Chapter 4

Numerical Solution of Non-Linear Partial Differential Equations Arising in Porous Media Using Operation Matrices

4.1 Introduction

To understand the physics of many complex physical problems in porous media, non-linear partial differential equations (NPDEs) plays a vital role, e.g., Burgers equation, Fisher equation, Huxley equation, Burgers-Fisher, and Burgers-Huxley. Burgers' equation introduced by a Dutch physicist J.H. Burgers to explain the nature of shock waves, traffic flow, and acoustic transmission. Later it is found that it is a fundamental NPDE appears in different fields of mathematics viz., fluid mechanics, non-linear acoustic, gas dynamics and traffic flow. It describes the discrepancy of vehicle density in highway traffic and the propagation of weak shock-waves in a fluid. Because of the non-linear convection term and the diffusion term with viscosity coefficient, Burgers equation roughly looks like Navier-Stokes equation. So it is considered as a simplified form of Navier-Stokes equation. It also arises when pure and contaminated water disperses in the longitudinal direction of the porous medium. This dispersion phenomenon may be miscible or immiscible fluid flow through the porous medium. When a fluid of lesser viscosity displaced a fluid flowing through a porous medium, then in place of usual displacement of the whole front protuberance occurred, which emit through the porous medium at a comparatively very high speed. This incident is

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known as instability phenomenon or fingering which yields to NPDE in Burgers equation form. In 1937, Fisher (1937) proposed a non-linear reaction-diffusion model to explain the spreading of a viral mutant in an infinitely long habitat. Later it is found that it has excellent applications in various fields like travelling wave behavior (Pablo and Sanchez (1998)) combustion (Aggarwal (1985)), autocatalytic chemical reactions (Aronson and Weinberger (1988)), gene propagation (Caonsa (1973)), tissue engineering (Maini et al. (2004)), and neurophysiology (Tuckwell (1988)). It also used as a mathematical model of reacting flow in the porous medium. Huxley equation is also a nonlinear reaction-diffusion model which also has important applications in various fields, e.g., biology, chemistry, fluid dynamics. The combined form of Burgers, Fisher and Huxley equations give nonlinear reaction-advection-diffusion models which are important NPDEs. In the fluid dynamic model, Burgers-Fisher equation has a sharp edge due to which many researchers study this model to understand the physical flows and calibrate the different numerical methods. It has various applications in number theory, gas dynamics, heat transfer, elasticity. Burgers-Fisher and Burgers-Huxley equations are also used as mathematical models of solute transport through the porous medium.

Wealthy literature is available in which these NPDEs are solved with various initial and boundary conditions representing different physical phenomena of nature. Some of those are explained here. The perturbation method has been applied to longitudinal and lateral dispersions in no uniform seepage flow through heterogeneous aquifer by Hunt (1978). Joshi et al. (2012) used the theoretical approach during the solution of Burgers' equation for longitudinal dispersion phenomena occurring in miscible phase flow through porous media. New integral transform with homotopy perturbation method is used by Kunjan and Twinkle (2015) to find the solution of Burgers' equation arising in the longitudinal dispersion phenomenon in fluid flow through porous media. Burgers' equation arising in longitudinal dispersion phenomena occurring in miscible phase flow through porous media has been solved by Meher and Mehta (2010) using Backlund transformation and by Patel and Mehta (2005) using Hope-Cole transformation. Many other researchers have also solved the Burgers' equation (Benton and Platzman (1972); Mittal and Singhal (1993); Kutluay et al. (1999); Ozis et al. (2003); Kutluay et al. (2004)). Due to various applications of Fisher equation, the solutions are given by many authors (Abtowitz and Zeppetella (1979); Wang (1988); Puri et al. (1989); Parekh and Puri (1990); Puri (1991); Tang and Weber (1991); Mavoungou and Cherruault (1994); Carey and Shen (1995); Qiu nad Sloan (1998); Al-Khalid (2001); Wazwaz and Gorguis (2004); Olmos and Shizgal (2006); Mittal and Jiwari (2009); Bastani and Salkuyeh (2012)). Time to time many researchers have solved the Burgers-Fisher as well as Burgers –Huxley equation (Wang et al. (1990); Wang and Lu (1990); Kaya and El-Sayed (2003); Ismail et al. (2004); Wazwaz (2005, 2008); Batiha et al. (2007, 2008); Babolian and Saeidian (2009); Olayiwola et al. (2010)). Due to the vast applications of NPDEs to explain the natural phenomena, lots of researchers are trying to get the solutions of these models from the last half-century, and it is an open field of research nowadays also.

In this chapter, a new numerical method based on spectral collocation approach is proposed to get the approximate solutions of considered NPDEs subject to initial and boundary conditions. For that, double shifted Chebyshev polynomials of the first-kind with spatial and temporal variables are used to approximate the solutions of the considered problems together to apply spectral collocation method in which shifted Chebyshev operational matrix in space as well as in temporal discretization are used for derivatives. The main advantage of choosing the double shifted Chebyshev polynomials of the first-kind is that the considered problems are directly converted in the system of non-linear equations instead of a system of ordinary differential equations as shown in last two chapters due to which there is no need to apply any finite difference scheme. Since it converts the considered problem into a system of nonlinear algebraic equations which are ultimately solved applying Newton's iterative method. The exponential convergence rate of the considered method is analyzed through the considered examples for both spatial and temporal discretization. To show the efficiency and accuracy of the considered method, it has applied to a number of physical problems, and a comparison of the approximate solutions with the existing analytical solutions present in literature are shown through the graphical presentations as well as tables.

