), slow diffusion ($d(c) = c^p, p > 0$), and other types of diffusion (Wazwaz (2001)) and $\xi(c)$ is the nonlinear reaction term. The simplest form of $\xi(c)$ is the so-called Fisher equation with $\xi(c) = c(1-c)$ which have been given by Fisher (1937) to depict the movement of a vital mutant in an infinitely long habitat and having great applications in various fields as discussed in Chapter 4. It also used as a mathematical model of reacting flow in the porous medium. If

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$\xi(c) = c(1-c)(c-1)$ then another simplest form of the equation (6.1) is known as Huxley equation which also has important applications in various fields, e.g., biology, chemistry, fluid dynamics. This model is often encountered during the modeling of various complex phenomena like heat and mass transfer, combustion theory, flow through porous media (Patel et al. (2013); Das et al. (2011); Kudryashov et al. (2013), Freitas et al. (2017)) and thus it is known as porous media equation. It represents the unsteady heat transfer where diffusivity is a power-law function of temperature (Polyanin and Zaitsev (2004)) and fluid dynamics of thin films (Witelski (1997)). If the diffusion term does not satisfy the condition for classical diffusion ($d(c) > 0$), equation (6.1) is known as degenerate parabolic differential equation (Witelski (1997)). J.D. Murray (1993) considered the diffusion coefficient as the function of the population in his model which shows that the population disperses faster in the lower density region as compared to the regions where population gets more crowded, and thus it represents the population pressure in biological systems.

Many analytical, as well as numerical methods, are exiting in the literature for the solution of different forms of equation (6.1). The Lie group similarity technique is used very frequently to get the exact solutions to these problems (Ames (1972)). To find another class of exact solutions, Bluman and Kumei (1989) have used the potential and nonlocal symmetries. E. A. Saied (1999) has solved the inhomogeneous nonlinear diffusion equation (NDE) by applying a non-classical method of Lie's generalization method. Saied and Hussein (1994) have been used the Lie similarity method to get the analytical solutions of inhomogeneous NDEs. In 1999, Changzheng (1999) had used a generalized conditional method to obtain the exact solution of NDE by reducing the considered problem in Fujita's equation. J. R. King (1991) investigated the local and nonlocal symmetries of two particular cases with $m = -4/3$ and $m = -2/3$. A. M.

Wazwaz (2001) has used the Adomian decomposition method (ADM) to get the solution of NDEs with power-law diffusivities, and he also used the He's variational iteration method to get the analytical solutions of linear and nonlinear diffusion problems in his other article (Wazwaz (2007)). S. Pamuk (2005) obtained the exact solution of the porous media equation by applying ADM. M. Sari (2009) used the compact finite difference method in space with third-order Runge-Kutta scheme in time to get the solutions of porous media equations.

Due to important applications of this model, the study of the solutions of different forms of this model has been carried out during last half century and still a deedful field of research to develop some better exact as well as numerical methods to approximate the solutions of these porous media equations.

In this chapter, Jacobi collocation technique discussed already in Chapter 5 is considered to get the numerical solution of equation (6.1) with $d(c) = c^m, m > 0$ and $\xi(c) = \lambda c(1-c)$ subject initial and boundary conditions. The results obtained for different particular cases are displayed through the graphical presentations and tabular forms.

6.2 Application of the Jacobi Collocation Technique During Solution of Porous Media Equation

Here, to show the importance of Jacobi operational matrix during the solution of porous media equation, the Jacobi collocation technique is applied to solve the porous media equation

$$\frac{\partial c(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(d(c)\frac{\partial c(x,t)}{\partial x} \right) + \xi(c), \quad 0 < x < L,\ 0 < t \le \tau, \tag{6.2}$$

with the boundary conditions

$$c(0,t) = f(t), \qquad 0 < t \le \tau, \tag{6.3}$$

$$c(L,t) = g(t), \qquad 0 < t \le \tau, \tag{6.4}$$

and initial condition

$$c(x,0) = h(x), \qquad 0 < x \le L, \tag{6.5}$$

Equation (6.2) with the use of initial condition (6.5) can be re-written in the form

$$\frac{\partial c(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(d(c)\frac{\partial c(x,t)}{\partial x} \right) - \xi(c) + c(x,0) - h(x), \tag{6.6}$$

which has to be solved along with the given boundary conditions.

