Chapter 5

Numerical Solution of Linear/Non-Linear Space Fractional Order Differential Equations Using Jacobi Polynomials

5.1 Introduction

The importance of fractional order partial differential equations (FPDEs) to describe the physical phenomena in various fields of science and engineering have already been discussed in Chapter 1. Lots of researchers have extended the classical methods in studying the initial and boundary value differential and integral equations of integer order to fractional order problems. Many researchers have constructed the operational matrix of integer as well as fractional order derivative and integration, which are used in many numerical methods such as tau method (Bhrawy (2015); Saadatmandi and Dehgan (2011); Bhrawy et al. (2016)), collocation method (Doha et al. (2011,2012)). The operational matrix for different orthogonal polynomials can be found in literature survey (Bhrawy (2015, 2016); Saadatmandi and Dehgan (2010, 2011, 2012); Bhrawy et al. (2016); Doha et al. (2011, 2012)).

To obtain the numerical solutions of FPDEs, a number of numerical methods are available in the literature. A spectral tau algorithm based on Jacobi operational matrix is used by Bhrawy et al. (2015) to solve the time fractional diffusion-wave equations. Saadatmandi and Dehgan (2011) used the shifted Legendre-tau approach to solve the space fraction diffusion equation subject to initial and boundary conditions with variable coefficients on a finite domain. A space-time Legendre spectral tau method is

---

The contents of this chapter are under review in **JAT – Journal of Approximation Theory, Elsevier.**

used by Bhrawy et al. (2016) to solve the two-sided space-time Caputo fractional diffusion-wave equations which are used in modeling practical phenomena of diffusion and wave in fluid flow, oil strata and others. Doha et al. (2011) have solved the linear and nonlinear multi-term FDEs using the shifted Chebyshev tau and collocation methods. In another article, Doha et al. (2012) derived the shifted Jacobi operational matrix of the fractional derivative to apply the spectral tau method to solve the numerical solution of general linear multi-term FDEs. For that, the authors have derived the shifted Chebyshev operational matrix of fractional order and used those in spectral methods for solving FDEs. Recently a Jacobi collocation method is derived by Bhrawy (2016) to solve the multi-dimensional non-linear fractional sub-diffusion problems. In 2015, Parvizi et al. (2015) used collocation method to solve the fractional order advection-diffusion equation with a nonlinear source term in which the space derivatives are replaced by the Riemann-Liouville derivatives, where the stability and convergence of the considered method were exhibited. Ren and Wang (2017) discussed a fourth-order extrapolated compact difference method for time-fractional convection-reaction-diffusion equations with spatially variable coefficients. In the year 2017, Wei (2017) presented and analyzed a new finite difference/local discontinuous Galerkin method for the fractional diffusion-wave equation. Nagy (2017) have proposed a new numerical scheme namely Sinc-Chebyshev collocation method to solve the time-fractional nonlinear Klein-Gorden equation in which fractional derivative is discussed in Caputo sense. Heydari et al. (2017) have proposed a new method based on the Legendre wavelets expansion together with operational matrices of fractional integration and derivatives of the basis functions to solve FPDEs with Dirichlet boundary conditions. Zheng et al. (2010) solved the symmetric FPDEs using Galerkin finite element approximation. A power penalty method for a 2D fractional partial differential linear complementarity problem has been solved by Chen and Wang (2017). Haar wavelet operational method is proposed by Wang et al. (2014) to solve FPDEs. During numerical solutions of FPDEs with variable coefficients using tau method, Chen et al. (2014) constructed generalized fractional-order Legendre functions and their operational matrices. But most of the methods discussed in literature generally talk about the solution of a particular type of problem or with a particular type of boundary conditions.

In this Chapter, to solve the linear/non-linear space fractional order partial differential equations subject to initial and any type of boundary conditions viz., Dirichlet, Neumann, and Robin type, the numerical algorithm used in last Chapter 4 is extended. For that, firstly shifted Jacobi polynomials as a basis function are used to approximate the solution of the considered problems together with spectral collocation method in which shifted fractional operational matrix in space, as well as temporal discretization, are used for derivatives. It converts the considered problem into the system of algebraic equations which can be solved easily. The exponential convergence rate of the proposed method is investigated through the illustrative examples for both spatial and temporal discretization. To show the efficiency and accuracy of the proposed method, it has applied to a number of physical problems, and a comparison of the approximate solution with the existing analytical solution present in literature are shown through the graphical presentations as well as tables.

5.2 Preliminaries

5.2.1 Shifted Jacobi Polynomials

The Jacobi polynomial can be defined for any arbitrary finite interval $[a,b]$ by using the definition given in Section 1.20 of Chapter 1. Let the shifted Jacobi polynomials $P_i^{(\alpha,\beta)}\left(\frac{2x-(a+b)}{(b-a)}\right)$ be denoted by $P_{b-a,i}^{(\alpha,\beta)}(x)$. The shifted Jacobi polynomials are also constituting a class of orthogonal polynomials as Jacobi polynomials with respect to the weight function $w_{b-a}^{(\alpha,\beta)}(x)=(b-x)^{\alpha}(x-a)^{\beta}$ on the interval $[a,b]$ given as

$$\int_a^b P_{b-a,i}^{(\alpha,\beta)}(x)\,P_{b-a,j}^{(\alpha,\beta)}(x)\,w_{b-a}^{(\alpha,\beta)}(x)\,dx=\delta_{ij}\chi_{b-a,j}^{(\alpha,\beta)}, \tag{5.1}$$

where δ_{ij} is the Kronecker function and

$$\chi_{b-a,j}^{(\alpha,\beta)}=\frac{(b-a)^{\alpha+\beta+1}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{(2j+\alpha+\beta+1)\Gamma(j+1)\Gamma(j+\alpha+\beta+1)}. \tag{5.2}$$

Since in most of the problems interval of interest is $[0,L]$, so changing the variable $z=\frac{2x}{L}-1$, it leads to the so-called shifted Jacobi polynomials denoted by $P_{L,i}^{(\alpha,\beta)}(x)$. It may also be obtained from the recurrence relation

$$P_{L,i+1}^{(\alpha,\beta)}(x)=(d_i x-e_i)P_{L,i}^{(\alpha,\beta)}(x)-c_i P_{L,i-1}^{(\alpha,\beta)}(x), \qquad i\geq 1, \tag{5.3}$$

where $d_i=\frac{(2i+\alpha+\beta+1)(2i+\alpha+\beta+2)}{L(i+1)(i+\alpha+\beta+1)}$,

$$e_i=\frac{(2i+\alpha+\beta+1)(2i^2+(1+\beta)(\alpha+\beta)+2i(\alpha+\beta+1))}{(i+1)(i+\alpha+\beta+1)(2i+\alpha+\beta)}.$$

This shifted Jacobi polynomials satisfy the following orthogonality condition with respect to the weight function $w_L^{(\alpha,\beta)}(x)=x^{\beta}(L-x)^{\alpha}$ on the interval $[0,L]$ as

$$\int_0^L P_{L,i}^{(\alpha,\beta)}(x)\,P_{L,j}^{(\alpha,\beta)}(x)\,w_L^{(\alpha,\beta)}(x)\,dx=\delta_{ij}\chi_{L,j}^{(\alpha,\beta)}, \tag{5.4}$$

where δ_{ij} is the Kronecker function and

$$\chi_{L,j}^{(\alpha,\beta)}=\frac{L^{\alpha+\beta+1}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{(2j+\alpha+\beta+1)\Gamma(j+1)\Gamma(j+\alpha+\beta+1)}. \tag{5.5}$$