4.2 Preliminaries

4.2.1 Shifted Chebyshev Polynomials of the First-Kind

In most of the numerical problems, the interval of interest is $[0, L]$. In order to use shifted Chebyshev polynomial in that interval, introducing the change of variable as discussed in Chapter 1 and denoted it by $T_{L,n}(x)$, which are satisfying the orthogonality condition as

$$\int_0^L T_{L,p}(x) T_{L,q}\, w_L(x)\, dx = \delta_{qp} \chi_q, \tag{4.1}$$

where $w_L(x) = 1 \big/ \sqrt{Lx - x^2}$ and $\chi_q = \dfrac{\varepsilon_q}{2}\pi,\ \varepsilon_0 = 2,\ \varepsilon_q = 1,\ q \geq 1.$

The analytical form of the shifted Chebyshev polynomials $T_{L,n}(x)$ is given by

$$T_{L,n}(x) = n \sum_{p=0}^{n} (-1)^{n-p} \frac{(n+p-1)!\, 2^{2p}}{(n-p)!(2p)!\, L^p} x^p, \tag{4.2}$$

where $T_{L,n}(0) = (-1)^n$ and $T_{L,n}(L) = 1.$

4.2.2 Shifted Chebyshev Spectral Collocation Method

Let $u(x) \in L^2_{w_L}(0, L)$ is expressed in terms of shifted Chebyshev polynomials as

$$u(x) = \sum_{i=0}^{\infty} c_i T_{L,i}(x), \tag{4.3}$$

where the coefficients c_i are given by

$$c_i = \frac{1}{\chi_i} \int_0^L w_L(x) u(x) T_{L,i}(x)\, dx,\ i = 0,1,\ldots; \tag{4.4}$$

Again for approximation, we have taken a finite number of terms, so the first $(M+1)$-terms of the shifted Chebyshev polynomials are taken during approximation. Thus we have

$$u_M(x) \simeq \sum_{i=0}^{M} c_i\, T_{L,i}(x). \tag{4.5}$$

The matrix representation of above equation is

$$u_M(x) \simeq C^T \Omega_{L,M}(x), \tag{4.6}$$

with

$$C^T \equiv [c_0, c_1, \ldots, c_M], \quad \Omega_{L,M}(x) \equiv [T_{L,0}(x), T_{L,1}(x), \ldots, T_{L,M}(x)]^T.$$

4.2.3 Derivatives

The first order derivative of the column vector $\Omega_{L,M}(x)$ is given as

$$\frac{d}{dx}\Omega_{L,M}(x) = D\Omega_{L,M}(x), \tag{4.7}$$

where D is the $(M+1)\times(M+1)$ order operational matrix of derivative given as

$$D = (d_{pq}) = \begin{cases} \dfrac{4p}{\varepsilon_q L}, & q = 0,1,\ldots,\ p = q + r, \begin{cases} r = 1,3,5,\ldots, M, & if\ M\ is\ odd, \\ r = 1,3,5,\ldots, M-1, & if\ M\ is\ even, \end{cases} \\ 0 & otherwise, \end{cases} \tag{4.8}$$

For even M, we have

$$D=\frac{2}{L}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 3 & 0 & 6 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 & . & . & . & 0 & 0 \\ 5 & 0 & 10 & 0 & 10 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ M-1 & 0 & 2M-2 & 0 & 2M-2 & . & . & . & 0 & 0 \\ 0 & 2M & 0 & 2M & 0 & . & . & . & 2M & 0 \end{pmatrix}, \qquad (4.9)$$

and for odd M , we have

$$D=\frac{2}{L}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 3 & 0 & 6 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 & . & . & . & 0 & 0 \\ 5 & 0 & 10 & 0 & 10 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 2M-2 & 0 & 2M-2 & 0 & . & . & . & 2M & 0 \\ M & 0 & 2M & 0 & 2M & . & . & . & 0 & 0 \end{pmatrix}, \qquad (4.10)$$

The higher order derivative of the column vector $\Omega_{L,M}(x)$ is given as

$$\frac{d^m}{dx^m}\Omega_{L,M}(x) = D^m\,\Omega_{L,M}(x), \qquad (4.11)$$

where m is a natural number and D^m denotes the mth-order derivative of $\Omega_{L,M}(x)$.

4.3 Application of the Spectral Method Based on Chebyshev Operational Matrix

In this section, Chebyshev operational matrix together with spectral collocation method is applied to solve the nonlinear differential equations subject to initial and boundary conditions with variable coefficients of the form

$$\frac{\partial u(x,t)}{\partial t} = \gamma(x,t)\frac{\partial^2}{\partial x^2}u(x,t) + \frac{\partial}{\partial x}\xi(u) + \zeta(u),\ a < x < b,\ t > 0, \tag{4.12}$$

or

$$u_t = \gamma u_{xx} + \xi(u)_x + \zeta(u), \qquad a < x < b,\ t > 0, \tag{4.13}$$

with the boundary conditions

$$u(a,t) = f(t), \qquad t > 0, \tag{4.14}$$

$$u(b,t) = g(t), \qquad t > 0, \tag{4.15}$$

and initial condition

$$u(x,0) = h(x), \qquad a < x < b, \tag{4.16}$$

where γ is the diffusivity, $\xi(u)$ and $\zeta(u)$ are the nonlinear expressions in terms of u. Here γ may be considered as constant to simplify the considered problems.

Equation (4.13) with the use of initial condition (4.16) can be rewritten as

$$u_t = \gamma u_{xx} + \xi(u)_x + \zeta(u) + u(x,0) - h(x), \qquad a < x < b,\ t > 0, \tag{4.17}$$

which has to solve along with the given boundary conditions.