To solve the problem (6.6) with the given boundary conditions, let us approximate the solution $c(x,t)$ as

$$c_{M,N}(x,t) = \sum_{p=0}^{M}\sum_{q=0}^{N} a_{pq}\, P_{\tau,p}^{(\alpha,\beta)}(t)\, P_{L,q}^{(\alpha,\beta)}(x) = \Omega_{\tau,M}^{T}(t)\, A\, \Omega_{L,N}(x), \tag{6.7}$$

where A is the $(M+1)\times(N+1)$ order matrix with the coefficients a_{pq}, given as

$$A = \begin{pmatrix} a_{00} & a_{01} & . & . & . & a_{0N} \\ a_{10} & a_{11} & . & . & . & a_{1N} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ a_{M0} & a_{M1} & . & . & . & a_{MN} \end{pmatrix}, \tag{6.8}$$

and T represents the transpose of the matrix.

The entries of the matrix A are obtained from

$$a_{pq} = \frac{1}{\chi_{\tau,p}^{(\alpha,\beta)}\,\chi_{L,q}^{(\alpha,\beta)}} \int_0^{\tau}\int_0^{L} c(x,t)\, P_{\tau,p}^{(\alpha,\beta)}(t)\, P_{L,q}^{(\alpha,\beta)}(x)\, w_{\tau}^{(\alpha,\beta)}(t)\, w_{L}^{(\alpha,\beta)}(x)\, dx\, dt,$$

$$p = 0,1,\ldots,M, \; q = 0,1,\ldots,N. \tag{6.9}$$

The standard order derivatives of the approximate solution (6.7) can be expressed as

$$\frac{\partial}{\partial t} c(x,t) \simeq \Omega_{\tau,M}^{T}(t) D_{\tau}^{T} A\, \Omega_{L,N}(x), \tag{6.10}$$

$$\frac{\partial^m}{\partial x^m} c(x,t) \simeq \Omega^T_{\tau,M}(t)\, A D_L^m \Omega_{L,N}(x), \tag{6.11}$$

$$c(x,0) \simeq \Omega^T_{\tau,M}(0)\, A \Omega_{L,N}(x). \tag{6.12}$$

Employing equations (6.10)-(6.12) in equation (6.6), we get

$$\begin{aligned} &\Omega^T_{\tau,M}(t)\left\lfloor D_\tau^T A - A D_L \left\{ d(\Omega^T_{\tau,M}(t) A\Omega_{L,N}(x))\Omega^T_{\tau,M}(t) A\Omega_{L,N}(x) \right\}\right\rfloor \Omega_{L,N}(x) \\ &\qquad = \Omega^T_{\tau,M}(0) A\Omega_{L,N}(x) - h(x) + \xi(\Omega^T_{\tau,M}(t) A\Omega_{L,N}(x)), \end{aligned} \tag{6.13}$$

with the boundary conditions

$$\begin{aligned} \Omega^T_{\tau,M}(t)\, A\Omega_{L,N}(0) &= f(t), \\ \Omega^T_{\tau,M}(t)\, A\Omega_{L,N}(L) &= g(t). \end{aligned} \tag{6.14}$$