The explicit analytic form of shifted Jacobi polynomials $P_{L,i}^{(\alpha,\beta)}(x)$ of degree i on the interval $[0,L]$ is given as

$$P_{L,i}^{(\alpha,\beta)}(x) = \sum_{p=0}^{i} (-1)^{i-p} \frac{\Gamma(i+\beta+1)\Gamma(i+p+\alpha+\beta+1)}{\Gamma(p+\beta+1)\Gamma(i+\alpha+\beta+1)\Gamma(i-p+1)\,p!\,L^p} x^p, \tag{5.6}$$

From this we get

$$P_{L,i}^{(\alpha,\beta)}(0) = (-1)^i \frac{\Gamma(i+\beta+1)}{\Gamma(\beta+1)\,i!},\ D^n P_{L,i}^{(\alpha,\beta)}(0) = (-1)^{i-n} \frac{\Gamma(i+\beta+1)\,(i+\alpha+\beta+1)_n}{L^n \Gamma(i-n+1)\,\Gamma(n+\beta+1)},\ i \ge n \in N, \tag{5.7}$$

$$P_{L,i}^{(\alpha,\beta)}(L) = \frac{\Gamma(i+\alpha+1)}{\Gamma(\alpha+1)\,i!},\ D^n P_{L,i}^{(\alpha,\beta)}(L) = \frac{\Gamma(i+\alpha+1)\,(i+\alpha+\beta+1)_n}{L^n \Gamma(i-n+1)\,\Gamma(n+\alpha+1)},\quad i \ge n \in N. \tag{5.8}$$

5.2.2 Function Approximation

Let $c(x)$ be a square integrable function with respect to the Jacobi weight function $w_L^{(\alpha,\beta)}(x)$ in the interval, i.e., $c(x) \in L^2_{w_L^{(\alpha,\beta)}}(0, L),$ then it can be expressed in terms of shifted Jacobi polynomials as

$$c(x) = \sum_{i=0}^{\infty} a_i P_{L,i}^{(\alpha,\beta)}(x), \tag{5.9}$$

where a_i's are the coefficients given by

$$a_i = \frac{1}{\chi_{L,i}^{(\alpha,\beta)}} \int_0^L w_L^{(\alpha,\beta)}(x)\, c(x)\, P_{L,i}^{(\alpha,\beta)}(x)\, dx,\ j = 0,1,\ldots; \tag{5.10}$$

As usual, the first $(M+1)$-terms of the shifted Jacobi polynomials are taken during approximation. Thus we have

$$c_M(x) \simeq \sum_{i=0}^{M} a_i\, P_{L,i}^{(\alpha,\beta)}(x), \tag{5.11}$$

which can be written in matrix form as

$$c_M(x) \simeq A^T \Omega_{L,M}(x), \tag{5.12}$$

with

$$A^T \equiv [a_0, a_1, \ldots, a_M],\ \Omega_{L,M}(x) \equiv [P_{L,0}^{(\alpha,\beta)}(x), P_{L,1}^{(\alpha,\beta)}(x), \ldots, P_{L,M}^{(\alpha,\beta)}(x)]^T. \tag{5.13}$$

5.2.3 The Jacobi Operational Matrix of Standard Order Derivative

The first order derivative of the column vector $\Omega_{L,M}(x)$ is given as

$$\frac{d}{dx}\Omega_{L,M}(x) = D\Omega_{L,M}(x), \tag{5.14}$$

where D is the $(M+1)\times(M+1)$ order operational matrix of derivative given by

$$D = (d_{pq}) = \begin{cases} \zeta(p,q), & p > q, \\ 0 & otherwise, \end{cases}$$

with

$$\zeta(p,q) = \frac{L^{\alpha+\beta}(p+\alpha+\beta+1)(p+\alpha+\beta+2)_q (q+\alpha+2)_{p-q-1}\Gamma(q+\alpha+\beta+1)}{\Gamma(p-q)\Gamma(2q+\alpha+\beta+1)} \times {}_3F_2\left(\begin{matrix} -p+1+q, & p+q+\alpha+\beta+2, & q+\alpha+1 \\ q+\alpha+2, & 2q+\alpha+\beta+2 & \end{matrix} ;1\right). \tag{5.15}$$

For even M, we have

$$\begin{pmatrix} 0 & 0 & 0 & . & . & . & 0 & 0 \\ \zeta(1,0) & 0 & 0 & . & . & . & 0 & 0 \\ \zeta(2,0) & \zeta(2,1) & 0 & . & . & . & 0 & 0 \\ \zeta(3,0) & \zeta(3,1) & \zeta(3,2) & . & . & . & 0 & 0 \\ . & . & . & .. & . & . & . & . \\ . & . & . & .. & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \zeta(M,0) & \zeta(M,1) & \zeta(M,2) & . & . & . & \zeta(M,M-1) & 0 \end{pmatrix}.$$

The higher order derivative of the column vector $\Omega_{L,M}(x)$ is given by

$$\frac{d^m}{dx^m}\Omega_{L,M}(x) = D^m\,\Omega_{L,M}(x), \tag{5.16}$$

where m is a natural number and D^m denotes the mth-order derivative of $\Omega_{L,M}(x)$.

5.2.4 The Jacobi Operational Matrix of Fractional Order Derivative

The Caputo fractional derivative of the order μ of the column vector $\Omega_{L,M}(x)$ is given by

$$D^{\mu}\Omega_{L,M}(x) \simeq {}_{L}D_{\mu}\Omega_{L,M}(x), \tag{5.17}$$

where ${}_{L}D_{\mu}$ denotes the $(M+1)\times(M+1)$ order Jacobi operational matrix of the fractional derivative of order μ in the interval $[0,L]$ and it is given as

$$ {}_{L}D_{\mu} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \Theta_{\mu}(\lceil \mu \rceil, 0) & \Theta_{\mu}(\lceil \mu \rceil, 1) & \cdots & \Theta_{\mu}(\lceil \mu \rceil, M) \\ \vdots & \vdots & \cdots & \vdots \\ \Theta_{\mu}(p,0) & \Theta_{\mu}(p,1) & \cdots & \Theta_{\mu}(p,M) \\ \vdots & \vdots & \cdots & \vdots \\ \Theta_{\mu}(M,0) & \Theta_{\mu}(M,1) & \cdots & \Theta_{\mu}(M,M) \end{pmatrix}, \tag{5.18}$$

where

$$\Theta_{\mu}(p,q) = \sum_{r=\lceil \mu \rceil}^{p} \frac{(-1)^{-r+p} L^{-\mu+1+\alpha+\beta}\Gamma(q+1+\beta)\Gamma(p+1+\beta)\Gamma(r+p+1+\alpha+\beta)}{\chi_{L,q}^{(\alpha,\beta)}(p-r)!\Gamma(q+1+\alpha+\beta)\Gamma(r+1+\beta)\Gamma(p+1+\alpha+\beta)\Gamma(r+1-\mu)} \times \sum_{s=0}^{q} \frac{(-1)^{q-s}\Gamma(q+s+1+\alpha+\beta)\Gamma(1+\alpha)\Gamma(r+s+1+\beta-\mu)}{s!(q-s)!\Gamma(s+1+\beta)\Gamma(s+r-\mu+2+\alpha+\beta)}. \tag{5.19}$$

It is to be noted that in ${}_{L}D_{\mu}$, the first $\lceil \mu \rceil$ rows are all zeros.