To solve the problem (4.17) with the given boundary conditions, let us approximate the solution $u(x,t)$ by $(M+1)\times(N+1)$ terms of shifted Chebyshev polynomials series as

$$u_{M,N}(x,t) = \sum_{p=0}^{M}\sum_{q=0}^{N} c_{pq}\, T_{\tau,p}(t) T_{L,q}(x) = \Omega_{\tau,M}^{T}(t)\, C\, \Omega_{L,N}(x), \tag{4.18}$$

where C is the $(M+1)\times(N+1)$ order matrix of the coefficients c_{pq}, given as

$$C = \begin{pmatrix} c_{00} & c_{01} & . & . & . & c_{0N} \\ c_{10} & c_{11} & . & . & . & c_{1N} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ c_{M0} & c_{M1} & . & . & . & c_{MN} \end{pmatrix}. \tag{4.19}$$

The entries of the matrix C are obtained from

$$c_{pq} = \frac{1}{\chi_{\tau,p}\chi_{L,q}} \int_0^{\tau}\int_0^{L} u(x,t) T_{\tau,p}(t) T_{L,q}(x) w_{\tau}(t) w_L(x)\, dx\, dt, \; p = 0,1,\ldots, M, \; q = 0,1,\ldots, N. \tag{4.20}$$

Derivative of the approximate solution can be expressed as

$$\frac{\partial}{\partial t} u(x,t) \simeq \Omega^T_{\tau,M}(t) D^T_{\tau} C \,\Omega_{L,N}(x), \tag{4.21}$$

$$\frac{\partial}{\partial x} u(x,t) \simeq \Omega^T_{\tau,M}(t) C D_L \,\Omega_{L,N}(x), \tag{4.22}$$

$$\frac{\partial^2}{\partial x^2} u(x,t) \simeq \Omega^T_{\tau,M}(t) C D^2_L \,\Omega_{L,N}(x), \tag{4.23}$$

$$u(x,0) \simeq \Omega^T_{\tau,M}(0) C \,\Omega_{L,N}(x). \tag{4.24}$$

Employing equations (4.21)-(4.24) in equation (4.17), we get

$$\begin{aligned} &\Omega^T_{\tau,M}(t)(D^T_{\tau} C - \gamma(x,t) C D^2_L)\Omega_{L,N}(x) - \xi(\Omega^T_{\tau,M}(t) C \,\Omega_{L,N}(x))_x \\ &\qquad = \Omega^T_{\tau,M}(0) C \,\Omega_{L,N}(x) - h(x) + \zeta(\Omega^T_{\tau,M}(t) C \,\Omega_{L,N}(x)), \end{aligned} \tag{4.25}$$

with the boundary conditions

$$\begin{aligned} &\Omega^T_{\tau,M}(t) C \,\Omega_{L,N}(a) = f(t), \\ &\Omega^T_{\tau,M}(t) C \,\Omega_{L,N}(b) = g(t). \end{aligned} \tag{4.26}$$

According to the spectral Chebyshev collocation method, equation (4.25) is to be satisfied at $(M+1)\times(N-1)$ collocation points, and the considered boundary conditions (4.26) are satisfied at $2(M+1)$ collocation points as follows

$$\begin{aligned} &\Omega^T_{\tau,M}(t_p)(D^T_{\tau} C - \gamma(x_q,t_p) C D^2_L)\Omega_{L,N}(x_p) - \xi(\Omega^T_{\tau,M}(t_p) C \,\Omega_{L,N}(x_q))_x \\ &\qquad = \Omega^T_{\tau,M}(0) C \,\Omega_{L,N}(x_q) - h(x_q) + \zeta(\Omega^T_{\tau,M}(t_p) C \,\Omega_{L,N}(x_q)), \end{aligned} \tag{4.27}$$

with the boundary conditions

$$\begin{aligned} &\Omega^T_{\tau,M}(t_p) C \,\Omega_{L,N}(a) = f(t_p), \\ &\Omega^T_{\tau,M}(t_p) C \,\Omega_{L,N}(b) = g(t_p). \end{aligned} \tag{4.28}$$

where t_p, $p = 0,1,\ldots,M$ are the roots of $T_{\tau,M+1}(t)$ and x_q, $q = 0,1,\ldots,N-2$ are the Gauss-Lobatto points.

From here we get the system of $(M+1)\times(N+1)$ non-linear algebraic equations in c_{pq} in which $(M+1)\times(N-1)$ equations come from equation (4.27), and $2(M+1)$ equations come from the equation (4.28). Now $(M+1)\times(N+1)$ equations are a non-linear system of algebraic equations, can be solved using Newton's iterative method for c_{pq}. Consequently $u_{M,N}(x,t)$ given in equation (4.18) can be calculated.

4.4 Illustrative Examples

To demonstrate the effectiveness and accuracy of the proposed method, some important nonlinear models are solved using the proposed method, and the results are compared with the existing exact solutions.

Example 1. Consider the Burgers equation

$$u_t = u_{xx} - uu_x, \qquad 0 < x < 1,\ t > 0,$$

with the boundary conditions

$$\left.\begin{aligned} u(0,t) &= \frac{1}{2} - \frac{1}{2}\tanh\left(-\frac{1}{8}t\right), \\ u(1,t) &= \frac{1}{2} - \frac{1}{2}\tanh\left\{\frac{1}{4}\left(1-\frac{1}{2}t\right)\right\}, \end{aligned}\right. \qquad t > 0,$$

and initial condition

$$u(x,0) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{4}\right), \qquad 0 < x < 1.$$

The exact solution of this problem is $u(x,t)=\frac{1}{2}-\frac{1}{2}\tanh\left\{\frac{1}{4}\left(x-\frac{1}{2}t\right)\right\}$ (Babolian and Saeidian (2009)). In Table 4.1, the absolute error $ER(x,t)=\max\limits_{\substack{0<x<1\\0\le t\le 1}}\left|c(x,t)-c_{M,N}(x,t)\right|$ for various values of M and N at $t=1$hr are presented. The plot of the approximate solution vs. *x* and *t* of this problem is shown in Fig. 4.1. Also in Fig. 4.2, the comparison of the approximate solution with the existing analytical solution vs. *x* and *t* is shown. The variations of absolute error vs. *x* and *t* are shown through Fig. 4.3. All the pictorial presentations are for $M=N=3$.