According to the spectral Jacobi collocation method, equation (6.13) is to be satisfied at $(M+1)\times(N-1)$ collocation points, and the considered boundary conditions (6.14) are satisfied at $2(M+1)$ collocation points as

$$\begin{aligned} &\Omega^T_{\tau,M}(t_p)\left[D_\tau^T A - A D_L \left\{ d(\Omega^T_{\tau,M}(t_p) A\Omega_{L,N}(x_q))\Omega^T_{\tau,M}(t_p) A\Omega_{L,N}(x_q) \right\}\right] \Omega_{L,N}(x_q) \\ &\qquad = \Omega^T_{\tau,M}(0) A\Omega_{L,N}(x_q) - h(x_q) + \xi(\Omega^T_{\tau,M}(t_p) A\Omega_{L,N}(x_q)), \end{aligned} \tag{6.15}$$

with the boundary conditions

$$\begin{aligned} \Omega^T_{\tau,M}(t_p)\, A\Omega_{L,N}(0) &= f(t_p), \\ \Omega^T_{\tau,M}(t_p)\, A\Omega_{L,N}(L) &= g(t_p), \end{aligned} \tag{6.16}$$

where t_p, $p = 0,1,...,M$ are the roots of $P^{(\alpha,\beta)}_{\tau,M+1}(t)$ and x_q, $q = 0,1,...,N-2$ are the Jacobi Gauss-Lobatto points. From here we get the nonlinear system of $(M+1)\times(N+1)$ algebraic equations in a_{pq} in which $(M+1)\times(N-1)$ equations arise from equation (6.15), and $2(M+1)$ equations arise from equation (6.16). This system can be solved using Newton iterative method for a_{pq}. As a result, $c_{M,N}(x,t)$ given in equation (6.7) can be calculated.

6.3 Solution of the Problem

To understand the physical behavior of reaction term and the role of diffusivity coefficients, let us reformulate the Porous-Fisher model by taking $d(c) = c^m$ and $\xi(c) = \lambda c(1-c)$ as

$$c_t = (c^m c_x)_x + \lambda c(1-c), \qquad 0 < x < 1,\, 0 < t \le \tau,$$

with the boundary conditions

$$c(0,t) = c(L,t) = 0, \qquad 0 < t \le \tau,$$

and the initial condition

$$c(x,0) = x, \qquad 0 < x < 1.$$

Case I: When $m = 0$, the considered Porous-Fisher equation reduces to a reaction-diffusion model of the form

$$c_t = c_{xx} + \lambda c(1-c).$$

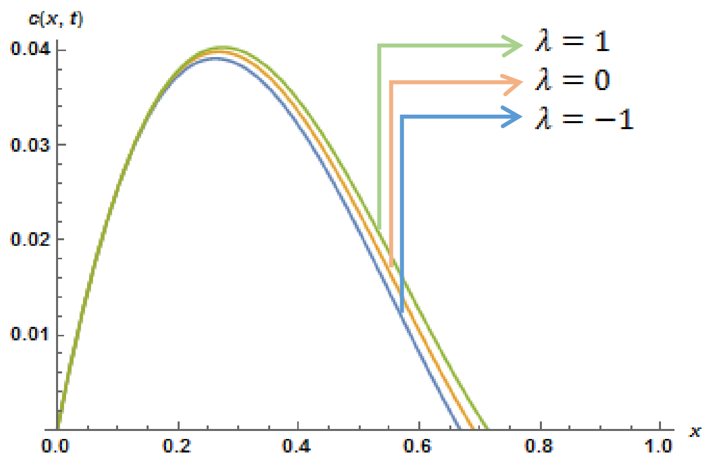

Fig. 6.1 Plots of the approximate solution vs. x when $\lambda = -1,0,1$ at $t = 1$hr for $M = N = 3$