5.3 Error Analysis

Theorem: Let $c(x,t)$ be the smooth function in $\Lambda \equiv [0,L]\times[0,\tau]$ and consider

$$\prod_{M,N}^{\alpha,\beta} = span\left\{P_{\tau,p}^{(\alpha,\beta)}(t)P_{L,q}^{(\alpha,\beta)}(x),\, p=0,1,\ldots,M,\, q=0,1,\ldots,N\right\}.$$

Suppose $c_{M,N}(x,t) \in \prod_{M,N}^{\alpha,\beta}$ is the best approximation of $c(x,t)$ and $\max_{(x,t)\in\Lambda}\left|\frac{\partial^{M+1}c(x,t)}{\partial t^{M+1}}\right| \le k_1$, $\max_{(x,t)\in\Lambda}\left|\frac{\partial^{N+1}c(x,t)}{\partial x^{N+1}}\right| \le k_2$, $\max_{(x,t)\in\Lambda}\left|\frac{\partial^{M+N+2}c(x,t)}{\partial t^{M+1}\partial x^{N+1}}\right| \le k_3$ with k_1, k_2, and k_3 are constants then

$$\left\|c(x,t) - c_{M,N}(x,t)\right\|_{\infty} \le k_1 \frac{(\tau/2)^{M+1}\Gamma(M+\alpha+2)}{\wp_M^{(\alpha,\beta)}\{(M+1)!\}^2\Gamma(\alpha+1)} + k_2 \frac{(L/2)^{N+1}\Gamma(N+\alpha+2)}{\wp_N^{(\alpha,\beta)}\{(N+1)!\}^2\Gamma(\alpha+1)}$$
$$+ k_3 \frac{(\tau/2)^{M+1}(L/2)^{N+1}\Gamma(M+\alpha+2)\Gamma(N+\alpha+2)}{\wp_M^{(\alpha,\beta)}\wp_N^{(\alpha,\beta)}\{(M+1)!\}^2\{(N+1)!\}^2(\Gamma(\alpha+1))^2},$$

where $\wp_M^{(\alpha,\beta)} = \frac{\Gamma(2M+\alpha+\beta+1)}{2^M M!\Gamma(M+\alpha+\beta+1)}$ is the leading coefficient of $P_{M+1}^{(\alpha,\beta)}(z)$.

Proof: See the article of Bhrawy and Zaky (2015).

5.4 Application of the Spectral Method Based on Jacobi Operational Matrix for FDEs

In this section, in order to show the importance of Jacobi operational matrix of fractional derivatives, the spectral collocation method is applied to solve the space fractional order linear/nonlinear reaction-advection-diffusion equations with variable coefficients of the form

$$\frac{\partial c(x,t)}{\partial t} = d(x,t)\frac{\partial^{\gamma} c(x,t)}{\partial x^{\gamma}} - v(x,t)\frac{\partial^{\eta} c(x,t)}{\partial x^{\eta}} - \xi(c(x,t),x,t),\ 0 < x < L,\ 0 < t \le \tau, \tag{5.20}$$

with any one of the boundary conditions:

(a) Dirichlet boundary conditions

$$c(0,t) = f(t), \qquad 0 < t \le \tau, \tag{5.21}$$

$$c(L,t) = g(t), \qquad 0 < t \le \tau, \tag{5.22}$$

(b) Neumann boundary conditions

$$\frac{\partial^{\eta} c(0,t)}{\partial x^{\eta}} = f(t), \qquad 0 < t \leq \tau, \tag{5.23}$$

$$\frac{\partial^{\eta} c(L,t)}{\partial x^{\eta}} = g(t), \qquad 0 < t \leq \tau, \tag{5.24}$$

(c) Mixed boundary conditions

$$c(0,t) = f(t), \qquad 0 < t \leq \tau, \tag{5.25}$$

$$\frac{\partial^{\eta} c(L,t)}{\partial x^{\eta}} = g(t), \qquad 0 < t \leq \tau, \tag{5.26}$$

along with initial condition

$$c(x,0) = h(x), \qquad 0 < x \leq L, \tag{5.27}$$

where $1 < \gamma \leq 2,\ 0 < \eta \leq 1,\ d$ is the diffusivity, v is the average fluid velocity, $\xi(c(x,t),x,t)$ is the linear/nonlinear reaction term. Here d, v may be constants which simplify the considered problems.

Equation (5.20) with the use of initial condition (5.27) can be rewritten in the form

$$\frac{\partial c(x,t)}{\partial t} = d(x,t)\frac{\partial^{\gamma} c(x,t)}{\partial x^{\gamma}} - v(x,t)\frac{\partial^{\eta} c(x,t)}{\partial x^{\eta}} + c(x,0) - h(x) - \xi(c(x,t),x,t), \tag{5.28}$$

which has to be solved along with the given boundary conditions.

To solve the problem (5.28) with the given boundary conditions, let us approximate the solution $c(x,t)$ by $(M+1)\times(N+1)$ terms of shifted Jacobi polynomials series as

$$c_{M,N}(x,t) = \sum_{p=0}^{M}\sum_{q=0}^{N} c_{pq}\, P_{\tau,p}^{(\alpha,\beta)}(t)\, P_{L,q}^{(\alpha,\beta)}(x) = \Omega_{\tau,M}^{T}(t)\, C\, \Omega_{L,N}(x), \tag{5.29}$$

where C is the $(M+1)\times(N+1)$ order matrix of the coefficients c_{pq}, given as

$$C=\begin{pmatrix} c_{00} & c_{01} & . & . & . & c_{0N} \\ c_{10} & c_{11} & . & . & . & c_{1N} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ c_{M0} & c_{M1} & . & . & . & c_{MN} \end{pmatrix}. \tag{5.30}$$

The entries of the matrix C are obtained from

$$c_{pq}=\frac{1}{\chi_{\tau,p}^{(\alpha,\beta)}\,\chi_{L,q}^{(\alpha,\beta)}}\int_0^\tau\int_0^L c(x,t)\,P_{\tau,p}^{(\alpha,\beta)}(t)\,P_{L,q}^{(\alpha,\beta)}(x)\,w_\tau^{(\alpha,\beta)}(t)\,w_L^{(\alpha,\beta)}(x)\,dx\,dt,$$
$$p=0,1,\ldots,M,\ q=0,1,\ldots,N. \tag{5.31}$$

The standard, as well as fractional order derivative of the approximate solution, can be expressed as

$$\frac{\partial}{\partial t}c(x,t)\simeq\Omega_{\tau,M}^T(t)\,{}_\tau D_1^T C\,\Omega_{L,N}(x), \tag{5.32}$$

$$\frac{\partial^\gamma}{\partial x^\gamma}c(x,t)\simeq\Omega_{\tau,M}^T(t)\,C\,{}_L D_\gamma\,\Omega_{L,N}(x), \tag{5.33}$$

$$\frac{\partial^\eta}{\partial x^\eta}c(x,t)\simeq\Omega_{\tau,M}^T(t)\,C\,{}_L D_\eta\,\Omega_{L,N}(x), \tag{5.34}$$

$$c(x,0)\simeq\Omega_{\tau,M}^T(0)\,C\,\Omega_{L,N}(x). \tag{5.35}$$

Employing equations (5.32)-(5.35) in equation (5.28), we get

$$\begin{aligned}&\Omega_{\tau,M}^T(t)({}_\tau D_1^T C-d(x,t)C\,{}_L D_\gamma+v(x,t)C\,{}_L D_\eta)\Omega_{L,N}(x)\\&\quad=\Omega_{\tau,M}^T(0)\,C\,\Omega_{L,N}(x)-h(x)+\xi(\Omega_{\tau,M}^T(t)\,C\,\Omega_{L,N}(x),x,t),\end{aligned} \tag{5.40}$$

with one of the boundary conditions:

(a) Dirichlet boundary conditions

$$\begin{aligned}\Omega_{\tau,M}^T(t)\,C\,\Omega_{L,N}(0)&=f(t),\\ \Omega_{\tau,M}^T(t)\,C\,\Omega_{L,N}(L)&=g(t).\end{aligned} \tag{5.41}$$

(b) Neumann boundary conditions

$$\begin{aligned}\Omega^T_{\tau,M}(t)C_L D_\eta \,\Omega_{L,N}(0), &= f(t),\\ \Omega^T_{\tau,M}(t)C_L D_\eta \,\Omega_{L,N}(L), &= g(t).\end{aligned} \tag{5.42}$$