Table 4.1 The absolute error $ER(x,1)$ with various choices of M and N

x	$M=N=3$	$M=N=5$	$M=N=7$
0.1	7.42e-04	6.32e-06	2.42e-11
0.2	9.93e-04	7.43e-06	4.23e-09
0.3	8.75e-04	7.12e-06	4.89e-08
0.4	5.11e-04	4.11e-06	3.12e-08
0.5	2.27e-05	3.12e-07	1.57e-11
0.6	4.67e-04	3.98e-06	3.67e-08
0.7	8.37e-04	5.24e-06	6.62e-08
0.8	9.64e-04	5.67e-06	7.54e-09
0.9	7.25e-05	6.52e-07	8.57e-11

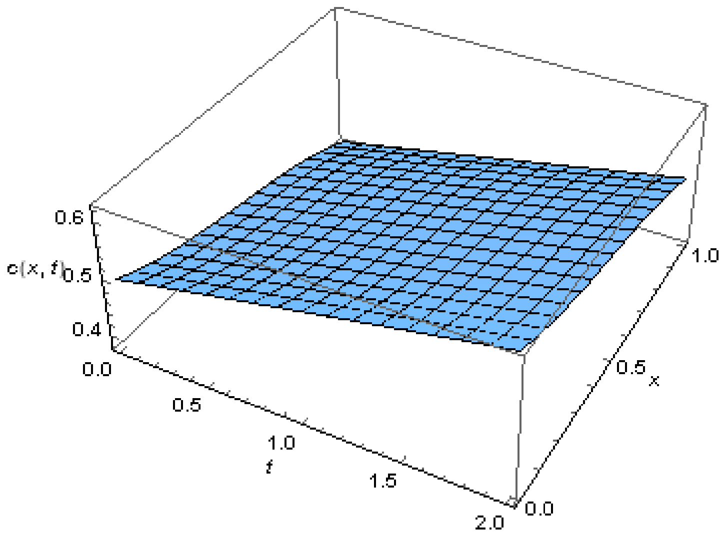

Fig. 4.1 Plot of the approximate solution of Example 1 vs. *x* and *t* for $M=N=3$

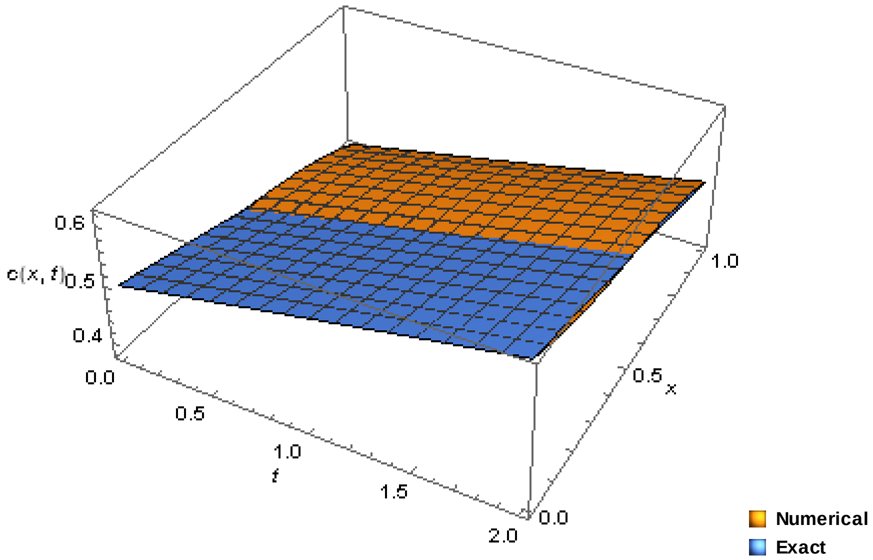

Fig. 4.2 Comparison between exact and approximate solutions of Example 1 vs. *x* and *t* for $M = N = 3$

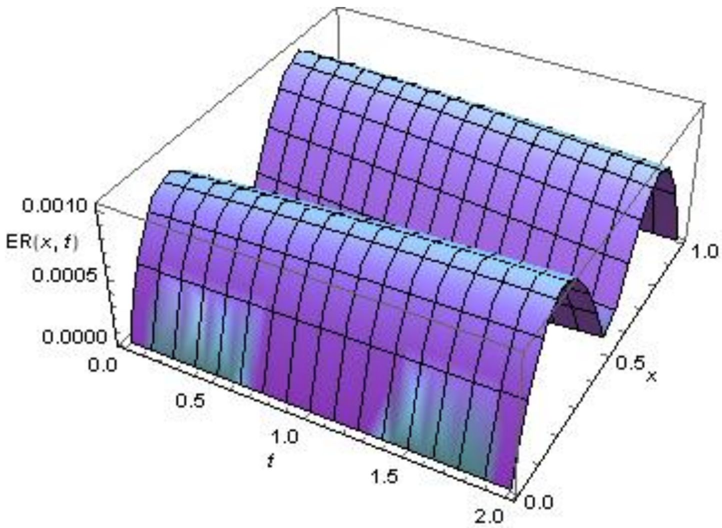

Fig. 4.3 Absolute errors of Example 1 vs. *x* and *t* for $M = N = 3$

Example 2. Consider the Fisher equation

$$u_t = u_{xx} + u(1-u), \qquad 0 < x < 1,\ t > 0,$$

with the boundary conditions

$$u(0,t) = \frac{1}{4}\left\{1 - \tanh\left(-\frac{5}{12}t\right)\right\}^2, \qquad t > 0,$$

$$u(1,t) = \frac{1}{4}\left\{1 - \tanh\left(\frac{1}{2\sqrt{6}}\left(1 - \frac{5}{\sqrt{6}}t\right)\right)\right\}^2, \qquad t > 0,$$

and initial condition

$$u(x,0) = \frac{1}{4}\left\{1 - \tanh\left(\frac{x}{2\sqrt{6}}\right)\right\}^2, \qquad 0 < x < 1.$$

The exact solution of this problem is $u(x,t) = \frac{1}{4}\left\{1 - \tanh\left(\frac{1}{2\sqrt{6}}\left(x - \frac{5}{\sqrt{6}}t\right)\right)\right\}^2$ (Babolian and Saeidian (2009)). The absolute errors are presented through Table 4.2 for various values of M and N at $t = 1$hr. The approximate solution of this problem vs. x and t is shown through Fig. 4.4. The comparison between the existing analytical solution and the approximate solution vs. x and t is displayed through Fig. 4.5. The variation of the absolute error vs. x and t is shown though Fig. 4.6. The graphs are drawn for $M = N = 3.$

