Chapter-2

Literature Review

2.1 Introduction

This chapter deals with evolution of fracture mechanics in general and the various parameters with modeling approach in particular with its applications and a brief historical literature review of the past work done. An attempt has been made to provide an insight into the use of classical fracture mechanics approach and deterministic methods for estimating the various fracture parameters, a central problem in computational fracture analyses. With respect to the state of the art to solve complex linear or nonlinear crack phenomena under mechanical, thermo-mechanical or electro-mechanical loads along with advanced damage model, the field of fracture mechanics has become sufficiently matured. Unfortunately, only a small fraction of those who have made major contributions to the area of fracture mechanics can be mentioned in the present context and it becomes more difficult to know which names to omit while space will prevent many others who have made significant contributions in this field. The author apologizes to those whose names are omitted.

The advent of fracture mechanics is generally attributed to the pioneering work of Inglis (1913) when he observed the infinite nature of stresses at the vertex of a degenerate ellipsoidal cavity. Consequent studies by other research workers have since then led to a better and deeper understanding of fracture phenomena. This in turn has led to the development of theories to explain and quantify the observed physical behavior of engineering structures during failure. Among the earliest fracture theories developed was

Linear Elastic Fracture Mechanics (LEFM), which was ultimately extended to take care of the more practical situation in terms of load, loading rate, material response etc.

The early development can be traced back to the work of Griffith (1920), when he formulated an energy equation to describe the propagation of cracks using the concept of critical energy release rate G_c. This fracture criterion, which is essentially a statement of the energy balance principle, indicates that crack propagation initiates when the gain in surface energy due to the increase in surface area equals the reduction in strain energy due to the displacement of the boundaries and the change in the stored elastic energy. This theory, however, is limited only to the failure analysis of homogeneous brittle materials such as glass, ceramics etc.

2.2 Linear Elastic Fracture Mechanics

Realizing this limitation, Orowan (1949) and Irwin (1957) proposed a modification of the theory, which can be used for engineering materials exhibiting limited ductility. The model is an extension of the energy formulation used by Griffith where plastic strain energy rate for crack propagation was added in the energy equation. Both of them recognized that the energy required to produce plastic strain at the crack tip is much greater than the surface energy needed to create new surfaces. It is through this work that the foundation of LEFM was established.

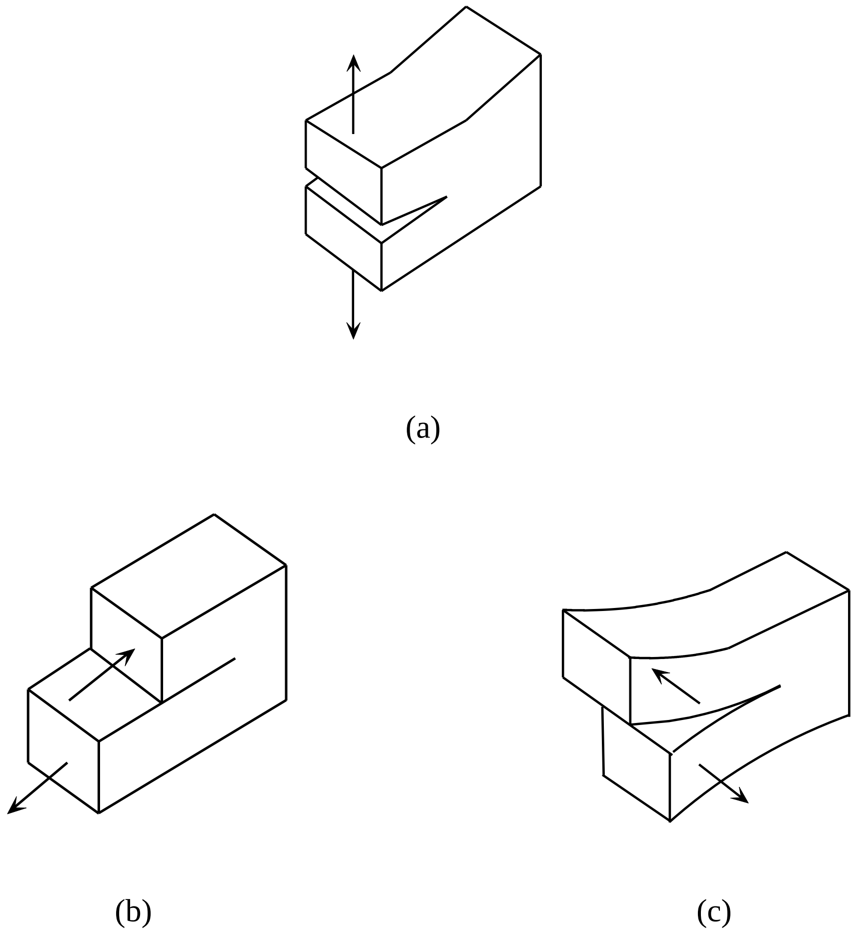

Figure 2.1: Various basic loading modes in fracture mechanics (a) Opening mode (b) Sliding mode (c) Tearing mode

Large magnitudes of stresses and their steeper gradients near the tip of a crack are of serious importance. The material near these areas become weak and fail to sustain the intense local stresses and as a result, further nucleation of micro-cracks occur which subsequently grow to attain critical dimensions. The investigation of near tip stress field and their intensity is of prime concern for fracture analyses. The stress field near the tip may be grouped into three basic types, each associated with a local mode of deformation, namely the opening mode (mode I), the sliding mode (mode II) and the tearing mode (mode III), Fig. 2.1. A relative motion due to symmetric separation characterizes the opening mode across the crack with respect to the crack surface. The sliding mode is characterized by the skew-symmetric motion (with respect to crack surface) of two points on each crack surface and passes one another in opposite direction normal to the crack front. The tearing mode is represented by the skew-symmetric motion (with respect to crack surface) and passes one another in opposite direction, parallel to the crack front in out of plane containing the crack surfaces. Any general deformation in the vicinity of the crack tip may be described by the proper combination of these three basic types and is known as mixed mode of fracture. The most direct approach of determining the stress and displacement fields associated with each mode follows from the method adopted by Irwin (1957) based on the formulations of Westergaard (1959). In engineering practices the importance of the opening mode far exceeds that of other modes. Whenever possible, we try to analyze the stress – strain fields about a single crack tip by means of linear elastic fracture mechanics (LEFM) because this is the simplest model and within LEFM any arbitrary crack tip stress state is considered to be linear elastic. Two important fracture parameters in LEFM are

- Stress Intensity Factor (SIF) and
- Energy Release Rate (G)