(c) Mixed boundary conditions

$$\begin{aligned}\Omega^T_{\tau,M}(t)C\,\Omega_{L,N}(0) &= f(t),\\ \Omega^T_{\tau,M}(t)C_L D_\eta \,\Omega_{L,N}(L), &= g(t).\end{aligned} \tag{5.43}$$

According to the spectral Jacobi collocation method, equation (5.40) is to be satisfied at $(M+1)\times(N-1)$ collocation points, and the considered boundary conditions are satisfied at $2(M+1)$ collocation points as

$$\begin{aligned}&\Omega^T_{\tau,M}(t_p)({}_\tau D_1^T C - d(x_q,t_p)C_L D_{2\mu} + v(x_q,t_p)C_L D_\mu)\Omega_{L,N}(x_q)\\ &\qquad = \Omega^T_{\tau,M}(0)C\,\Omega_{L,N}(x_q) - h(x_q) + \xi(\Omega^T_{\tau,M}(t_p)C\,\Omega_{L,N}(x_q), x_q, t_p),\end{aligned} \tag{5.44}$$

with one of the boundary conditions:

(a) Dirichlet boundary conditions

$$\begin{aligned}\Omega^T_{\tau,M}(t_p)C\,\Omega_{L,N}(0) &= f(t_p),\\ \Omega^T_{\tau,M}(t_p)C\,\Omega_{L,N}(L) &= g(t_p).\end{aligned} \tag{5.45}$$

(b) Neumann boundary conditions

$$\begin{aligned}\Omega^T_{\tau,M}(t_p)C_L D_\mu \,\Omega_{L,N}(0), &= f(t_p),\\ \Omega^T_{\tau,M}(t_p)C_L D_\mu \,\Omega_{L,N}(L), &= g(t_p).\end{aligned} \tag{5.46}$$

(c) Mixed boundary conditions

$$\begin{aligned}\Omega^T_{\tau,M}(t_p)C\,\Omega_{L,N}(0) &= f(t_p),\\ \Omega^T_{\tau,M}(t_p)C_L D_\mu \,\Omega_{L,N}(L), &= g(t_p),\end{aligned} \tag{5.47}$$

where t_p, $p = 0,1,\ldots,M$ are the roots of $P^{(\alpha,\beta)}_{\tau,M+1}(t)$ and x_q, $q = 0,1,\ldots,N-2$ are the Jacobi Gauss-Lobatto points.

From here, the system of $(M+1)\times(N+1)$ linear/non-linear algebraic equations in c_{pq} are obtained, in which $(M+1)\times(N-1)$ equations come from equation (5.44) and $2(M+1)$ equations come from one of the equation (5.45) or (5.46) or (5.47). Now if the system of $(M+1)\times(N+1)$ equations is a linear system of algebraic equations, then it can be solved easily but if the system of $(M+1)\times(N+1)$ equations is a non-linear system of algebraic equations, can be solved using Newton's iterative method for c_{pq}. Consequently $c_{M,N}(x,t)$ given in equation (5.29) can be calculated.

5.5 Illustrative Examples

To illustrate the effectiveness and accuracy of the proposed method, some examples are carried out in this section. In general, to solve the problem, we construct the proposed algorithm with general Jacobi parameters (α and β) which takes the particular values to achieve the approximate solution. During numerical computations, the values of Jacobi parameters are taken as, and $M=N=3.$ Comparison of the results obtained by the proposed method with the existing exact solution shows that the present method is very effective, convenient and reliable.

Example 1. Consider the following space fractional reaction-advection-diffusion problem

$$\frac{\partial c}{\partial t}=\Gamma(0.2)x^{0.8}\frac{\partial^{1.8}c}{\partial x^{1.8}}-4.5\frac{\partial c}{\partial x}+5e^{-t}(x^2+3.5), \qquad 0<x<1,\ 0<t\le\tau,$$

with the boundary conditions

$$c(0,t)=c(1,t)=0, \qquad 0<t\le\tau,$$

and initial condition

$$c(x,0)=5x(1-x), \qquad 0<x<1.$$

The exact solution of this problem is $c(x,t)=5e^{-t}x(1-x)$ (Parvizi et al. (2015)). In Table 5.1, the absolute error $ER(x,t)=\max_{\substack{0<x<1\\0\le t\le 1}}\left|c(x,t)-c_{M,N}(x,t)\right|$ for different values of M and N are presented at $t=1$hr. In Fig. 5.1, the approximate solution of this problem vs. x is plotted and compared with the existing analytical solution at $t=1$. The variations of absolute error vs. x at $t=1$ are shown through Fig. 5.2. All the plots are drawn for $M=N=3$.

Table 5.1 The absolute error $ER(x,1)$ with various choices of M and N

x	$M=N=3$	$M=N=5$	$M=N=7$
0.1	9.02e-04	7.86e-06	1.72e-09
0.2	1.40e-03	1.20e-05	4.42e-08
0.3	1.59e-03	1.72e-05	2.89e-07
0.4	1.52e-03	2.32e-05	3.54e-07
0.5	1.28e-03	4.28e-04	6.24e-08
0.6	9.42e-04	8.52e-06	5.22e-07
0.7	5.68e-04	2.38e-05	4.65e-07
0.8	2.37e-04	4.27e-04	4.33e-09
0.9	2.39e-05	4.97e-07	3.49e-11

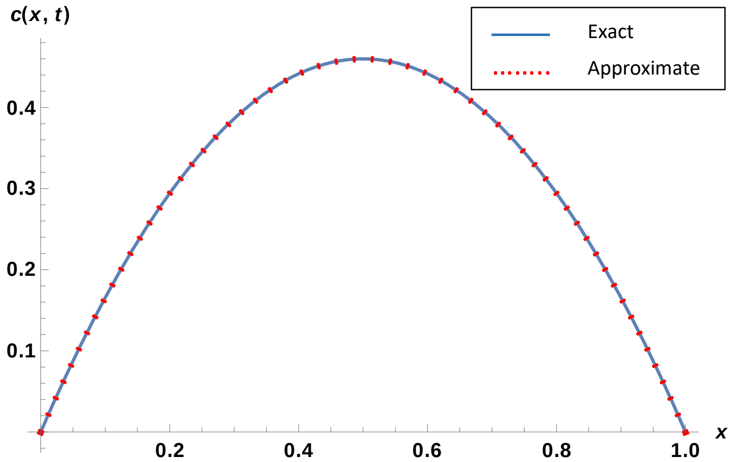

Fig. 5.1 Comparison between exact and approximate solutions of Example 1 vs. x for $M=N=3$ at $t=1$hr

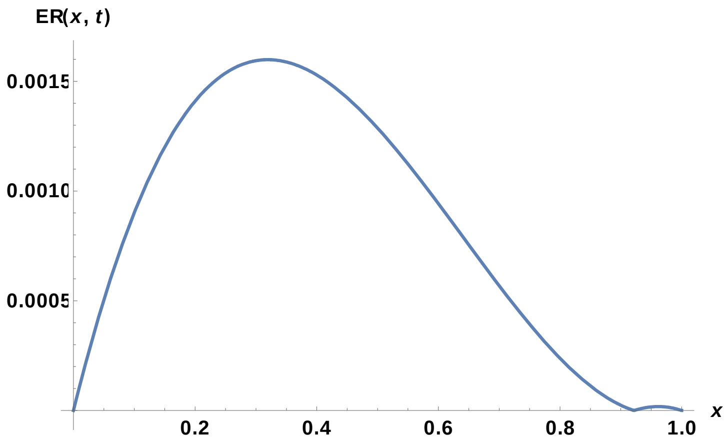

Fig. 5.2 Variation of absolute error of Example 1 vs. x for $M = N = 3$ at $t = 1$hr

