APPENDIX

A) Static Condensation

In static condensation process a reduced set of equations are achieved by eliminating selected set of degree of freedom $\{\Delta_b\}$ and remaining only the desired degree of freedom $\{\Delta_c\}$. The mathematical description of static condensation is as follows.

The structural equilibrium equations are of the form,

$$[K]\{\Delta\} = \{P\} \tag{1}$$

Which, may be written as

$$\begin{bmatrix} K_{bb} & K_{bc} \\ K_{cb} & K_{cc} \end{bmatrix} \begin{Bmatrix} \Delta_b \\ \Delta_c \end{Bmatrix} = \begin{Bmatrix} P_b \\ P_c \end{Bmatrix} \tag{2}$$

We aim to eliminate degree of freedom $\{\Delta_b\}$

Now expending the upper part of the equation (2), we have

$$[K_{bb}]\{\Delta_b\} + [K_{bc}]\{\Delta_c\} = \{P_b\} \tag{3}$$

$$\{\Delta_b\} = [K_{bb}]^{-1}[\{P_b\} - [K_{bc}]\{\Delta_c\}] \tag{4}$$

Equation (4) may also be written as

$$\Delta_b = K_{bb}^{-1}(P_b - K_{bc}\Delta_c) \tag{5}$$

Now expending the lower portion of equation (2)

$$K_{cb}\Delta_b + K_{cc}\Delta_c = P_c \tag{6}$$

On putting the value of Δ_b from equation (5) into (6), we have

$$K_{cb}\left[K_{bb}^{-1}[P_b - K_{bc}\Delta_c]\right] + K_{cc}\Delta_c = P_c \tag{7}$$

$$-K_{cb}K_{bb}^{-1}K_{bc}\Delta_c + K_{cc}\Delta_c = P_c - K_{cb}K_{bb}^{-1}P_b \tag{8}$$

$$[-[K_{cb}][K_{bb}]^{-1}[K_{bc}] + K_{cc}]\{\Delta_c\} = \{P_c\} - [K_{cb}][K_{bb}]^{-1}\{P_b\} \tag{9}$$

So with the above definitions, we can write the following condensed set of equations

$$[\widehat{\mathrm{K}}_{\mathrm{cc}}]\{\Delta_{\mathrm{c}}\} = [\widehat{\mathrm{P}}_{\mathrm{c}}] \qquad (10)$$

Where

$\widehat{\mathrm{K}}_{\mathrm{cc}} = -\mathrm{K}_{\mathrm{cb}}\mathrm{K}_{\mathrm{bb}}\mathrm{K}_{\mathrm{bc}} + \mathrm{K}_{\mathrm{cc}}$

$\widehat{\mathrm{P}}_{\mathrm{c}} = \mathrm{P}_{\mathrm{c}} - \mathrm{K}_{\mathrm{cb}}\mathrm{K}_{\mathrm{bb}}\mathrm{P}_{\mathrm{b}}$

Equation (10) is the condensed set of equations. In the above set of condensed equations $\{\Delta_{\mathrm{c}}\}$ have been retained and $\{\Delta_{\mathrm{b}}\}$degree of freedom have been eliminated.

Now from equation (10), the condensed DOF $\{\Delta_{\mathrm{c}}\}$ are determined by inversion

$$\{\Delta_{\mathrm{c}}\} = [\widehat{\mathrm{K}}_{\mathrm{cc}}]^{-1}[\widehat{\mathrm{P}}_{\mathrm{c}}] \qquad (11)$$

The eliminated degree of freedom can be determined by putting the value of $\{\Delta_{\mathrm{c}}\}$ from equation (11) into equation (4)

$$\{\Delta_{\mathrm{c}}\} = [\mathrm{K}_{\mathrm{bb}}]^{-1}\left[\{\mathrm{P}_{\mathrm{b}}\} - [\mathrm{K}_{\mathrm{bc}}][\widehat{\mathrm{K}}_{\mathrm{cc}}]^{-1}\{\widehat{\mathrm{P}}_{\mathrm{c}}\}\right] \qquad (12)$$

Equation (12) is the recovery of eliminated degree of freedom.

B) Cholesky Decomposition

Cholesky decomposition is a special version of LU decomposition tailored to handle symmetric matrices more efficiently. For a symmetric matrix *A*, by definition,$a_{\mathrm{ij}} = a_{\mathrm{ji}}$. LU decomposition is not efficient enough for symmetric matrices. The computational load can be halved using Cholesky decomposition.

Using the fact that *A* is symmetric,

$$A = LL' \qquad (1)$$

Where *L* is a lower triangular matrix as

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & l_{33} & 0 & 0 \\ l_{\mathrm{n}1} & l_{\mathrm{n}2} & l_{\mathrm{n}3} & \cdots & l_{\mathrm{nn}} \end{bmatrix} \qquad (2)$$

With Cholesky decomposition, the elements of L are evaluated as follows:

$$l_{\mathrm{kk}} = \sqrt{a_{\mathrm{kk}} - \sum_{\mathrm{j}=1}^{\mathrm{k}-1} l_{\mathrm{kj}}^2} \quad \text{Where } \mathrm{k} = 1, 2, \ldots\ldots, n \qquad (3)$$

$$l_{\mathrm{ki}} = \frac{1}{l_{\mathrm{ii}}}\left(a_{\mathrm{ki}} - \sum_{\mathrm{j}}^{\mathrm{i}-1} l_{\mathrm{ij}} l_{\mathrm{kj}}\right) \quad \text{With i=1, 2,……,k-}1 \qquad (4)$$

Cholesky decomposition is evaluated column by column (Starting from the first column) and, in each row, the elements are evaluated from top to bottom. That in each column the diagonal element is evaluated first using equation (3) (the elements above the diagonal are zero) and then the other elements in the same row are evaluated next using equation (4). This is carried out for each column starting from the first one.

Once the Cholesky decomposition of A is done, the set of linear equations $Ax = b$ in the unknown vector x may be written as $LL'x = b$. Now, writing y for $L'x$, we get $Ly = b$, which may be solved for y, then $y = L'x$ is solved for x.

Note that if the value within the square root in equation (3) is negative, Cholesky decomposition will fail. However, this will not happen for positive semi definite matrices. However Cholesky decomposition is a good way to test for the positive semi definiteness of symmetric matrices.