CHAPTER
3

Chapter 3. PID Controller Design Techniques

This chapter is divided into four sections. Conventional techniques of PID controller design are discussed in the first section. Internal Model Control (IMC) based PID controller design techniques are discussed in the second section. Generalized procedure for development of (linearized) state space and transfer function models of nonlinear multivariable processes, useful in controller design, is described in section three. Finally, the techniques of process identification are described in section four.

3.1 Conventional PID Controller Design Techniques

The conventional PID controller design techniques can be broadly classified as:

a) Closed loop tuning methods, and

b) Open loop tuning methods.

3.1.1 Ziegler-Nichols Closed loop Method

The closed loop tuning method is based on the frequency response analysis of linear systems, when subjected to a periodic input (forcing function). The Ziegler-Nichols closed loop tuning method is also known as the ultimate gain method or the continuous cycling method (Ziegler & Nichols, 1993). In this method, the process is operated in closed loop using Proportional control action only. As the value of the proportional gain, K_C is increased; the closed loop response becomes more and more oscillatory. At the critical value of proportional gain, $K_C=K_{Cu}$ (known as the ultimate gain), the closed loop system exhibits continuous sustained oscillations and any further increase in the proportional gain K_C leads to instability. The time period of sustained oscillations (corresponding to $K_C= K_{Cu}$) is known as the critical or ultimate period of oscillation, P_u.

Based on the ultimate gain and period of oscillation, Ziegler-Nichols suggested a set of PID controller tuning parameters, as shown in Table 3.1

Table 3. 1 PID Controller Parameters: Ziegler-Nichols Closed loop Method

Controller	**Ziegler-Nichols (Z-N) Tuning Parameters**		
	K_C	τ_I	τ_D
P only	$0.5\ K_{Cu}$	-	-
PI	$0.45\ K_{Cu}$	$P_u/1.2$	-
PID	$0.6\ K_{Cu}$	$P_u/2$	$P_u/8$

3.1.2 Tyreus-Luyben Closed loop Method

Tyreus and Luyben (Luyben & Luyben, 1997) proposed slightly modified PID controller tuning parameters, originally based on the principle of continuous cycling method of Ziegler-Nichols. Their parameters show slight improvement over the Ziegler-Nichols parameters, with regard to handling process uncertainties and less oscillatory closed loop performance (Bequette, 2003).

Table 3. 2 PID Controller Parameters: Tyreus-Luyben

Controller	**Tyreus-Luyben Tuning Parameters**		
	K_C	τ_I	τ_D
PI	$K_{Cu}/3.2$	$2.2\ P_u$	-
PID	$K_{Cu}/2.2$	$2.2\ P_u$	$P_u/6.3$

3.1.3 Ziegler-Nichols Open loop Method

In order to overcome the limitations of closed loop PID control parameters, Ziegler and Nichols suggested PID controller tuning parameters listed in Table 3.3, on the basis of open loop step response of an identified Integrator Plus Dead Time (IPDT) process. They extended their results to an identified First Order Plus Dead Time (FOPDT) process as well. One quarter decay ratio performance criterion was selected. This method is also called the Process Reaction Curve (PRC) method (Ziegler & Nichols, 1993)

Table 3. 3 PID controller tuning parameters based on Ziegler-Nichols open loop method

Controller	**Ziegler-Nichols Open loop Tuning parameters**			
	K_C		τ_I	τ_D
	Integrator Plus Dead Time (IPDT) Process $g_p(s) = \frac{K}{s} e^{-\theta s}$	First Order Plus Dead Time (FOPDT) Process $g_p(s) = \frac{K_p}{(\tau_p s + 1)} e^{-\theta s}$		
P only	$\frac{1}{K\theta}$	$\frac{\tau_p}{K_p\theta}$	-	-
PI	$\frac{0.9}{K\theta}$	$\frac{0.9\,\tau_p}{K_p\theta}$	$3.3\,\theta$	-
PID	$\frac{1.2}{K\theta}$	$\frac{1.2\,\tau_p}{K_p\theta}$	$2\,\theta$	$0.5\,\theta$

3.1.4 Cohen-Coon Open loop Method

Cohen and Coon (Cohen & Coon, 1953) suggested the PID controller tuning parameters (Table3.4) based on the open loop step response of First Order Plus Dead Time (FOPDT) process model and one quarter decay ratio performance criterion. They further suggested that the open loop step response of any higher order system may be approximated by a First Order Plus Dead Time (FOPDT) process model and the PID controller is appropriately tuned based on the identified First Order Plus Dead Time (FOPDT) process model.