Example 2. Consider the following space fractional advection-diffusion problem

$$\frac{\partial c(x,t)}{\partial t} = d\frac{\partial^{\gamma} c(x,t)}{\partial x^{\gamma}} - \frac{\partial^{\eta} c(x,t)}{\partial x^{\eta}}, \qquad 0 < x < 1,\ t > 0,$$

with the boundary conditions

$$\begin{aligned} c(0,t) &= e^{(d+1)t}, \\ c_x(1,t) &= -e^{-1+(d+1)t}, \end{aligned} \qquad t > 0,$$

and initial condition

$$c(x,0) = e^{-x}, \qquad 0 < x < 1.$$

The exact solution of this problem is $c(x,t) = e^{-x+(d+1)t}$ for $\gamma = 2$ and $\eta = 1$ (Bastani and Salkuyeh (2012)). During the solution of this problem the value of diffusivity constant is taken as $d = 1$. In Table 5.2, the absolute error already defined in Example 1 is presented for different values of M and N at $t = 1$hr. The approximate solution of this problem vs. x and t for $\gamma = 2$ and $\eta = 1$ is shown through Fig. 5.3. The comparison between the existing analytical solution and the approximate solution vs. x and t for $\gamma = 2$, $\eta = 1$ is displayed through Fig. 5.4. The absolute error vs. x and t for $\gamma = 2$, $\eta = 1$ is shown though Fig. 5.5. The approximate solution vs. x is displayed through Fig. 5.6 for different values of γ, η at $t = 1$.

Table 5.2 The absolute error $ER(x,1)$ with various choices of M and N

x	$M=N=3$	$M=N=5$	$M=N=7$
0.1	7.42e-05	6.48e-07	4.82e-10
0.2	6.29e-05	5.23e-07	4.76e-09
0.3	4.13e-05	4.38e-06	3.92e-09
0.4	1.44e-05	2.52e-06	2.84e-08
0.5	1.28e-05	2.78e-07	2.26e-08
0.6	3.52e-05	1.42e-06	3.84e-11
0.7	4.78e-05	1.98e-06	4.83e-10
0.8	4.53e-05	1.34e-06	5.57e-11
0.9	2.24e-05	1.32e-07	1.01e-12

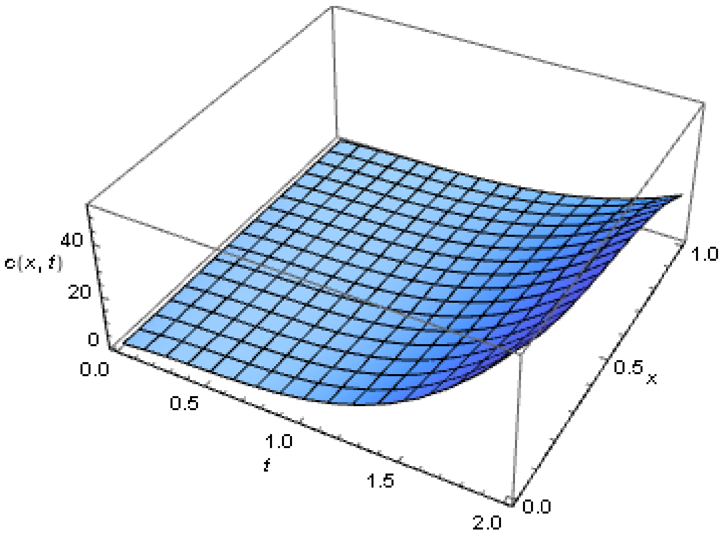

Fig. 5.3 Plot of the approximate solution of Example 2 vs. *x* and *t* for $M=N=3,\ \gamma=2$ and $\eta=1$

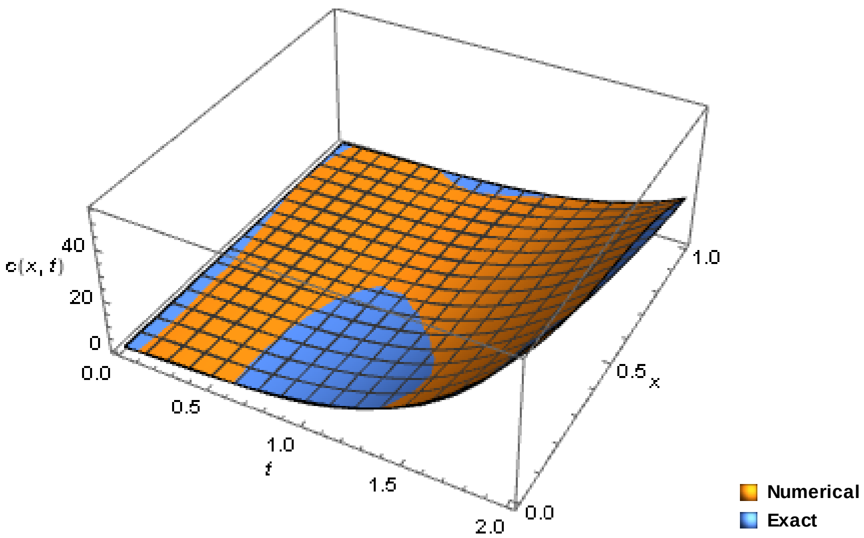

Fig. 5.4 Comparison between the exact and approximate solution of Example 2 vs. *x* and *t* for $M=N=3,\ \gamma=2$ and $\eta=1$

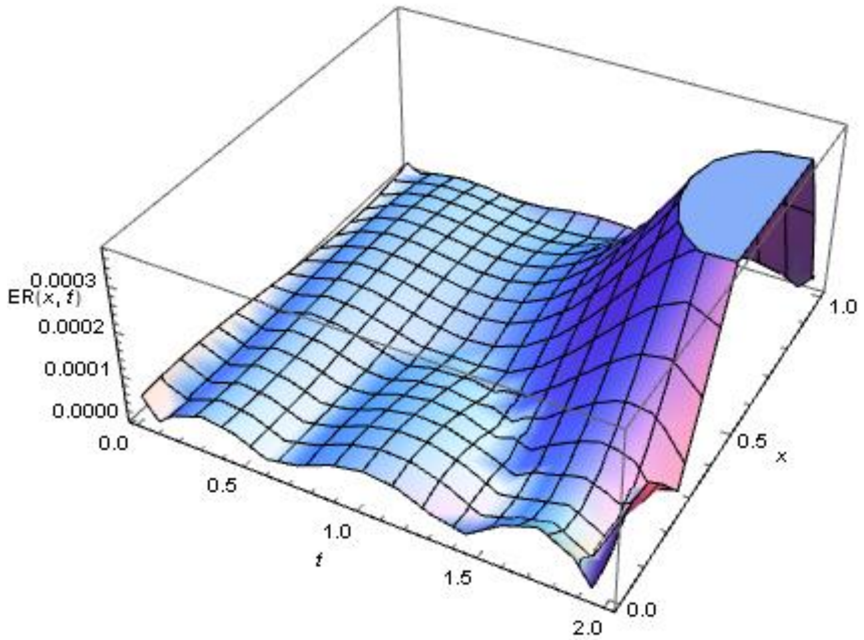

Fig. 5.5 Variation of absolute error of Example 2 vs. *x* and *t* for $M = N = 3, \gamma = 2$ and $\eta = 1$