Table 4.2 The absolute error $ER(x,1)$ with various choices of M and N

x	$M = N = 3$	$M = N = 5$	$M = N = 7$
0.1	8.93e-04	7.48e-07	9.57e-10
0.2	1.17e-03	2.35e-07	8.23e-09
0.3	1.02e-03	1.08e-06	4.32e-09
0.4	5.81e-04	3.54e-06	1.94e-09
0.5	6.50e-06	2.78e-08	1.06e-12
0.6	5.93e-04	5.24e-06	2.54e-11
0.7	1.03e-03	2.98e-06	6.52e-10
0.8	1.18e-03	7.34e-06	4.55e-11
0.9	8.91e-04	7.42e-07	6.24e-12

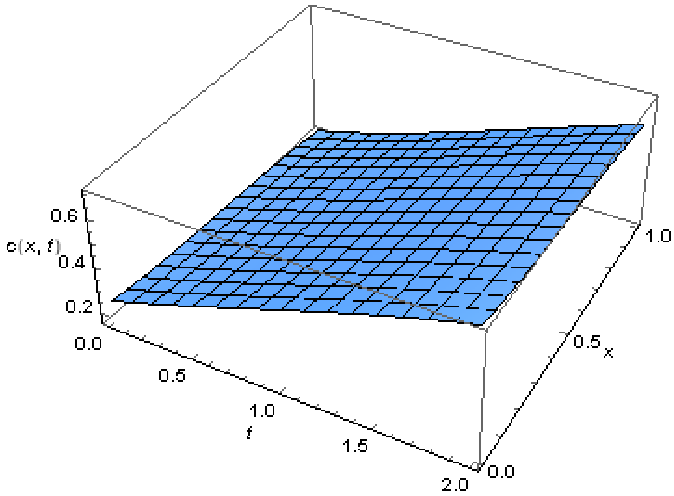

Fig. 4.4 Plot of the approximate solution of Example 2 vs. x and t for $M = N = 3$

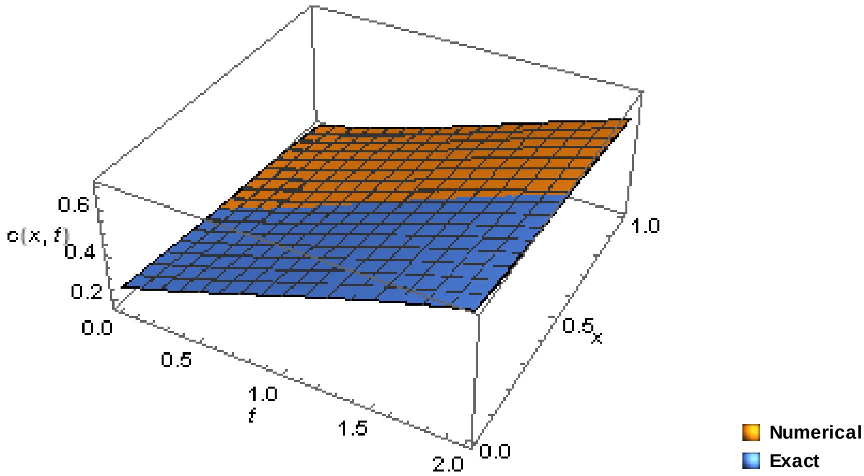

Fig. 4.5 Comparison between the exact and approximate solution of Example 2 vs. x and t for $M = N = 3$

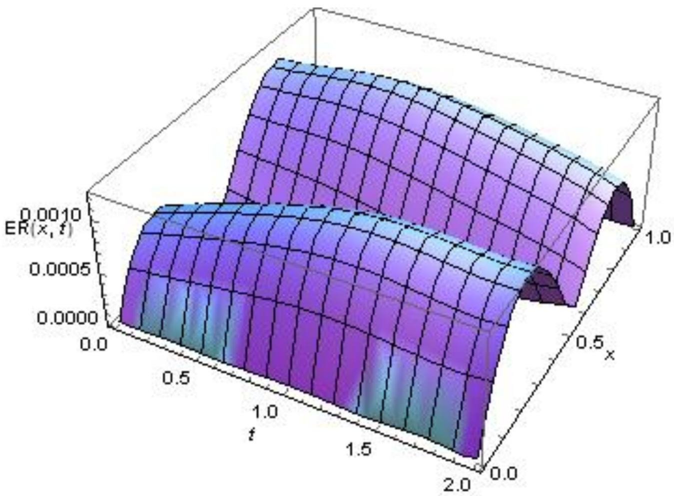

Fig. 4.6 Absolute errors of Example 2 vs. x and t for $M = N = 3$

Example 3. Consider the Huxley equation

$$u_t = u_{xx} + u(1-u)(u-1), \qquad 0 < x < 1,\ t > 0,$$

with the boundary conditions

$$\begin{aligned} u(0,t) &= \frac{1}{2} + \frac{1}{2}\tanh\left(-\frac{1}{4}t\right), \\ u(1,t) &= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}\left(1-\frac{1}{\sqrt{2}}t\right)\right), \end{aligned} \qquad t > 0,$$

and initial condition

$$u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right), \qquad 0 < x < 1.$$

The exact solution of this problem is $u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}\left(x-\frac{1}{\sqrt{2}}t\right)\right)$ (Babolian and Saeidian (2009)). The absolute errors for above example are given in Table 4.3 for various values of M and N at $t = 1$hr. The approximate solution and its comparison with the exact solution vs. x and t are presented in Fig. 4.7 and Fig. 4.8, respectively. The variation of absolute error vs. x and t is displayed through Fig. 4.9. The graphs are drawn for $M = N = 3$.