2.2.1 Stress Intensity Factor (SIF)

For each mode, the equations of stress-fields satisfying the boundary condition of having stress-free crack faces with far-field boundary conditions are available (Westergaard, 1959; Irwin, 1957; Sneddon, 1946; Williams, 1939). Neglecting the higher order terms, the two-dimensional near field solutions of stresses for stationary crack in an isotropic material in mode I are expressed as

$$\sigma_{\mathrm{ij}} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r_{\mathrm{t}}}} f_{\mathrm{ij}}(\theta) \tag{2.1}$$

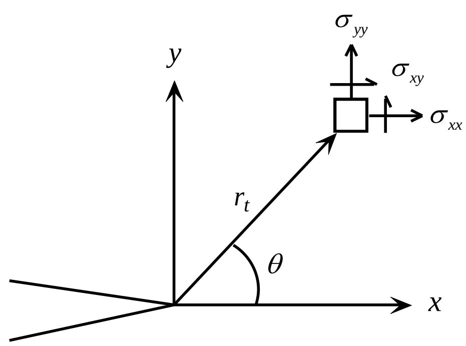

Figure 2.2: Coordinate system at a crack tip region

Where, σ_{ij} are the stress tensor, r_{t} (R) and θ denote the coordinates of a point (Fig. 2.2) with origin at the crack tip, K_{I} the mode I SIF and $f_{\mathrm{ij}}(\theta)$ are expressed as

$$f_{\mathrm{xx}}(\theta) = \cos\theta/2\,(1 - \sin\theta/2\sin 3\theta/2) \tag{2.2}$$

$$f_{\mathrm{yy}}(\theta) = \cos\theta/2\,(1 + \sin\theta/2\sin 3\theta/2) \tag{2.3}$$

$$f_{\mathrm{xy}}(\theta) = f_{\mathrm{yx}} = \sin\theta/2\cos\theta/2\cos 3\theta/2 \tag{2.4}$$

For plane stress case, $\sigma_{\mathrm{zz}} = 0$ and for plane strain this quantity becomes, $\sigma_{\mathrm{zz}} = \nu(\sigma_{\mathrm{xx}} + \sigma_{\mathrm{yy}})$. Associated displacements are

$$u_{\mathrm{xx}} = \frac{K_{\mathrm{I}}}{\mu}\sqrt{\frac{R}{2\pi}}\cos\theta/2\,(1 - 2\nu + \sin^2\theta/2) \tag{2.5}$$

$$u_{\mathrm{yy}} = \frac{K_{\mathrm{I}}}{\mu}\sqrt{\frac{R}{2\pi}}\sin\theta/2\,(2 - 2\nu + \cos^2\theta/2) \tag{2.6}$$

In plane strain, u_{zz} is zero and in plane stress it becomes $\frac{\nu}{E}(\sigma_{xx}+\sigma_{yy})$, while $\mu = E/2(1+\nu)$ represents the shear modulus. Thus the above feature indicates that the stress and deformation fields in the vicinity of crack tip may be uniquely defined by a single parameter K, the SIF. At the very crack tip, i.e., $r_t \rightarrow 0$, the stress becomes infinite and then, SIF becomes a measure of singularity at the crack tip. In order to give a proper dimension to the stresses, K must be proportional to the square root of length; the only characteristic length is the crack size *'2a'*. For an infinite plate with applied stress σ in the y-direction, K takes the form, $K = Y\sigma\sqrt{a}$, where Y is a constant. Whether the state of stress is attained by longer crack (larger 'a') or higher value of stresses (larger σ) is immaterial, only it indicates similar deformation fields near the crack tip for identical K factors. Failures occur when this deformation or the state of stress exceeds a critical limit denoted by K_{IC}, which is accepted as the fracture resistance of material against failure and is often known as the fracture toughness of the material.

2.2.2 Energy Release Rate (G)

Griffith (1920) introduced the concept of strain energy release rate G in the theory of energy balance of a cracked body of brittle materials. In a regular body, the strain energy release rate may be defined in terms of the potential energy U as

$$G = -\frac{\partial U}{\partial a} \tag{2.7}$$

$$U = \int_V W dv - \int_{S_t} \bar{T}.u ds \tag{2.8}$$

Where, V is the volume of the body, W the strain energy density, S_t, the part of the body subjected to surface traction $\bar{T}$ and u the displacement vector.

Irwin (1962) interpreted G as the crack driving force. Irwin (1960) and Orowan (1949) modified the original theory of Griffith (1920) to include the plastic energy dissipation rate due to yielding of material at the crack tip region. In order to relate the SIF with G, Irwin considered a small extension of the crack in x-direction and calculated the energy released by the displacement of the crack surface in y-direction to obtain $K^2 = E'G$, where E' equals to E in plane stress and $E/(1-\nu^2)$ for plane strain. Therefore, in mode I for plane stress or plane strain, the strain energy release rate become $EG = K_\mathrm{I}^2$ or $EG = (1-\nu^2)K_\mathrm{I}^2$, respectively.

2.2.3 Small Scale Yielding (SSY)

As in materials with finite strength, there cannot be the existence of a stress singularity at the crack tip (predicted by LEFM), the highly stressed material at the tip yields and plastically deforms. In LEFM approach, the plastic zone size at the crack tip is assumed to be small enough to be completely surrounded by an annulus where stresses are described by the K-field. Nonetheless, because a K-dominated annulus completely bounds the crack tip, K is still a single parameter description of the crack tip stress state. This concept suggests that despite the crack tip yielding, LEFM can still describe crack tip fields in such materials. This is known as small scale yielding SSY (Freund, 1998; Freund and Rosakis, 1992).

2.3 Elastic Plastic Fracture Mechanics

So far it has been implicitly assumed that the crack tip plasticity is extremely small such that LEFM may be suitably employed in fracture analyses. Undeniably, plastic deformation is a fact that occurs at the crack tip in all engineering materials and their alloys. The importance of the area of EPFM was appreciated when it was recognized that considerable amount of plastic deformation occurs in most of the materials prior to fracture. The crack growth in these materials involves plastic strain fields near the crack tip and fracture occurs after a critical amount of plastic strain is reached at certain locations. As one moves away from the crack tip, the stresses decay rapidly and reach the nominal value. This implies that the plastic deformation is confined to a region near the tip, which is thus enclosed by the surrounding material remaining in the elastic state. Therefore, large stress gradients near the tip regions produce a sizeable plastic zone rendering the elastic solution inadmissible for accurate representation of crack tip parameters. The characterization of the crack tip stress fields by a parameter estimated without the direct use of stresses within the uncertain yielded region will naturally provide a more practical and accurate fracture criterion. Further, the plastic deformation near the tip blunts the initial sharp tip making it a notch of finite root radius. When real cracks grow, the crack front are not straight even in 2-D problems and become non planar. This situation invariably develops some additional complexities in correctly evaluating the fracture parameters in yielded regime.