Table 3. 4 PID controller tuning parameters suggested by Cohen and Coon Open loop Method

Identified process	**First order plus dead time (FOPDT) model** $g_p(s) = \frac{K_p}{(\tau_p s + 1)} e^{-\theta s}$		
Controller	**Cohen and Coon Open loop Tuning Parameters**		
	K_C	τ_I	τ_D
P only	$\frac{\tau_p}{K_p\theta}\left[1 + \frac{\theta}{3\tau_p}\right]$	-	-
PI	$\frac{\tau_p}{K_p\theta}\left[0.9 + \frac{\theta}{12\tau_p}\right]$	$\frac{\theta\left[30 + 3\,\theta/\tau_p\right]}{9 + 20\,\theta/\tau_p}$	-
PID	$\frac{\tau_p}{K_p\theta}\left[\frac{4}{3} + \frac{\theta}{4\tau_p}\right]$	$\frac{\theta\left[32 + 6\,\theta/\tau_p\right]}{13 + 8\,\theta/\tau_p}$	$\frac{4\,\theta}{11 + 2\,\theta/\tau_p}$

3.2 Internal Model Control (IMC) based Controller Design Techniques

The Internal Model Control (IMC) based controller design technique involves the following steps:

a) Development of process (transfer function) model, $\bar{g}_p(s)$

The method of development of state space model and transfer function model of a general multivariable nonlinear process is elaborately described in Section 3.3.

b) Dead time/Time delay Approximation

Presence of dead time/time delay term introduces nonlinearity in a process transfer function. Hence it is important to appropriately approximate the dead time/time delay term (Mikleš & Fikar, 2007). Following are the three commonly used methods of approximation:

i. First order Pade' approximation:

Approximation of dead time/time delay term up to linear term gives:

$$e^{-\theta s} \approx \frac{-0.5\theta s + 1}{0.5\theta s + 1} \quad \text{Eq. 3. 1}$$

First order Pade' approximation introduces one additional pole and one additional zero (numerator dynamics) in a process transfer function. The additional pole is obtained by the solution of denominator polynomial as

$$0.5\theta s + 1 = 0 \quad \text{Eq. 3. 2}$$

$$p = \frac{-2}{\theta} \quad \text{Eq. 3. 3}$$

Since, the pole is located on the Left Half Plane (LHP), the first order Pade' approximation does not introduce any instability in the process. Similarly, the additional zero is obtained by the solution of numerator polynomial as

$$(-0.5\theta s + 1) = 0 \qquad \text{Eq. 3. 4}$$

$$z = \frac{2}{\theta} \qquad \text{Eq. 3. 5}$$

Since, the zero is located on the Right Half Plane (RHP), the first order Pade' approximation leads to single inverse response in open loop.

ii. Second order Pade' approximation:

Approximation of dead time/time delay term up to quadratic term gives:

$$e^{-\theta s} \approx \frac{\frac{\theta^2 s^2}{12} - \frac{\theta s}{2} + 1}{\frac{\theta^2 s^2}{12} + \frac{\theta s}{2} + 1} \qquad \text{Eq. 3. 6}$$

Second order Pade' approximation introduces two additional poles and two additional zeros (numerator dynamics) in a process transfer function. The additional poles are obtained by the solution of denominator quadratic polynomial (roots of s) as:

$$\frac{\theta^2 s^2}{12} + \frac{\theta s}{2} + 1 = 0 \qquad \text{Eq. 3. 7}$$

$$p_{1,2} = \frac{-3}{\theta} \pm \frac{\sqrt{3}}{\theta} j \qquad \text{Eq. 3. 8}$$

Since, the two poles are complex conjugates with negative real part, both the poles are open loop stable and hence the second order Pade' approximation does not introduce any instability in the process. Similarly, the two additional zeros are obtained by the solution of numerator quadratic polynomial (roots of s) as:

$$\frac{\theta^2 s^2}{12} - \frac{\theta s}{2} + 1 = 0 \qquad \text{Eq. 3. 9}$$

$$z_{1,2} = \frac{3}{\theta} \pm \frac{\sqrt{3}}{\theta} j \qquad \text{Eq. 3. 10}$$