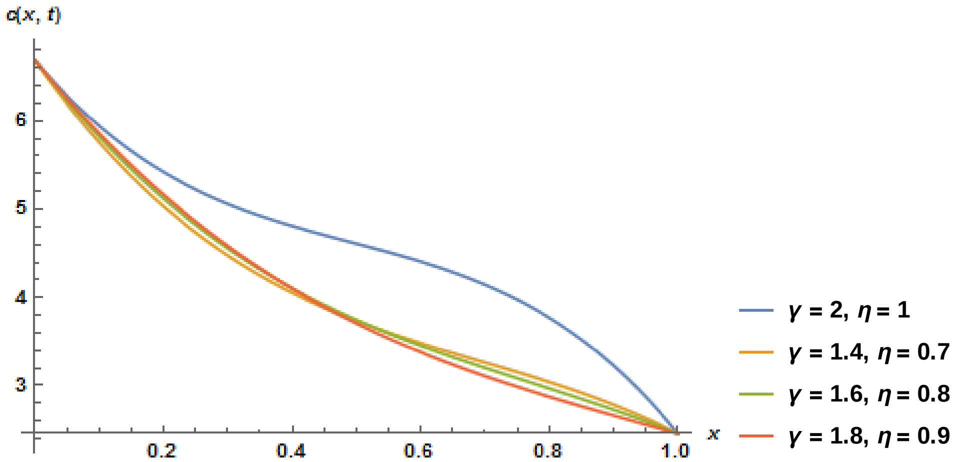

Fig. 5.6 Plots of the approximate solution of Example 2 vs. *x* for different value of γ, η and $M = N = 3$ at $t = 1$hr

Example 3. Consider the following problem

$$\frac{\partial c}{\partial t} = \Gamma(1.2)x^{1.8}\frac{\partial^{1.8} c}{\partial x^{1.8}} - c^3 + c^2 + 5e^{-t}(-x^9 + 3x^8 - 3x^7 + 2x^5 - x^4 + 6x^3 - 3x^2),$$
$$(x,t) \in \Omega = [0,1]\times[0,\tau],$$

with the boundary conditions

$$c(0,t) = c(1,t) = 0, \qquad 0 < t \le \tau,$$

and initial condition

$$c(x,0) = x^2 - x^3, \qquad 0 < x < 1.$$

The exact solution of this problem is $c(x,t) = 5e^{-t}(x^2 - x^3)$ (Parvizi et al. (2015)). The absolute error for above example is tabulated in Table 5.3 for different values of M and N at $t = 1$hr. A comparison of the approximate solution using the proposed method with existing exact solution vs. x and the variations of absolute error vs. x at $t = 1$hr are displayed through Figures 5.7 and 5.8, respectively.

Table 5.3 The absolute error $ER(x,1)$ with various choices of M and N

x	$M = N = 3$	$M = N = 5$	$M = N = 7$
0.1	2.96e-04	3.32e-05	3.52e-08
0.2	7.34e-04	2.40e-05	2.56e-08
0.3	1.23e-03	2.76e-06	4.38e-07
0.4	1.72e-03	2.22e-05	1.54e-09
0.5	2.11e-03	2.28e-04	1.86e-08
0.6	2.34e-03	3.52e-06	5.72e-09
0.7	2.32e-03	4.58e-05	6.52e-07
0.8	1.97e-03	1.24e-04	4.52e-09
0.9	1.22e-03	1.97e-07	7.54e-12

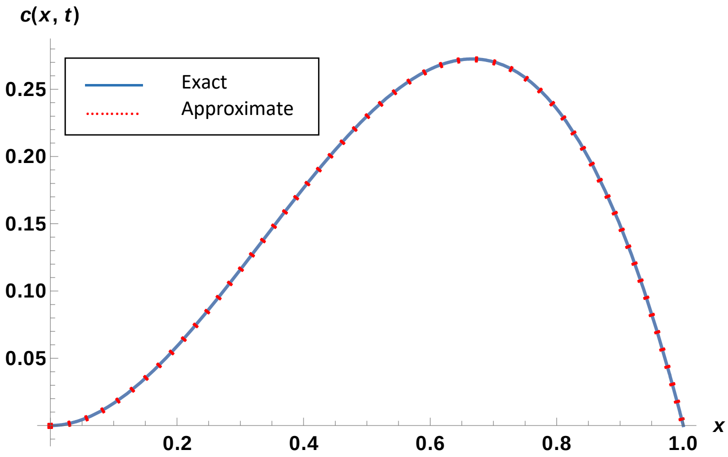

Fig. 5.7 Comparison between exact and approximate solutions of Example 3 vs. x for $M = N = 3$ at $t = 1$hr

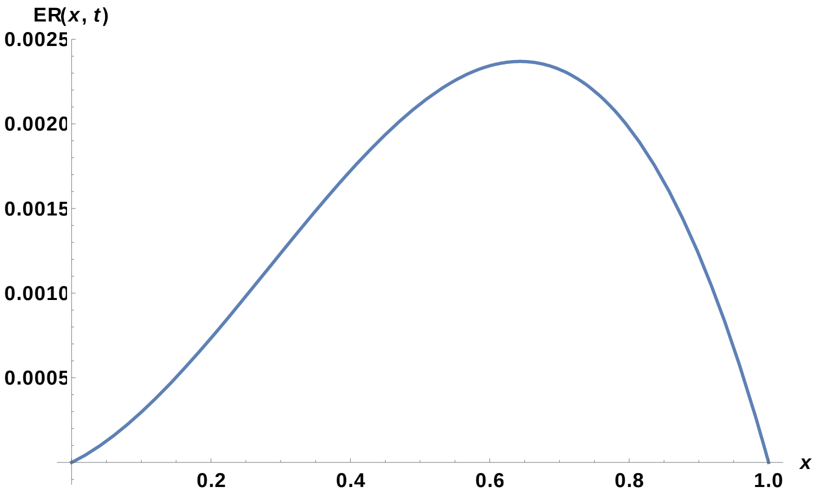

Fig. 5.8 Variation of absolute error of Example 3 vs. x for $M = N = 3$ at $t = 1hr$

Example 4. Consider the following space fractional Fisher's type problem

$$\frac{\partial c(x,t)}{\partial t} = \frac{\partial^{\gamma} c(x,t)}{\partial x^{\gamma}} + \lambda c(1-c), \quad 0 < x < L,\, 0 < t \le \tau,$$

where $1 < \gamma \le 2$, with the boundary conditions

$$c(0,t) = \frac{1}{\left(1 + e^{-5/6\lambda t}\right)^2}, \qquad 0 < t \le \tau,$$

$$c(1,t) = \frac{1}{\left(1 + e^{\sqrt{\lambda/6} - 5/6\lambda t}\right)^2}, \qquad 0 < t \le \tau,$$

and initial condition

$$c(x,0) = \frac{1}{\left(1+e^{\sqrt{\lambda/6}\,x}\right)^2}, \qquad 0 < x < L.$$

This problem has the exact solution $c(x,t) = \frac{1}{\left(1+e^{\sqrt{\lambda/6}\,x - 5/6\lambda t}\right)^2}$ for $\gamma = 2$ (Bastani and Salkuyeh (2012)). During the solution of this problem, $\lambda = 1$ is considered. The absolute error is shown in Table 5.4 for different values of M and N at $t = 1$hr. The approximate solution of this problem vs. *x* and *t* is depicted through Fig. 5.9 for $\gamma = 2$. In Fig. 5.10, the comparison between the existing analytical solution and the approximate solution vs. *x* and *t* for $\gamma = 2$ is shown. The variation of absolute error vs. *x* and *t* for $\gamma = 2$ is shown through Fig. 5.11 and the approximate solutions for different values of γ at $t = 1$hr are presented through Fig. 5.12.