Interests in the fracture mechanics of ductile materials arose out of the research conducted by Dugdale (1960) and Barenblatt (1962). Dugdale proposed a simple model, the strip yield model, to deal with plasticity at the crack tip. A key assumption in the model states that stress values at the crack tip are limited by the yield strength of the material and the yielding is confined to a narrow band along the crack line.

Of the various concepts developed over the years, the J contour integral and the crack tip opening displacement (CTOD) have drawn the attention of yielding fracture mechanics. They form the main stream of fracture parameters when plastic deformations are considered in the analyses and are widely used in computational fracture mechanics.

2.3.1 The Path Independent *J* integral

During the analysis of lattice defects, Eshelby (1956) deduced a surface integral representing the force on an elastic singularity or in-homogeneity giving rise to conservation law in regular elasto-static fields in the presence of infinitesimal deformation. The path independent *J* integral introduced by Rice (1968) is such a parameter for linear or nonlinear elastic material. For two-dimensional stress field, the expression for *J* is

$$J = \int_{\Gamma} \left(W n_1 - T_{\mathrm{i}} \frac{\partial u_{\mathrm{i}}}{\partial x} \right) d\Gamma \qquad (2.9)$$

Where, Γ is the contour enclosing the crack tip, *W* the strain energy density, *n* the unit outward normal on Γ, T_i the traction vector on Γ and u_{i} is the displacement vector. The value of J is independent of the choice of Γ. Rice (1968) has shown that the J integral may be interpreted as the potential energy difference between two identically loaded bodies with slight differing in crack sizes and may be equated to $J = -\partial U / \partial a$. Hence, J is equal to *G* for linear elastic material and may be correlated to the K_{I} by

$$J = \frac{K_{\mathrm{I}}^2}{E} \qquad \text{for plane stress} \qquad (2.10)$$

$$J = \frac{(1-\nu^2)}{E} K_{\mathrm{I}}^2 \qquad \text{for plane strain} \qquad (2.11)$$

The report of Rice's work came at a time when elastic-plastic analyses of crack problems were gaining importance. The derivation of *J* integral is strictly valid for linear or nonlinear elastic material. Materials exhibiting elastic non-linearity in terms of load-displacements during loading, traces the original loading path at the time of unloading. By idealizing elasto-plastic deformation as nonlinear elastic, Rice provided the basis for extending fracture mechanics methodology well beyond the validity limits of LEFM.

The details of the stress and strain fields associated with the dominant singularity governing the plastic behavior at a crack tip are presented for condition of plane stress and plan strain for cracks in both far tensile and far shear fields, (Hutchinson, 1968). Fundamentals of the phenomenological theory of nonlinear fracture mechanics are available in Hutchinson (1983).

2.3.2 Crack Tip Opening Displacement (CTOD)

When Wells (1961) attempted to measure mode I critical SIF, K_{IC} values in a number of structural steels, he found that these materials were too tough to be characterized by LEFM. Looking at the fractured test specimens, Wells observed that crack faces had moved away from each other before fracture and plastic deformation had blunted an initially sharp crack, Fig. 2.3. The amount of crack blunting increased in proportion to the material toughness. Through this observation Wells put forward the opening at the crack tip as a measure of fracture toughness and today we call this parameter as CTOD.

One can estimate the CTOD by solving for the displacement at the physical crack tip, assuming $a + r_{\mathrm{y}}$ as an effective crack length

$$u_{\mathrm{y}} = \frac{\kappa+1}{2\mu} K_{\mathrm{I}} \sqrt{\frac{r_y}{2\pi}} = \frac{4}{E'} K_{\mathrm{I}} \sqrt{\frac{r_y}{2\pi}} \qquad (2.12)$$

Where μ is shear modulus, $\kappa = 3 - 4\nu$ (plane strain) and $\kappa = (3-\nu)/(1+\nu)$ (plane stress), r_{y} is the Irwin plastic zone correction, E' is the effective Young's modulus and a is the half crack length. The Irwin plastic zone correction for plane stress is

$$r_{\mathrm{y}} = \frac{1}{2\pi}\left(\frac{K_{\mathrm{I}}}{\sigma_{\mathrm{ys}}}\right)^2 \tag{2.13}$$

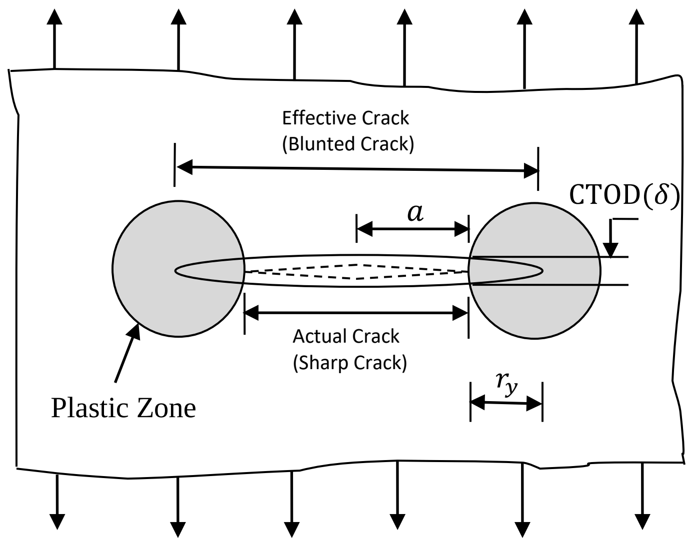

Figure 2.3: Crack tip opening displacement

Therefore in the limit of small scale yielding the CTOD (δ)can be related to the mode I SIF as

$$\delta = 2u_{\mathrm{y}} = \frac{4}{\pi}\frac{K_{\mathrm{I}}^2}{\sigma_{\mathrm{ys}}E} \tag{2.14}$$

Alternatively, CTOD can be related to the energy release rate by

$$\delta = \frac{4}{\pi}\frac{G}{\sigma_{\mathrm{ys}}} \tag{2.15}$$

On the other hand using the strip-yield model the relationship of CTOD with K_{I} and G can be written as

$$\delta = \frac{K_{\mathrm{I}}^2}{\sigma_{\mathrm{ys}}E} = \frac{G}{\sigma_{\mathrm{ys}}} \tag{2.16}$$

In strip-yield model, it is assumed that stress state is plane stress in nature and the material is non hardening. The actual relationship between CTOD and K_{I} and G depends on stress state and strain hardening. The more generalized form of this relationship can be expressed as

$$\delta = \frac{K_{\mathrm{I}}^2}{m\sigma_{\mathrm{ys}}E'} = \frac{G}{m\sigma_{\mathrm{ys}}} \tag{2.17}$$

Where m is a dimensionless constant that is approximately 1.0 for plane stress and 2.0 for plane strain.