Since, the two zeros are complex conjugates with positive real part, the two RHP zeros lead to double invers response in open loop.

iii. Taylor series approximation:

Approximation of dead time/time delay term up to linear term in Taylor series gives:

$$e^{-\theta s} \approx (1 - \theta s) \qquad \text{Eq. 3. 11}$$

The Taylor series approximation introduces no additional pole but only an additional zero (numerator dynamics) in a process transfer function. The additional zero is obtained by the solution of polynomial:

$$(1 - \theta s) = 0 \qquad \text{Eq. 3. 12}$$

$$z = \frac{1}{\theta} \qquad \text{Eq. 3. 13}$$

Since, the zero is located on the Right Half Plane (RHP), the Taylor series approximation leads to single inverse response in open loop.

c) Design of open loop model based controller, $q(s)$

The block diagram of open loop model based controller is shown in Figure 3.1.

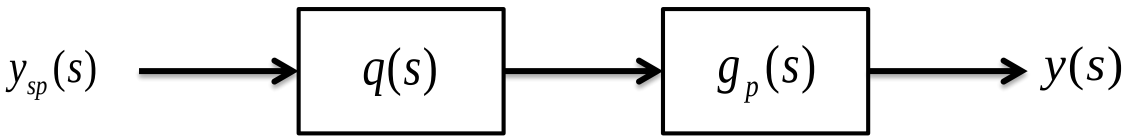

Figure 3. 1 Block diagram of open loop model based controller

The design of open loop model based controller involves obtaining a controller transfer function, $q(s)$, that provides an output $y(s)$ with desirable characteristics (fast response, minimum or no overshoot, zero offset, fewer or no oscillations etc.); subject to step change in set point. The model based controller, $q(s)$ is defined in terms of the inverse of process model. Based on the dynamic control law, the open loop model based controller is expressed as:

$$q(s) = \bar{g}_{p-}^{-1}(s) f(s) \qquad \text{Eq. 3. 14}$$

Where, $f(s)$ represents the Internal Model Control (IMC) filter. Proper selection of IMC filter $f(s)$ is very important in order to make the controller $q(s)$ proper and avoid internal instability (Morari & Zafiriou, 1989).

d) Selection of Internal Model Control (IMC) filter, $f(s)$

The two forms of IMC filter that are commonly used in controller design are:

i. Low pass filter, expressed as:

$$f(s) = \frac{1}{(\lambda s + 1)^n} \qquad \text{Eq. 3. 15}$$

ii. Modified filter, expressed as:

$$f(s) = \frac{(\gamma s + 1)}{(\lambda s + 1)^n} \qquad \text{Eq. 3. 16}$$

e) Factorization of process (transfer function) model

Since the model based controller $q(s)$ makes use of the inverse of process model, it is important to factorize the process model into invertible part, $\bar{g}_{p-}(s)$ and non-invertible part, $\bar{g}_{p+}(s)$ and use only the invertible part, $\bar{g}_{p-}(s)$ in controller design. A process model is factorized as:

$$\bar{g}_p(s) = \bar{g}_{p-}(s)\ \bar{g}_{p+}(s) \quad \text{Eq. 3. 17}$$

f) Design of closed loop IMC based PID controller

The open loop model based controller, $q(s)$ is effective only if the process model is perfect and there are no disturbances affecting the process. In practice, however, there is a mismatch between the actual process output and its model prediction. Moreover, the process may suffer from unmeasured disturbances/uncertainties. A closed loop control system is capable of overcoming the above shortcomings of an open loop control system. The block diagram of open loop model based control system is therefore modified into a closed loop system, as shown in Figure 3.2