Case II: When $m = 1$, the considered Porous-Fisher equation becomes a non-linear reaction-convection-diffusion model of the form

$$c_t = (cc_x)_x + \lambda c(1-c),$$

or $$c_t = cc_{xx} + c_x^2 + \lambda c(1-c).$$

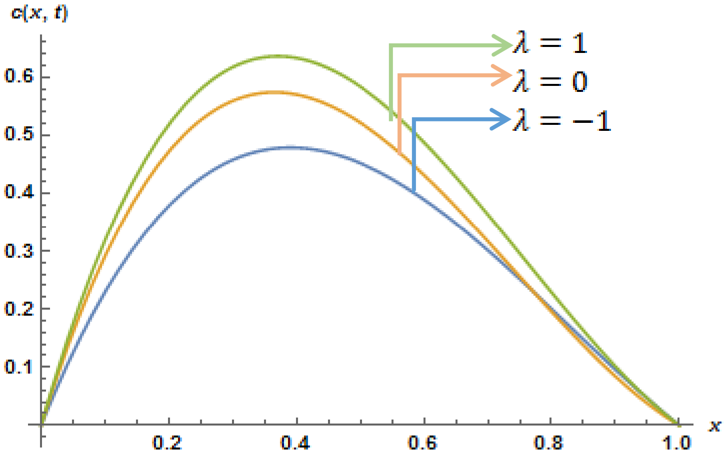

Fig. 6.2 Plots of the approximate solution vs. x when $\lambda = -1,0,1$ at $t = 1$hr for $M = N = 3$

Case III: When $m = 2$, the considered Porous-Fisher equation becomes a highly nonlinear reaction-convection-diffusion model of the form

$$c_t = (c^2 c_x)_x + \lambda c(1-c),$$

or $$c_t = c^2 c_{xx} + 2cc_x^2 + \lambda c(1-c).$$

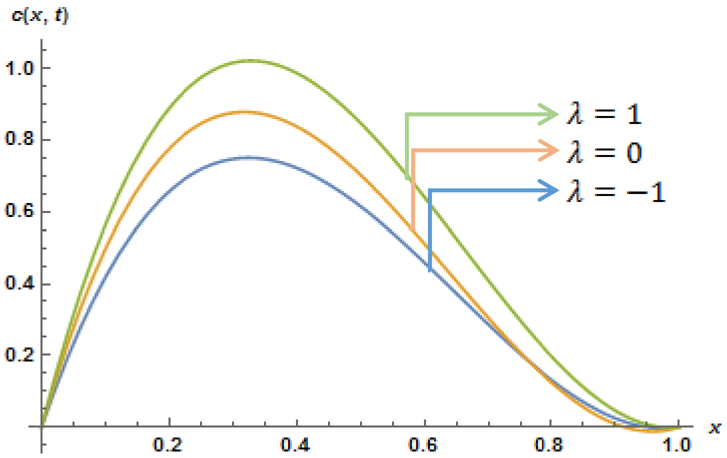

Fig. 6.3 Plots of the approximate solution vs. x when $\lambda = -1,0,1$ at $t = 1$hr for $M = N = 3$

6.4 Numerical Results and Discussion

Numerical values of the field variable $u(x,t)$ for various values of x at time $t=1$hr are calculated. During numerical computation, the parameter values of Jacobi polynomials are taken as $\alpha=\beta=0$ and also the values of M,N are taken as $M=N=3$. It is observed from Fig. 6.1 that for Case I, the sub-diffusions are observed and the overshoots decrease as the value of λ is decremented by 1. The fact is that due to the presence of sink term $(\lambda=-1)$, the height of the overshoot decreases as compared to the case of source term $(\lambda=1)$. The similar natures are found for the Case II and Case III which are depicted through Figures 6.2 and 6.3. It is seen from the figures that if the power of nonlinearity increases in the diffusion term the probability density function increases with the increase in x.

6.5 Conclusions

In the present scientific contribution, a method is proposed to solve NPDEs encountered in porous media. The considered NPDEs are converted into a system of nonlinear algebraic equations using shifted Jacobi polynomials together with shifted Jacobi operational matrix. The Newton iterative method is used in the solution of nonlinear algebraic equations. The author firmly believes that the present demonstration of simple, efficient and reliable method towards the solutions of NPDEs with variable coefficients will be appreciated by the researchers working in the area of modeling of NPDEs.