2.3.3 Relationship between *J* and CTOD

For linear elastic material behavior, the relationship between CTOD and G is provided in the last subsection. As $J = G$ for linear elastic conditions, these equations also explain, in the limit of small-scale yielding, the relationship between CTOD and J. That is

$$J = m\sigma_{\mathrm{ys}}\delta \tag{2.18}$$

Shih (1981) provided the evidence that a unique J - CTOD relationship applies well beyond the validity limits of LEFM. He evaluated the crack tip displacements implied by the HRR solution and later related the crack tip displacement to J and flow properties. According to Shih the J – CTOD relationship is

$$\delta = \frac{d_{\mathrm{n}}J}{\sigma_0} \tag{2.19}$$

Where d_{n} is a dimensionless constant, given by

$$d_{\mathrm{n}} = \frac{2\tilde{u}_{\mathrm{y}}(\pi,n)\left[\frac{\alpha\sigma_0}{E}\left\{\tilde{u}_{\mathrm{x}}(\pi,n)+\tilde{u}_{\mathrm{y}}(\pi,n)\right\}\right]^{1/n}}{I_{\mathrm{n}}} \tag{2.20}$$

For linear elastic and elastic-plastic materials with small-scale yielding, the applied J , i. e. J_{app} is given by

$$J_{\mathrm{app}} = (1-\nu^2)K_{\mathrm{I}}^2/E \qquad (2.21)$$

From the calculated values of J-integral i. e. J_{app} from finite elements method, the studies of Rice and Tracey (1969) indicates through plots that a liner relationship arises between CTOD and J_{app} quite early in the loading history and this linear relation should continue as long as small deformation yielding conditions do not exceed.

Rice in 1973 suggested a relation

$$\delta_{\mathrm{t}} = 0.55 J_{\mathrm{app}}/\bar{\sigma}_{\mathrm{flow}} \qquad (2.22)$$

Where $\bar{\sigma}_{\mathrm{flow}}$ represents the flow stress in tension at an equivalent shear strain of $N/(1+N)$, or

$$\delta_{\mathrm{t}} = 0.55 J_{\mathrm{app}} \left[\frac{2}{\sqrt{3}}(1+\nu)(1+N)\sigma_0/NE\right]^N/\sigma_0 \qquad (2.23)$$

Where, σ_0 is the reference stress. This agrees with calculations by Tracey (1973) who found that the relation $\delta_{\mathrm{t}} = 0.54 J_{\mathrm{app}}/\bar{\sigma}_{\mathrm{flow}}$ arose for small-scale yielding in his finite element solutions for both hardening and non-hardening materials.

Depending on results using hardening singularity element with path independence J and the non-hardening singularity element with path dependent J, Tracey in 1976 suggested the following relation

$$\delta_{\mathrm{t}} = 0.54(1+N)J_{\mathrm{app}}/\bar{\sigma}_{\mathrm{flow}} \qquad (2.24)$$

Rice in 1968 suggested $\delta_{\mathrm{t}} = 0.67 J_{\mathrm{app}}/\sigma_0$ and Rice-Johnson (1970) used the non-hardening limit of the HRR singularity to obtain $\delta_{\mathrm{t}} = 0.79 J_{\mathrm{app}}/\sigma_0$. The experimental results of Robinson and Tetelman (1974) gave the relation $\delta_{\mathrm{t}} = J_{\mathrm{app}}/\sigma_0$, a somewhat larger δ_{t} value that would be predicted from all the analyses discussed previously.

2.3.4 Plastic Zone Shape

The plastic zone size and shape at the crack tip is important in the fracture behavior of non-brittle materials and often provides information for the path invariance character of J contour integrals. In the determination of plastic zone size, the yield criterion employed in a particular analysis is also important. The shape of the plastic zone depends on the stress field obtained from any elastic-plastic analysis in a body. For a two-dimensional deformation field the equation for plastic zone boundary as a function of K_{I}, defined by a radius at any point in case of straight crack is

$$r_{\mathrm{pz}}(\theta) = \frac{K_{\mathrm{I}}^2}{4\pi\sigma_{\mathrm{y}}^2}\left[(1-2\nu)^2(1+\cos\theta) + \frac{3}{2}sin^2\theta\right] \text{ for plane strain} \qquad (2.25)$$

$$r_{\mathrm{pz}}(\theta) = \frac{K_{\mathrm{I}}^2}{4\pi\sigma_{\mathrm{y}}^2}\left[(1+\cos\theta) + \frac{3}{2}sin^2\theta\right] \qquad \text{for plane stress} \qquad (2.26)$$

The above expressions have been derived using *von Mises* yield criterion, Anderson (2005).

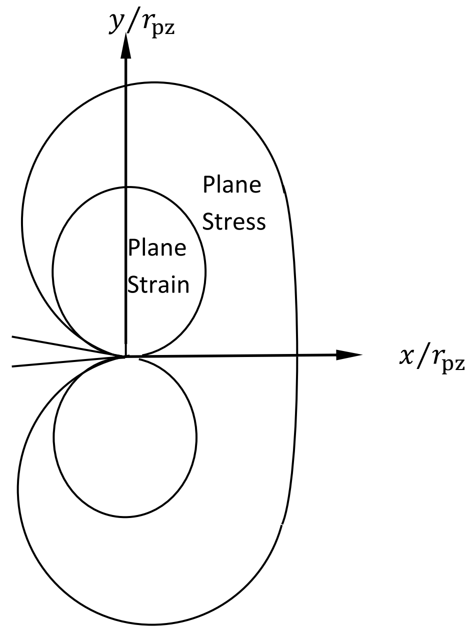

Figure 2.4: Comparison of plane stress and plane strain plastic zone boundaries

2.4 Fracture Mechanics of Pressure Sensitive Yielding Materials

Generally it is assumed that hydrostatic pressure has no effect on material plastic deformation and therefore, effect of plastic dilatancy is neglected in classical plasticity theory. Classical plasticity theories are mainly applicable to dense materials. In contrast, rocks, concretes, soils, and other porous materials exhibit pressure sensitive yielding and plastic volumetric deformation. In recent times, materials like toughened structural polymers, metallic foam, plastics and transformation toughened ceramics, vertically aligned carbon nanotubes (VACNTs) have attracted tremendous research attention due to some of their outstanding mechanical properties and experimental results on the mechanical behavior of these materials support a constitutive description that also accounts for pressure-sensitive yielding. Therefore, in such cases the J_2 flow theory becomes inadequate. For these materials, volumetric strains are usually closely related to the pressure sensitivity. It is considered that pressure sensitive yielding and plastic volumetric strains of these materials stem from a variety of factors like basic flow mechanism, cavitation, crazing in glassy polymers and several other interacting micro-mechanisms (Altenbach and Ochsner, 2014).