Table 5.4 The absolute error $ER(x,1)$ with various choices of M and N

x	$M = N = 3$	$M = N = 5$	$M = N = 7$
0.1	1.24e-04	7.24-06	5.42e-09
0.2	1.64e-04	2.23-05	1.29e-08
0.3	1.43e-04	3.42e-05	6.54e-08
0.4	8.14e-05	7.22e-05	2.74e-09
0.5	7.27e-07	8.29e-06	1.21e-08
0.6	8.27e-05	4.32e-06	3.76e-08
0.7	1.44e-04	1.84-05	5.23-08
0.8	1.64e-04	1.02e-05	9.33e-10
0.9	1.24e-04	2.32e-06	5.75e-09

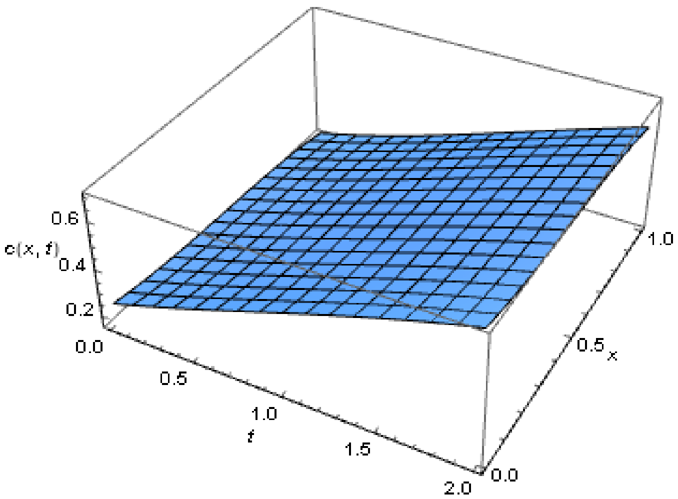

Fig. 5.9 Plot of the approximate solution of Example 4 vs. *x* and *t* for $M = N = 3,\ \gamma = 2$ and $\lambda = 1$

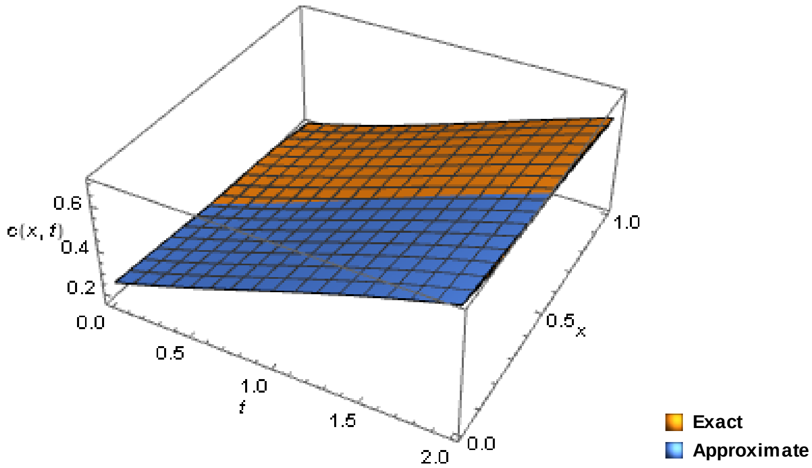

Fig. 5.10 Comparison between exact and approximate solutions of Example 4 vs. *x* and *t* for $M = N = 3,\ \gamma = 2$ and $\lambda = 1$

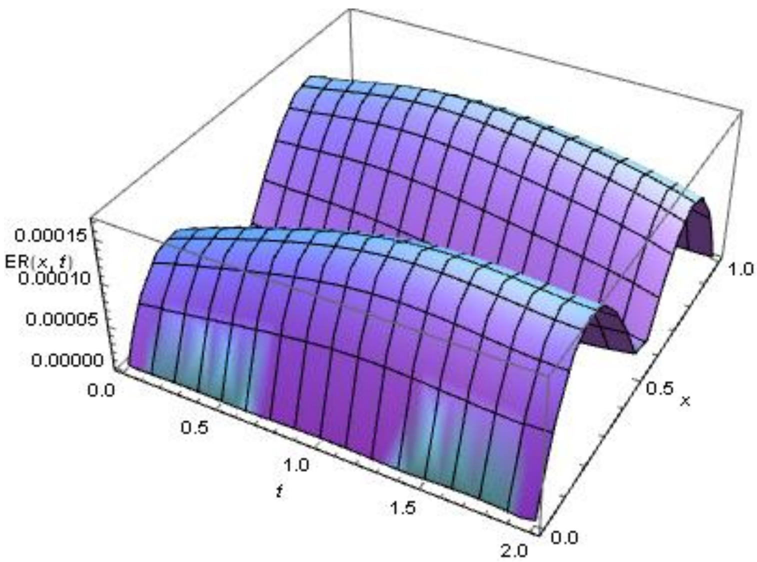

Fig. 5.11 Variation of absolute error of Example 4 vs. *x* and *t* for $M = N = 3,\ \gamma = 2$ and $\lambda = 1$

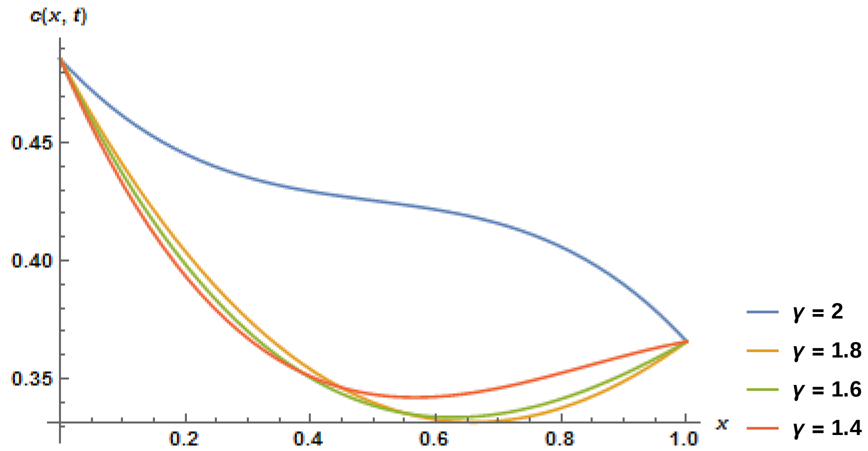

Fig. 5.12 Plots of the approximate solution of Example 4 vs. x for different value of γ and $M = N = 3$ at $t = 1$hr

Example 5. Consider the following nonlinear space fractional Burger-Fisher problem

$$\frac{\partial c(x,t)}{\partial t} = \frac{\partial^{\gamma} c(x,t)}{\partial x^{\gamma}} + c(x,t)\frac{\partial^{\eta} c(x,t)}{\partial x^{\eta}} + c(x,t)(1-c(x,t)), \quad 0 < x < L,\ 0 < t \le \tau,$$

with the boundary conditions

$$c(0,t) = \frac{1}{2} + \frac{1}{2}\tanh\left\{\frac{5}{8}t\right\}, \qquad 0 < t \le \tau,$$

$$c(1,t) = \frac{1}{2} + \frac{1}{2}\tanh\left\{\frac{1}{4}\left(1 + \frac{5}{2}t\right)\right\}, \qquad 0 < t \le \tau,$$

and initial condition

$$c(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left\{\frac{x}{4}\right\}, \qquad 0 < x < L.$$

The exact solution of this problem is $c(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left\{\frac{1}{4}\left(x + \frac{5}{2}t\right)\right\}$ for $\gamma = 2$ and $\eta = 1$ (Babolian and Saeidian (2009)). In Table 5.5, the absolute error is presented for different values of M and N at $t = 1$hr. The approximate solution of this problem and the comparison of that with the exact solution vs. x and t are shown through the Fig. 5.13 and Fig. 5.14, respectively for $\gamma = 2$, $\eta = 1$. The variations of absolute error vs. x

and t for $\gamma = 2$, $\eta = 1$ are presented through Fig. 5.15. The approximate solutions for different values of γ, η at $t = 1$hr are shown through Fig. 5.16.