Table 4.3 The absolute error $ER(x,1)$ with various choices of M and N

x	$M = N = 3$	$M = N = 5$	$M = N = 7$
0.1	1.16e-03	1.27e-05	5.42e-08
0.2	1.57e-03	2.40e-06	1.12e-10
0.3	1.41e-03	1.76e-06	1.94e-07
0.4	8.79e-04	4.22e-06	3.32e-11
0.5	1.46e-04	3.58e-07	1.86e-11
0.6	5.98e-04	5.52e-06	5.72e-09
0.7	1.16e-03	4.38e-05	7.42e-10
0.8	1.38e-03	2.14e-06	6.54e-09
0.9	1.05e-03	1.10e-07	2.06e-12

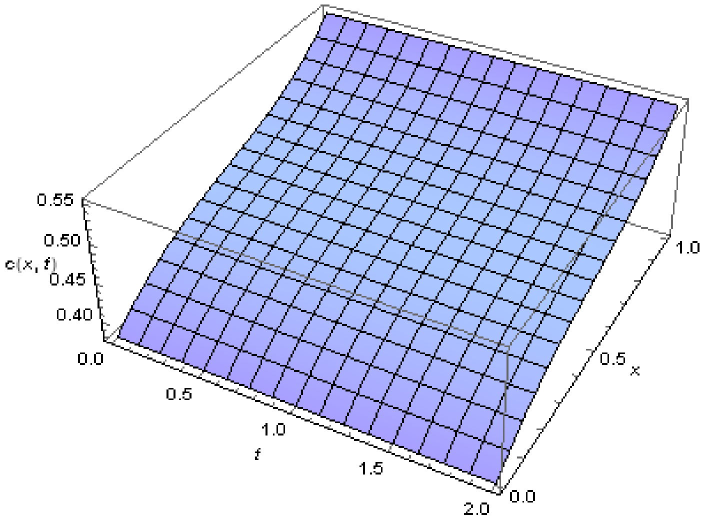

Fig. 4.7 Plot of the approximate solution of Example 3 vs. *x* and *t* for $M = N = 3$

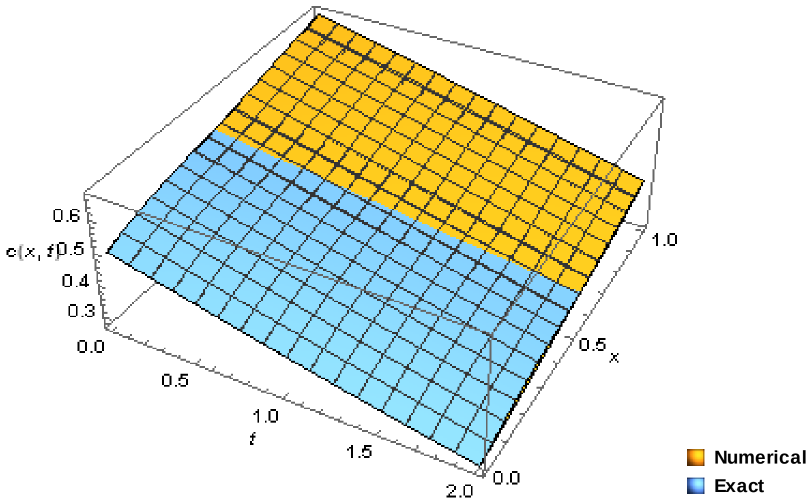

Fig. 4.8 Comparison between the exact and approximate solution of Example 3 vs. *x* and *t* for $M = N = 3$

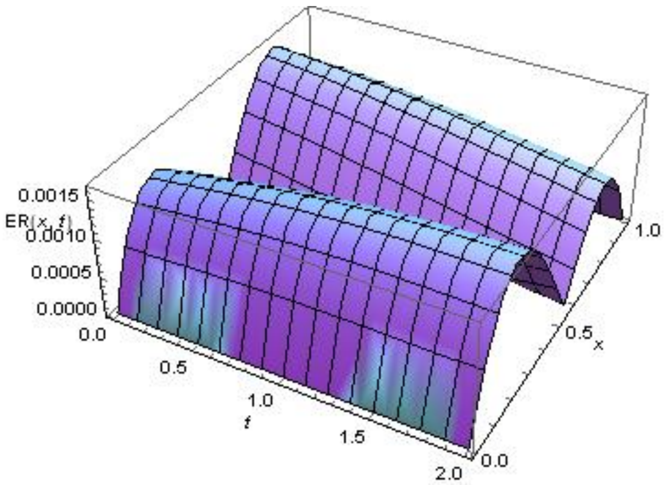

Fig. 4.9 Absolute errors of Example 3 vs. *x* and *t* for $M = N = 3$

Example 4: Consider the Burgers-Fisher equation

$$u_t = u_{xx} + uu_x + u(1-u),\ 0 < x < 1,\ t > 0,$$

with the boundary conditions

$$\left.\begin{aligned} u(0,t) &= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{5}{8}t\right), \\ u(1,t) &= \frac{1}{2} + \frac{1}{2}\tanh\left\{\frac{1}{4}\left(1+\frac{5}{2}t\right)\right\}, \end{aligned}\right. \qquad t > 0,$$

and initial condition

$$u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{4}\right), \qquad 0 < x < 1.$$

This problem has the exact solution $u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left\{\frac{1}{4}\left(x+\frac{5}{2}t\right)\right\}$ (Babolian and Saeidian (2009)). The absolute errors are given in Table 4.4 for various values of M and N at $t = 1$hr. The approximate solution of this problem vs. x and t is presented through Fig. 4.10. In Fig. 4.11, the comparison between the existing analytical solution and the approximate solution vs. x and t is shown. The variation of absolute error vs. x and t is shown through Fig. 4.12. All the plots are drawn for $M = N = 3$.