2.4.1 Studies based on Small Deformation Formulation

Over the years, a lot of research works have been carried out for stationary and growing cracks in plastically compressible and pressure sensitive solids by several researchers.

Using finite element analysis, Rudnicki and Rice (1975) investigated the conditions for localization of deformation in pressure sensitive dilatants materials. This localization of deformation into a shear band may be considered as a result of instability in the constitutive description of homogeneous deformation. It is argued in their analyses that there is presence of vertex type yield surface as well as non-normality flow rule during the stages of deformation of these pressure sensitive materials. Both the yield surface and nor-normality flow rule are shown to be destabilizing and they strongly influence the ensuing predictions for localization.

Biaxial tension tests on glassy bisphenol A-polycarbonate was performed by Carapellucci and Yee (1986) and they found that a modified Mises yield criterion, where the yielding is dependent on the hydrostatic stress, fitted their experimental data properly. Sue and Yee (1989) examined the toughening mechanisms involved in a multiphase alloy of Nylon 6, 6 / Polyphenylene oxide, and it was observed that there was a significant amount of plastic volumetric change in the composite material owing to the formation of crazes at large strain.

The pressure sensitive yielding phenomenon is as well observed in transformation toughened ZrO_2-containg ceramics (Chen and Reyes-Morel, 1986; Reyes-Morel and Chen, 1988).

Hwang and Luo (1988) provided the solutions for near-tip fields for steadily growing cracks in elastic perfectly - plastic compressible material. The study carried out by Spitzig and Richmond (1979) illustrates that for polyethylene and polycarbonate type of polymeric materials, the flow stress is significantly dependent on hydrostatic stress.

Based on the Drucker-Prager yield criterion, Li and Pan (1990), Chang et. al. (1997) studied the effects of pressure-sensitive yielding on crack-tip fields for power law hardening and perfectly plastic materials with K_{I} field boundary conditions. It has been observed that the HRR – type asymptotic crack-tip fields do exist for power law hardening materials. Their study also demonstrates that the pressure sensitivity can lower the hydrostatic and effective stress directly ahead of the crack tip.

For elastic-plastic pressure sensitive dilatants materials, Dong and Pan (1991) explored the crack-tip fields by finite element methods under plane strain and small scale yielding conditions. Their results indicate that the HRR-type asymptotic crack-tip solutions exist beyond the limiting value of the pressure sensitivity parameter for each hardening exponent in case of power law hardening materials.

The analytical solution of near tip stress fields for perfectly plastic pressure-sensitive material under plane stress condition was given by Li (1992). Asymptotic crack tip fields based on Drucker-Prager yield criterion have also been presented by Yuan and Lin (1993) and Yuan (1994/1995). Bigoni and Radi (1993) addressed to the study of mode I steady state crack propagation in elasto-plastic pressure sensitive solids with the Drucker-Prager yield condition and linear isotropic hardening.

Stoughton and Yoon (2004) proposed a non-associated flow rule based on a pressure sensitive yield criterion with isotropic hardening. The importance of their work is that the model distorts the shape of the yield function in tension and compression, fully accounting for strength differential effect.

The active yield surface equation during the loading / unloading process based on the Drucker - Prager yield function and a newly developed anisotropic hardening rule are illustrated in Choi and Pan (2009). The results based on their finite element analysis show that the critical locations for fatigue crack initiation, according to the stress distributions for pressure sensitive materials, agree well with the experimental observations.

2.4.2 Studies based on Large Deformation Formulation

Several experimental observations indicate that a hydrostatic tension plays a key role in the nucleation of crack in polymeric materials (Ishikawa et. al., 1977; Nimmer and Woods, 1992; Narisawa and Yee, 1993). A significant advancement in modeling the large inelastic deformation of amorphous polymers by shear yielding has been made by Parks et. al. (1984); Boyce et. al. (1988); Arruda and Boyce (1993); Wu and Van der Giessen (1993). These modeling are able to describe the experimental observations.

Using finite element finite strain analysis in front of a crack with a blunt notch, Lai and Van der Giessen (1997) reported the crack tip plastic zone and near tip fields in viscoplastic amorphous glassy polymers. Their results show a completely different kind of plastic zone in front of the crack tip characterized by softening immediately after yield, and there after followed by progressive strain hardening. It is also observed in their work that the distribution of hydrostatic stress is closely related to the pattern of the plastic deformation ahead of the crack tip.

Plane strain mode I crack growth resistance of metallic foams has been modeled by a cohesive zone model embedded within a compressible elastic plastic solid by Chen et.al (2001). Through their finite element finite deformation study under small scale yielding, the authors conclude that the cohesive zone model embedded within a plastically compressible elastic plastic solid has considerable promise to predict the crack growth resistance of metallic foams.

The deformation and failure response of a blunt notched specimen geometry and a sharp notched specimen geometry of polycarbonate have been analyzed by Gearing and Anand (2004). It is shown in their results that the constitutive model and failure criteria, when calibrated correctly, are capable to predict quantitatively the ductile fracture response of blunt-notched specimen as well as the competition between the ductile and brittle failure mechanisms in more sharply notched specimen.

Under mode I loading and small scale yielding conditions, the effects of crack-tip shape on the stress and deformation fields in front of blunted cracks in amorphous glassy polymers are numerically investigated by Li et. al. (2008).

Contrary to the polymers where softening takes place immediately after the yield, it has also been observed in the literature that some plastic materials like foams and VACNTs exhibit a hardening-softening-hardening type response. For example, very recently, using finite element method, the finite strain deformation behavior of a class of compressible elastic-viscoplastic solids with a hardening-softening-hardening variation of flow strength was studied at length for various kinds of geometries in Hutchens et al. (2011), Needleman et al. (2012), Mohan et al. (2013), Khan et al. (2017). It was found that the plastic compressibility coupled with such hardening-softening-hardening response leads to localized deformation.

2.5 Fracture Mechanics of Plastically Incompressible Materials: Crack Tip Blunting and Fields

In quantitative fracture mechanics, the crack tip surrounding stress and deformation fields provide a useful framework for assessing the fracture behavior, which are generally expressed in various formats in elastic and inelastic deformation ranges. In the studies of ductile fracture, crack tip blunting and fields have been studied by a number of investigators. In plastically incompressible solids, the material is characterized by J_2 flow theory.

The asymptotic crack-tip field for various power law hardening materials has been illustrated by Hutchinson (1968), Rice and Rosengren (1968). The path independent J -integral and the approximate analysis of strain concentration by notches and cracks were provided by Rice (1968).