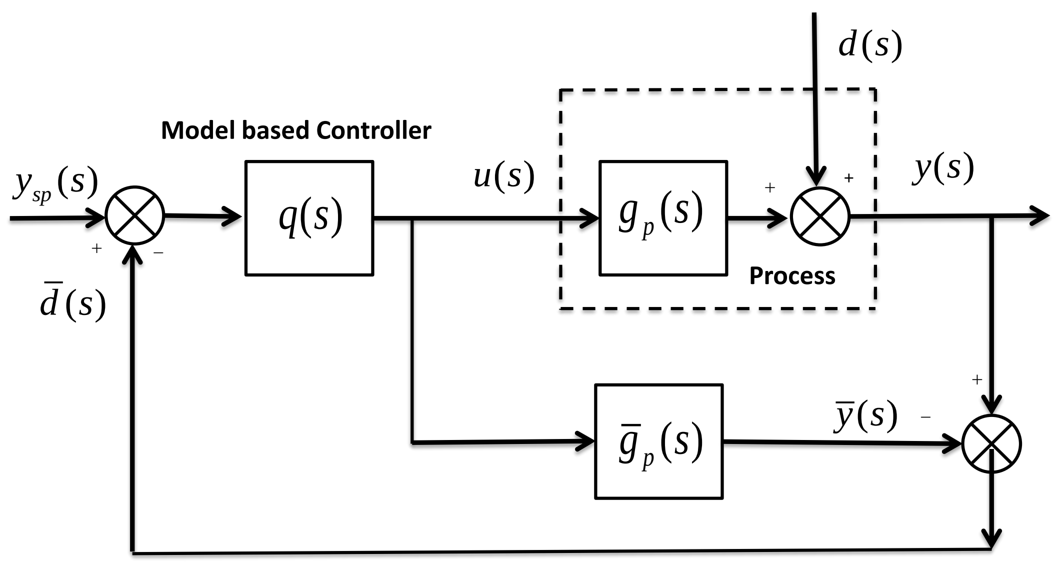

Figure 3. 2 Closed loop Model based Controller

The closed loop model based controller is further rearranged as a standard Internal Model Control (IMC) based PID feedback controller (Yesil et al., 2007), as shown in Figure 3.3.

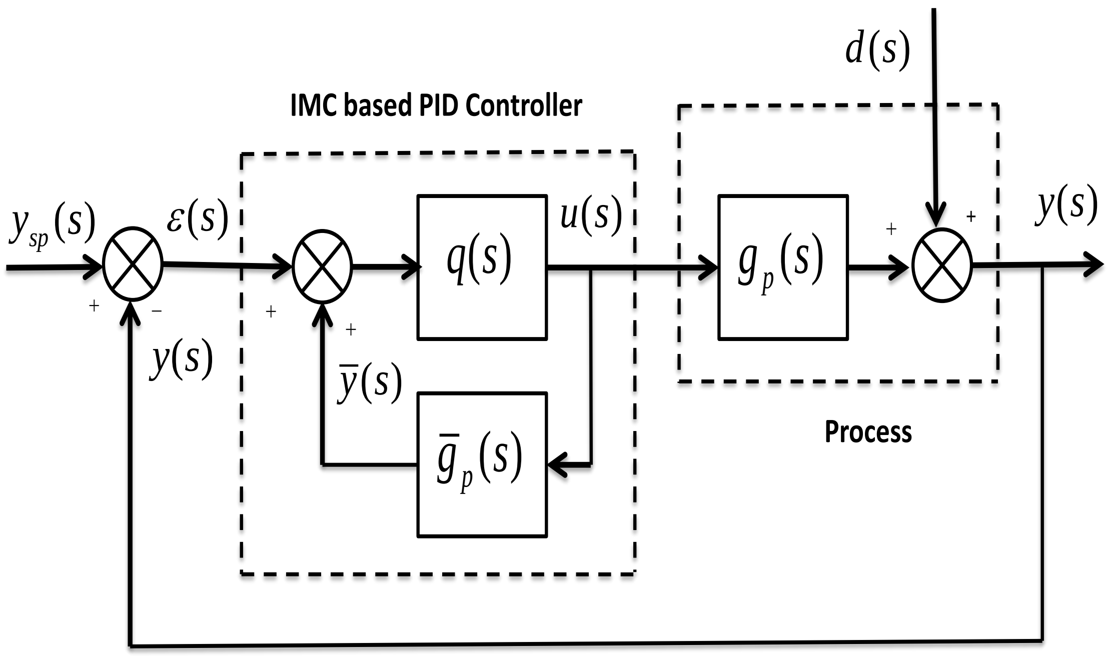

Figure 3. 3 Closed loop Block Diagram of IMC based Feedback Controller

Upon simplification of the closed loop block diagram shown in Figure 3.3, the Internal Model Control (IMC) based PID controller relationship is obtained as:

$$g_c(s) = \frac{u(s)}{\varepsilon(s)} = \frac{q(s)}{[1 - q(s)\bar{g}_p(s)]} \quad \text{Eq. 3. 18}$$

The IMC based PID controller derived in Equation 3.18 is finally compared with a standard PID controller, represented as:

$$g_c(s) = K_c(1 + \frac{1}{\tau_I s} + \tau_D s) = K_c \frac{(\tau_I \tau_D s^2 + \tau_I s + 1)}{\tau_I s} \quad \text{Eq. 3. 19}$$

The block diagram of Internal Model Control (IMC) based PID controller in standard form is shown in Figure 3.4.