Table 4.4 The absolute error $ER(x,1)$ with various choices of M and N

x	$M = N = 3$	$M = N = 5$	$M = N = 7$
0.1	1.43e-04	2.21e-05	1.11e-09
0.2	1.94e-04	2.34e-06	1.26e-08
0.3	8.85e-05	3.42e-06	6.15e-10
0.4	1.13e-04	8.12e-07	7.14e-09
0.5	3.54e-04	8.19e-06	8.21e-09
0.6	5.73e-04	4.26e-06	3.56e-11
0.7	7.12e-04	5.24e-05	3.53e-08
0.8	7.13e-04	6.22e-07	4.23e-10
0.9	5.15e-04	4.27e-07	6.45e-11

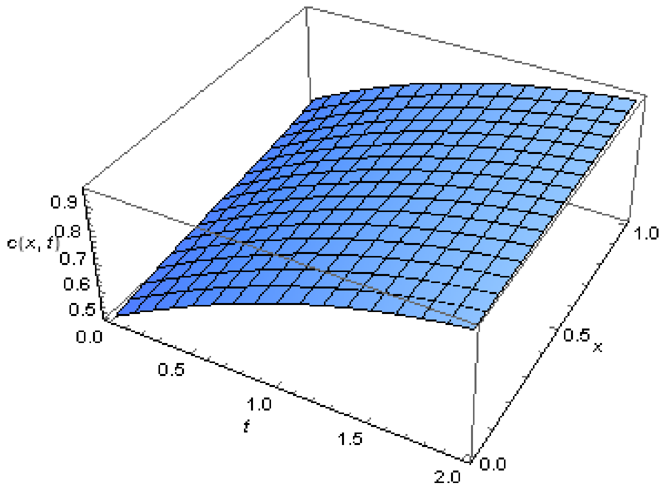

Fig. 4.10 Plot of the approximate solution of Example 4 vs. *x* and *t* for $M = N = 3$

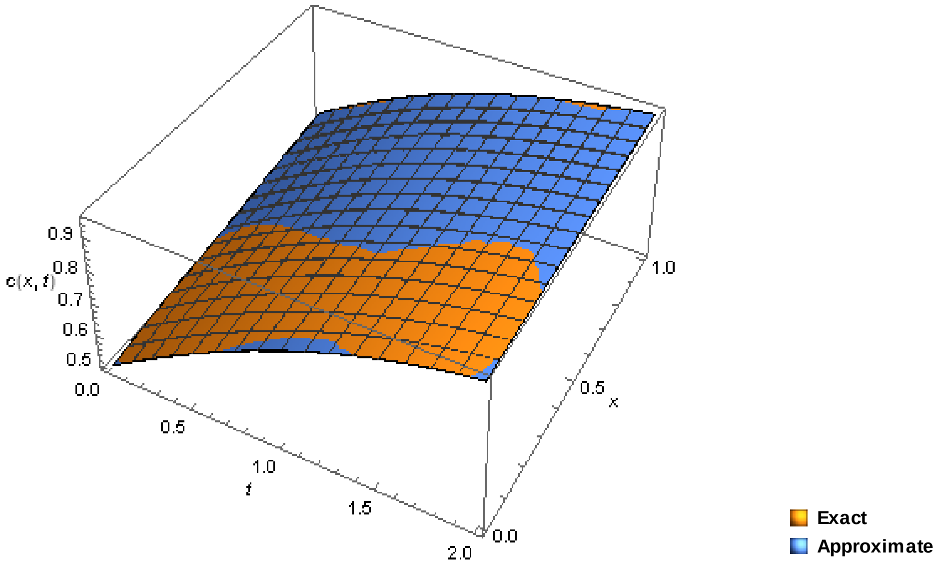

Fig. 4.11 Comparison between the exact and approximate solution of Example 4 vs. *x* and *t* for $M = N = 3$

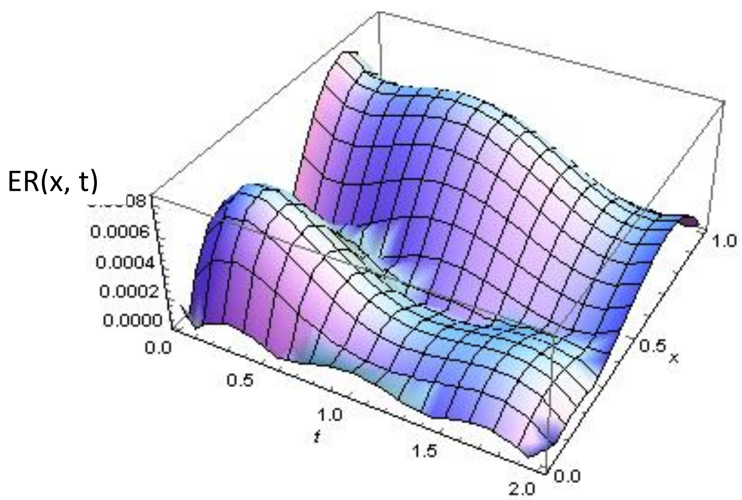

Fig. 4.12 Absolute errors of Example 4 vs. *x* and *t* for $M = N = 3$

Example 5. Consider the Burgers-Huxley equation

$$u_t = u_{xx} + uu_x + u(1-u)(u-1), \quad 0 < x < 1,\ t > 0,$$

with the boundary conditions

$$\begin{aligned} u(0,t) &= \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{3}{8}t\right), \\ u(1,t) &= \frac{1}{2} - \frac{1}{2}\tanh\left\{\frac{1}{4}\left(1+\frac{3}{2}t\right)\right\}, \end{aligned} \qquad t > 0,$$

and initial condition

$$u(x,0) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{4}\right), \qquad 0 < x < 1.$$

The exact solution of this problem is $u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left\{\frac{1}{4}\left(x+\frac{3}{2}t\right)\right\}$ (Babolian and Saeidian (2009)). In Table 4.5, the absolute errors are presented for different values of M and N at $t = 1$hr. The approximate solution of this problem and the comparison with the exact solution vs. x and t are shown through the Fig. 4.13 and Fig. 4.14, respectively. The variation of absolute error vs. x and t is presented through Fig. 4.15. All the plots are drawn for $M = N = 3$.