The role of large crack tip geometry changes in plane strain fracture was illustrated by Rice and Johnson (1970). It has been described there that the inclusion of large geometry change effects in crack tip stress analysis reveals fairly different characteristics from those predicted through small deformation formulation based conventional analysis.

In case of plane strain large deformation, analyses of the stress and strain fields around smoothly-blunting crack tips subjected to mode I loads, have been carried out by use of finite element method for both hardening and non-hardening elastic-plastic solids (McMeeking, 1977). The results include the crack tip shape and near tip deformation field and CTOD has been related to a parameter of the applied load, the *J*-integral.

Relationships between the *J*-integral and the crack opening displacement for stationary and growing cracks are obtained by exploiting the dominance of the Hutchinson-Rice-Rosengren (HRR) singularity in the crack tip region (Shih, 1981).

Based on micro-mechanical considerations, Hutchinson (1983) provided a phenomenological constitutive behavior for the steady creep of polycrystalline materials undergoing creep constrained grain boundary cavitation. In order to reveal the role of cavitation on near tip-behavior, singular stress and strain-rate fields at the tip of a stationary plane strain crack were obtained.

For power law hardening orthotropic materials, Pan and Shih (1986) provided the solution for crack tip fields. By using slip line theory, McClintock (1971) showed various types of crack tip blunting for an initially sharp crack tip.

Exact solutions to the asymptotic problem of a two-dimensional crack propagating steadily and quasi-statically in a linear hardening material under various modes of loading have been presented by Amazigo and Hutchinson (1977), Castaneda (1987). Some issues arising in the coupling of finite strain plasticity theory and finite element formulations are discussed by Needleman (1985).

The effect of notch root radius on the material fracture toughness was investigated by Nishida et. al. (1994) and Damani et. al. (1996). Using Gurson's constitutive model and employing finite deformation plasticity theory, plane strain finite element analysis of a ductile three-point bend fracture specimen and a single edge notched specimen subjected to mode I quasistatic and dynamic loading is conducted by Basu and Narasimhan (1996), Narasimhan and Basu (1999), Basu and Narasimhan (2000). In this study, the interaction between the notch tip and a pre-nucleated hole ahead of it is modeled and several dynamic analyses are performed considering rate-independent as well as rate-dependent material behavior and with different impact speeds.

Constraint effects on ductile fracture process of microvoid growth and coalescence near a notch tip in a ductile material under mode I and mixed mode loading (involving modes I and II) are investigated by Roy and Narasimhan (1999). A comparison of Mode I and Mode III results for the elastic stress distribution in the immediate vicinity of a blunt notch was made by Smith (2004).

Taking the advantage of some analytical formulations which are able to describe stress distributions ahead of different shaped notches with end holes, Lazzarin et. al. (2011) discusses the form and the significance of the notch stress intensity factors with reference to in-plane shearing loading.

2.6 Fracture Mechanics under Fatigue Loading

A component, which fails at a high constant (monotonic) load, may fail under substantially smaller fatigue load. Flaws and crack like defects may be present anywhere in a structural component when the component taken into service. Based on the assumed initial flaw size, the time taken for this initial flaw to grow and reach a critical size is estimated. Over the years, numerous models have been developed to reveal the near-tip deformation, stress and strain fields under cyclic loading. Also, several realistic analyses of fatigue crack propagation have been performed numerically using various elasto-plastic constitutive models. A considerable amount of literature on this topic is available in Rice (1967), Hertzberg (1989), Kanninen and Popelar (1985), Suresh (1991).

2.6.1 Studies based on Small Deformation Formulation

In an elastic-perfectly plastic solid, Fleck (1986) modeled the fatigue crack growth using a two-dimensional finite element analysis. Plane deformations were enforced and using node release technique, crack advance was simulated. The analysis suggests that there is occurrence of crack closure for a transient period of crack growth and a residual wedge of material is being left on the crack flanks near the initial location of crack tip.

A model for the elastic-plastic finite element simulation of fatigue crack growth and crack closure is presented and evaluated by McClung and Sehitoglu (1989). Their results demonstrate that for reliable analysis, careful attention must be given to a series of critical decisions about mesh and model design. Of particular concern is the refinement of mesh along the crack line and the extension of crack growth beyond initial crack length.

An elastic-plastic finite element simulation of fatigue crack propagation, which accounts for plasticity-induced crack closure, has been used in order to study the size of the forward/ reversed plastic zones at the crack tip (McClung, 1991).

High resolution experimental measurements of near tip fatigue crack parameters are directly compared with elastic-plastic finite element simulations of growing fatigue cracks by McClung and Davidson (1991). Numerical and experimental results are shown to be qualitatively and often quantitatively consistent.

Ellyin and Wu (1992) proposed one elastoplastic constitutive model and they also implemented it in a finite element program to study crack front behavior under variable loading. Their study represents stress, strain and displacement distribution along a stationary crack front for constant amplitude cyclic loading with an overload cycle. The analysis predicts a decreased tensile stress and damage accumulation following an overload cycle.

Using an improved node release scheme, a comprehensive elastic-plastic constitutive model is employed in a finite element analysis of fatigue crack closure (Wu and Ellyin, 1996). Residual tensile deformation and residual compressive stress are found to be two major factors in determining the crack opening stress.

The role of kinematic hardening in strain localization phenomena in a single crystal under cyclic loading has been investigated in Flouriot et. al. (2003). Ratchetting phenomena are shown to take place in some of the localization bands.

2.6.2 Studies based on Large Deformation Formulation

Continuous efforts have been made to find the key results of fatigue crack growth for small deformation formulation as well as to incorporate large-deformation formulations into the modeling of fatigue cracking.

For isotropic materials, finite element simulations of the crack propagation using smooth plastic blunting and re-sharpening under cyclic loading have been described by Gu and Ritchie (1999), Tvergaard and Hutchinson (2002). In order to avoid severe distortions of the elements at the crack tip, the above simulations were carried out for reasonably large deformations at the crack tip for two and three cycles, respectively.

Considering large crack tip geometry changes, various finite element analyses of plane strain mode I tensile crack in elastic-plastic solids under cyclic loading were performed to elucidate the peculiarities of the near-tip large deformation, strain and stress fields (Toribio and Kharin, 1998, 2000, 2001, 2006, 2007, 2009). In these studies, the effects of load range, load ratio and overload on the near tip plastic zones, crack tip deformations, strains and stresses were illustrated. Also the role of other effects like effect of mesh sensitivity, role of strain hardening etc have been studied.

Recently, using repeated remeshing techniques during the stages of plastic deformation, fatigue crack growth calculations for many load cycles of an isotropic material model have been done in order to investigate the steady state crack growth, crack closure etc in Tvergaard (2004), Hunnell and Kujawski (2009), Levkovitch et al., (2005).