Table 5.5 The absolute error $ER(x,1)$ with various choices of M and N

x	$M = N = 3$	$M = N = 5$	$M = N = 7$
0.1	1.43e-04	2.21e-05	1.11e-09
0.2	1.94e-04	2.34e-06	1.26e-08
0.3	8.85e-05	3.42e-06	6.15e-10
0.4	1.13e-04	8.12e-07	7.14e-09
0.5	3.54e-04	8.19e-06	8.21e-09
0.6	5.73e-04	4.26e-06	3.56e-11
0.7	7.12e-04	5.24e-05	3.53e-08
0.8	7.13e-04	6.22e-07	4.23e-10
0.9	5.15e-04	4.27e-07	6.45e-11

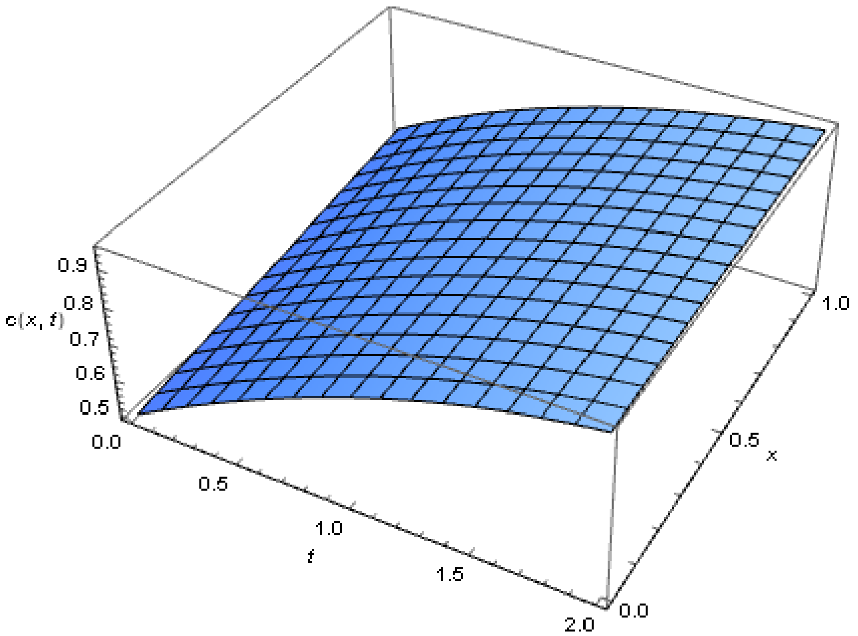

Fig. 5.13 Plot of the approximate solution of Example 5 vs. x and t for $M = N = 3$, $\gamma = 2$ and $\eta = 1$

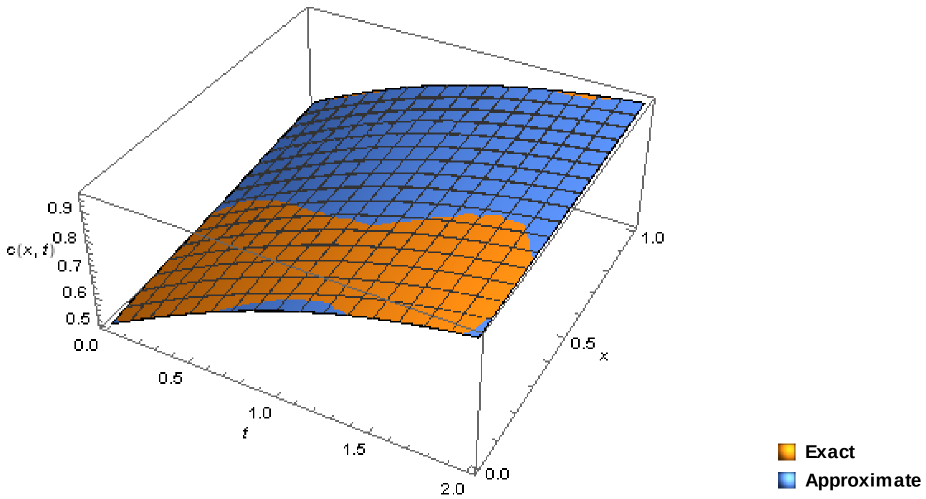

Fig. 5.14 Comparison between exact and approximate solutions of Example 5 vs. *x* and *t* for $M = N = 3,\ \gamma = 2$ and $\eta = 1$

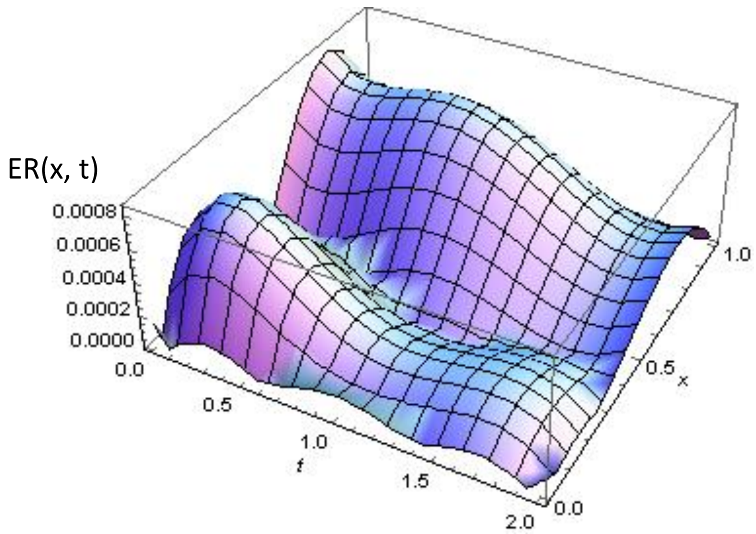

Fig. 5.15 Variation of absolute error of Example 5 vs. *x* and *t* for $M = N = 3,\ \gamma = 2$ and $\eta = 1$

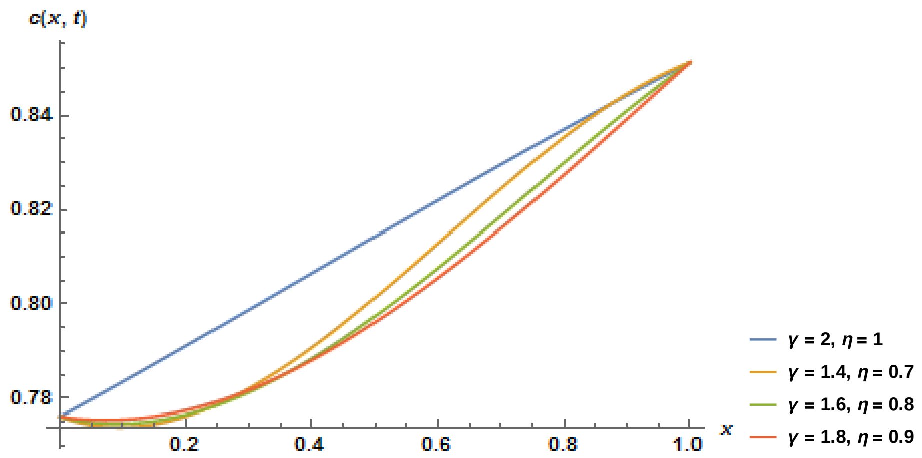

Fig. 5.16 Plots of the approximate solution of Example 5 vs. *x* for different value of γ, η and $M = N = 3$ at $t = 1$hr

5.6 Conclusions

In the present chapter, a method is proposed to solve a class of fractional partial differential equations viz., advection-diffusion equations with linear/non-linear reaction terms subject to initial and boundary conditions. The considered problems are converted into a system of algebraic equations using shifted Jacobi polynomials together with shifted Jacobi operational matrix. The Newton iterative method is used during the solution of nonlinear algebraic equations. The author is optimist that the present demonstration of simplicity, efficiency, and reliability of the proposed method towards the solutions of a number of linear/nonlinear FPDEs subject to initial and boundary conditions with constant or variable coefficients will be appreciated by the researchers working in the area of modeling of fractional order systems. To validate the efficiency of the proposed method a comparative study of each problem with the existing result is evaluated numerically through error analyses, which is displayed graphically.