Table 4.5 The absolute error $ER(x,1)$ with various choices of M and N

x	$M = N = 3$	$M = N = 5$	$M = N = 7$
0.1	3.83e-04	2.34e-07	3.24e-10
0.2	4.76e-04	3.35e-07	3.87e-09
0.3	3.42e-04	4.47e-06	4.25e-10
0.4	6.32e-05	4.35e-07	5.64e-09
0.5	2.77e-04	5.28e-07	6.21e-09
0.6	5.97e-04	6.23e-07	2.57e-10
0.7	8.13e-04	6.87e-06	8.64e-09
0.8	8.43e-05	7.15e-06	7.36e-10
0.9	6.05e-05	5.24e-07	5.27e-11

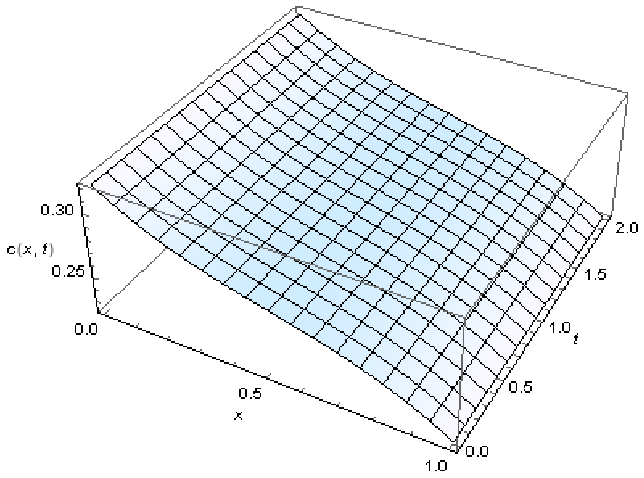

Fig. 4.13 Plot of the approximate solution of Example 5 vs. *x* and *t* for $M = N = 3$

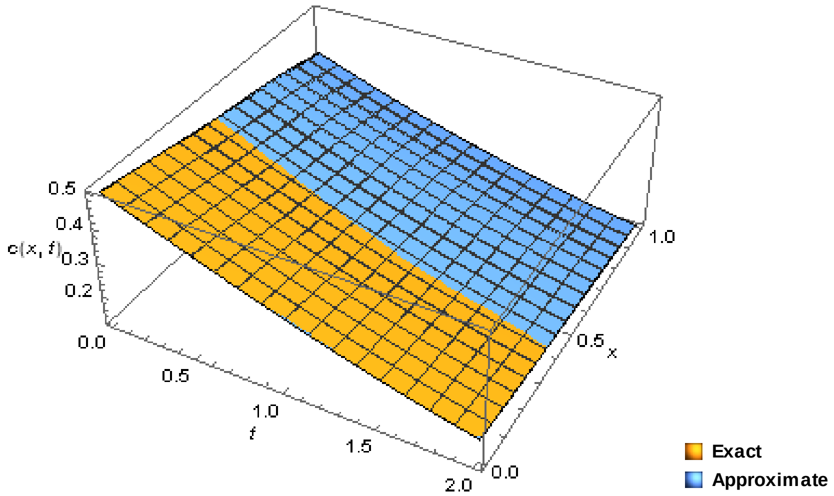

Fig. 4.14 Comparison between the exact and approximate solution of Example 5 vs. *x* and *t* for $M = N = 3$

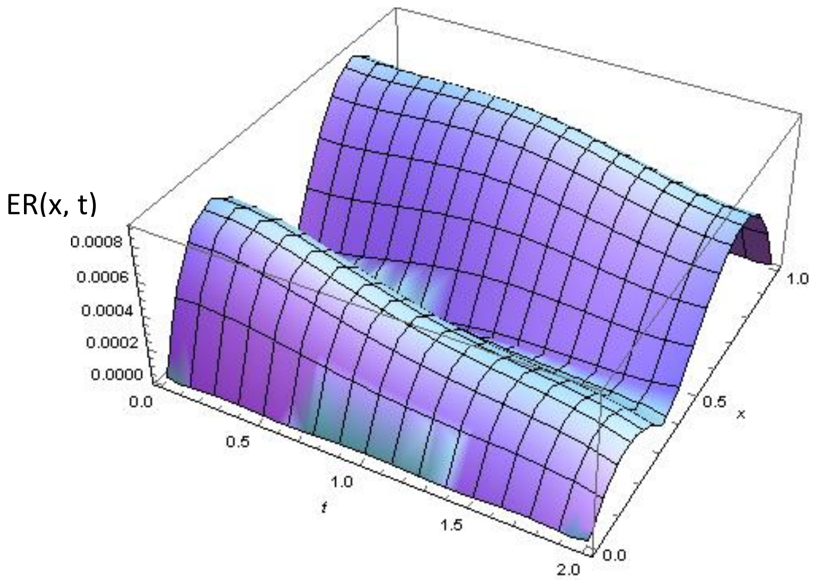

Fig. 4.15 Absolute errors of Example 5 vs. *x* and *t* for $M = N = 3$

4.5 Conclusions

In the present chapter, a method is proposed to solve some NPDEs subject to initial and boundary conditions. The considered problems are converted into a system of algebraic equations using shifted Chebyshev polynomials together with shifted Chebyshev operational matrix. The Newton iterative method is used during the solution of nonlinear algebraic equations. The author strongly believes that the present demonstration of efficient and reliable method towards the solutions of a number of NPDEs subject to initial and boundary conditions with constant or variable coefficients will be appreciated by the researchers working in the area of modeling of nonlinear physical and engineering problems. Also, this method can be extended to solve time, space and space-time fractional order differential equations by considering the fractional order operational matrix of derivative, which is discussed in next chapter. To validate the efficiency of the proposed method a comparative study of each problem with the existing analytical result is evaluated numerically through error analysis which is displayed graphically. The error analysis shows the exponential convergence rate of the proposed method.