Using micromechanics based constitutive equation, plane strain finite deformation finite element calculations of mode I crack growth under small scale creep conditions are carried out in Wen et. al. (2017). Attention is confined to circumstances where the sole damage mechanism is the nucleation and growth of cavities on grain boundaries.

2.7 Fracture Mechanics of Elastic-Plastic Solids with Plastic Non-Normality

In the theories of plasticity of metals, it is assumed that the principal directions of plastic strain increment and deviatoric stress tensor are same and further this assumption is put into a relation which we call the associated flow rule. Plasticity theories of rocks also make use of a similar concept with the difference that the requirement of pressure-dependence of the yield surface entails a relaxation of the above assumption. Here, it is characteristically presumed that the plastic strain increment and the normal to the pressure-dependent yield surface have the same direction and under this condition the flow rule is set to be normality flow rule otherwise it is called non-normality flow rule. In all reported literature for material modeling, associated flow rule has been used despite the fact that the reliable modeling of such materials, where non-normality appears as a result of a dilatancy effect, induces the necessary introduction of the non-associative flow rule concept. Also in crystal plasticity for making the predictions of the model more accurate, a deviation from the normality rule is necessary. Some literature on this is available in Drucker and Prager (1952), Drucker et. al. (1957), Roscoe et. al. (1958), Mroz (1963), Mandel (1966), Nova and Wood (1978), Calvetti et. al. (2003), Nova (2004), Sumelka and Nowak (2016).

Using finite element analysis, Rudnicki and Rice (1975) investigated the conditions for localization of deformation in pressure sensitive materials like rocks. It is argued in their analyses that there must be presence of vertex type yield surface as well as non-normality flow rule during the stages of deformation of these pressure sensitive materials.

For ductile pressure sensitive materials, finite element simulations of crack growth have been performed by Aoki et. al. (1984 and 1987), Needleman and Tvergaard (1987 and 1992) in which a non-associative elasto-plastic formulation has been adopted based on the Gurson (1977) model.

Bigoni and Radi (1993) analyzed the quasi-static steady state crack propagation in a pressure sensitive elastic plastic solid with the use of volumetric non-associative flow law. Their study concludes that the singularity of the near tip fields appears to be primarily ruled by the flow-rule, and not by the gradient of yield surface.

Considering J_2 -plasticity or the Mises yield surface but allowing for deviation from normality flow rule, asymptotic analysis near the crack tip was presented by Nemat-Nasser and Obata (1990).

Needleman et. al. (2012), Singh and Khan (2018), has illustrated a few results for plastically compressible hardening-softening-hardening solids where the constitutive relation exhibits plastic non-normality.

2.8 Crack Growth Modeling Strategies

During the past few decades, many methods for crack propagation/growth simulation have been developed such as the node release technique, the extended finite element method (X-FEM), the cohesive zone model (CZM), , crack tip blunting model etc (Kim et al., 2012; Gu and Ritchie, 1999; Tvergaard, 2004) etc. For ready reference, brief descriptions of these methods are outlined below.

(a) Node Release Technique

For crack propagation, the easiest finite element method based on the Lagrangian approach uses a node release technique (Charalambides and McMeeking, 1987). In this procedure, at each growth step, the crack is enforced to move forward the entire distance between the two nodes of an element edge. Here, the two nodes are separated once a certain value, for instance, the stress or the stain attains a critical/limiting value at that node. Even though, the node release technique is simple to use but it cannot simulate a true discontinuity. In this method the mesh topology is repeatedly modified during the course of simulation.

(b) Extended Finite Element Method

The extended finite element method (X-FEM) was originally proposed by Belytschko and Black (1999). They presented this method for enriching the finite element approximations so that the crack growth problems can be solved with minimum remeshing. The essential idea in this method is to add discontinuous enrichment functions to the finite element approximations. In X-FEM, we first create a standard finite element mesh for the problem without accounting for the geometric entity. Then the crack is introduced separately to the mesh by enriching the standard displacement approximation with additional functions. Afterward, a much more well-designed technique was introduced by adapting an enrichment which includes the asymptotic near tip field as well as a Heavyside function H(x), (Moes et. al., 1999; Dolbow et. al., 2001). This method has also some limitations like spontaneous multiple crack initiation, branching, coalescence and the complicated numerical integration.

(c) Cohesive Zone Model

Barenblatt (1959) and Dugdale (1960) first proposed independently the cohesive zone model (CZM) which is a partially mechanistic approach to model the fracture process. In the development of this model, Barenblatt aimed to eliminate the stress singularity associated with LEFM using a cohesive fracture theory and Dugdale aimed to estimate the plastic zone size at the crack tip using a strip yield zone. The cohesive zone model is known as an effective method for crack initiation and propagation. The cohesive zone model is easy to implement in a finite element code and estimates the accurate results. The CZM is also useful to model the interface (Elices et. al, 2001: Alfano et al, 2009).

The CZM corresponds to the interaction between two separating surfaces which are assumed to exist ahead of the crack tip and the constitutive behaviour of the surfaces is called as the traction-separation law. This traction-separation law relates the stresses and displacements between the two separating surfaces. When the stress at the notch tip reaches the maximum stress in the traction-separation law, then only the cohesive zone is initiated. After the initiation, the stress is reduced at the notch tip with increase of separation of the surfaces in accordance with a pre-defined softening function. After complete separation of the two surfaces the stress falls to zero at the notch tip and subsequently fracture is initiated.

(d) Crack-Tip Blunting Model

Numerically, the most rigorous way for crack propagation modeling in ductile metals in Paris regime is to simulate the plastic sliding-off process at the tip of an advancing crack, (Laird and Smith, 1962). According to this model investigation focuses solely on simulation of fatigue crack growth due to crack tip plasticity only. In this mechanism of fatigue crack growth, crack growth is associated with crack-tip blunting under the maximum load and re-sharpening of the crack tip takes place under minimum load. In the loading and unloading stages in a single load cycle, the amount of displacement in the y direction of the crack surfaces with reference to initially sharp crack tip location (or as per the 90^0 intercept construction method) and the displacement in the x direction from the initial crack tip location obtained from the numerical calculations provide the basis of such model.

2.9 Computational Methods in Fracture Mechanics

The quantitative estimation of various fracture characterizing parameters involves estimation of the stress and strain fields associated with a loaded cracked body. Various methods exist to evaluate the stress and strain fields and some of the methods relevant in the context of fracture mechanics are illustrated below.

2.9.1 Methods of Stress Calculations

The differential equation for stress fields is derived using the equilibrium and compatibility conditions for any elastic problem. The different methods of finding the stress distribution in the context of fracture mechanics may be broadly grouped into the following categories

- Analytical methods
- Numerical methods

2.9.1.1 Analytical Methods

Analytical solutions of fracture problems are limited to idealized situations wherein the domain is considered to be infinite and the boundary conditions are relatively simple. The available solutions are restricted to only simple geometries due to the mathematical complexities. For a number of simple geometries having cracks located at different positions, the appropriate stress functions are available from the methods proposed by Westergaard (1959) or Mukhelishivilis (1953). Ever since the solutions for crack problems of Inglis (1913) numerous mathematical techniques have been developed for the analysis of cracked bodies under different conditions. For the calculation of SIF under elastic stress state, the continuum-based method was extensively used. Some of the significant techniques among them are Direct Method, Conformal Mapping, Polynomial Approximation, Laurent Series Expansion, Asymptotic Expansion, and Integral Transform techniques etc. These techniques are well suited for infinite fields.

The analytical complexity in solving the general class of crack problems with arbitrary geometry and complicated boundary conditions has resulted in the widespread use of numerical techniques. However, although it offers sufficiently accurate results even for complicated problems but not necessarily provide a fast and cheap computational means.

2.9.1.2 Numerical Methods

These methods use much simpler mathematical treatments and find applications in most practical problems. In general, these methods begin with an indirect solution under integral sign and obtain a direct numerical solution of the governing partial differential equation. The solution method reduces the integral equation into an infinite set of linear equations involving an infinite number of unknowns. The infinite set is then truncated in a suitable manner to obtain a finite set of algebraic equations valid at discrete set of points. The solution of this set of algebraic equations by suitable numerical scheme results in the direct numerical solution of the problem.

The main disadvantage in these methods is that the solutions are cumbersome and relatively slower compared to analytic solution for a similar problem, if available. The most popular methods are

- Finite Difference Method (FDM)
- Finite Element Method (FEM)
- Boundary Element Method (BEM)
- Meshless Method

(a) Finite Difference Method (FDM)

The finite difference method uses finite differential representation (Taylors series) of a function in a local domain and solves the differential field equations by considering values of the mesh points throughout the domain of interest to get an approximate solution. This results in a large system of algebraic equations and requires the considerable computational power for obtaining the solution. To obtain accurate results for the finite difference method, a good understanding of the effects of mesh discretization is also important.

(b) Finite Element Method (FEM)

In fracture mechanics, the cracked body is represented by a set of elements interconnected by nodal points. The mechanical behavior of elements is either known explicitly in terms of the force-displacements of the individual nodes constituting an element or may be derived assuming a stress-strain relation and performing numerical integration over the element volume. Once the displacement variations are known, the strain distributions may be obtained and subsequently using stress-strain relation the stress field is derived for a specific load. The development in the variety of elements has made the FEM technique suitable for modeling 2-D, 3-D, or quasi-three dimensional rotational symmetric structures. In addition, different types of material behavior like linear elastic, nonlinear elastic, elastic-plastic or thermal problems in fracture mechanics studies are easily handled. In fracture mechanics analyses, two approaches are generally followed - those based on conventional elements and those based on *singularity* elements.

The conventional finite elements suffer from the inability to represent satisfactorily the finite or infinite singularity in stress or strain concentrations. Several special elements have been developed and found extensive use in fracture analyses to represent the crack tip stress field more accurately. These elements include square root type singularity incorporated in their formulation stage (Byskov,1970; Barsoum, 1976).

Most of the finite element based computations undertaken for verification of path independence of *J* contour integral used small deformation theory which implies that the stresses and strains are computed from the original shape of the body and the shape is not redefined as the deformation proceeds. Incremental laws of plasticity with *von Mises* yield criteria are used with or without work hardening. Some of the earlier works in this regard were reported, Marcal and King (1969), Swedlow et al. (1965) and Sumpter and Turner (1976). All these computations ignored the paths very close to the crack tip to avoid the intense gradient of field quantities and *J* evaluated along different paths remained fairly constant within the limits of numerical accuracy. Considering paths involving different regions like wholly within plastic, partly plastic and partly elastic or completely within elastic regions, the constancy of the integral values was lost. Bakker (1982) used quarter point singularity element for 3-D case to show *J* values to be path independent and compared with experimental results yielding fairly good agreement. Virtual crack extension method modified by Bakker (1983) also improved results compared to other reported earlier literature.

(c) Boundary Element Method (BEM)

The boundary element method has emerged as a powerful alternative to FEM. The most important feature of the BEM is that it reduces the dimensionality of the problem by one, resulting in a much smaller system of equations and considerable reduction of data required for the analyses. The partial differential equation describing the material behavior within the domain and on the boundaries are transformed to an integral equation over the boundary of a domain and boundary of the domain must be sub-divided into segment, known as boundary elements. The method is based on Betti's reciprocal theorem and limited by the use of a reference loading with known solution in stress analyses. The above integral equations are subsequently evaluated by numerical means. The method is relatively easy for simpler geometries but to model a complicated cracked structure it is necessary to divide it into simpler sub-regions, more or less like FEM. With this method solutions at the interior points of the domain are not available and have to be evaluated from the solutions obtained at the boundary (Banerjee, 1981; Becker, 1992).

(d) Meshless Method

The mesh free method establishes a system of algebraic equations for the whole problem domain without the use of any predefined mesh. The set of nodes may be scattered in the domain or boundaries without forming any mesh and therefore, no relationship between the nodes is required for the interpolation of field quantities. In reality, it requires background cells for the integration of system matrices derived for the problem domain. For instance, in the stress analysis of a single domain with areas of stress concentration, one can easily include additional points without worrying about their relationship with the other existing nodes. In crack growth problems, nodes may be easily added around the crack tip to capture the desired accuracy in field variables and in fracture mechanics some work is reported in Belytschko et al (1994a, 1994b). There are a number of mesh free methods suitable for different applications and the technique is still continuously developing.

2.10 Concluding Remarks

Considering the potential applications of the relatively new materials mentioned above and their limited exploration till now as available from the literature review, the present author believes that there is a need to explore the behavior of notches and cracks for different strain hardening and hardening-softening-hardening materials following isotropic, plastically compressible rate dependent elastic-viscoplastic constitutive relation under various loading conditions. Moreover when the material is plastically compressible and hardness function is of the hardening-softening-hardening type, the application of different type of loadings may provide some new insights of the crack growth, near crack-tip deformation, stress and strain fields. Also the effect of non-normality may provide some interesting results. To the best of the author's knowledge such studies are not reported in the literatures so far. In order to investigate exhaustively the behavior of such newer materials with appropriate constitutive equation under various loading conditions, the necessary foundations in continuum mechanics approach along with the quasi-static convected coordinate formulation is briefly revisited and outlined in the next chapter.