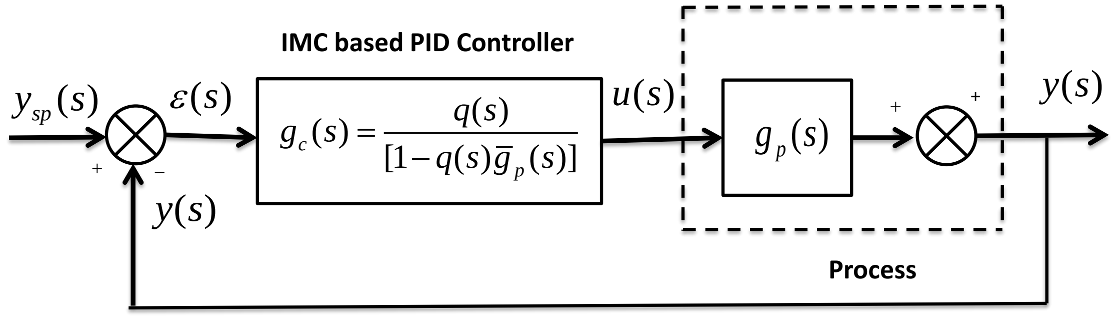

Figure 3. 4 Closed loop block diagram of IMC based PID controller

g) Closed loop transfer functions of IMC based PID controller

Based on the closed loop block diagram of Internal Model Control (IMC) based PID controller shown in Figure 3.4, the two closed loop transfer functions for set point and load change are derived as:

i. Close loop transfer function for set point change:

$$g_{sp}(s) = \frac{y(s)}{y_{sp}(s)} = \frac{q(s)g_p(s)}{1 + q(s)[g_p(s) - \bar{g}_p(s)]} \quad \text{Eq. 3. 20}$$

ii. Close loop transfer function for load change:

$$g_{load}(s) = \frac{y(s)}{d(s)} = \frac{[1 - q(s)\bar{g}_p(s)]}{1 + q(s)[g_p(s) - \bar{g}_p(s)]} \quad \text{Eq. 3. 21}$$

As a special case, when the process model is perfect (no mismatch with actual process) and there are no disturbances acting on the process, the IMC based PID controller behaves like an open loop model based controller, as shown in Figure 3.1.

The case of perfect model is mathematically expressed as:

$$g_p(s) = \bar{g}_p(s) \quad \text{Eq. 3. 22}$$

Substitution of equation 3.22 into equation 3.20 yields:

$$g_{sp}(s) = \frac{y(s)}{y_{sp}(s)} = q(s)g_p(s)$$ Eq. 3. 23

The block diagram of closed loop transfer functions is shown in Figure 3.5

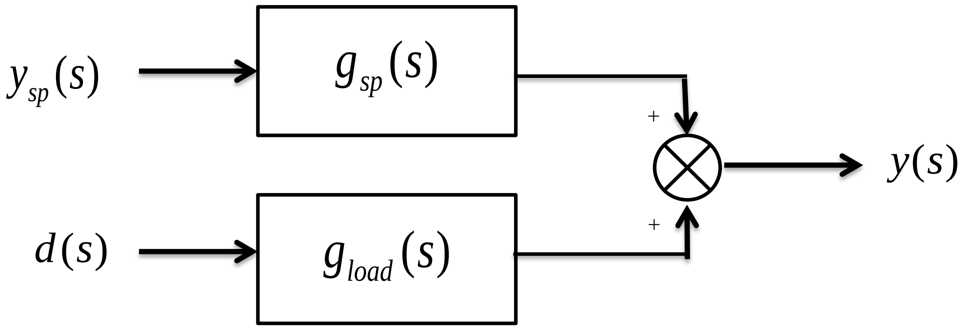

Figure 3. 5 Block Diagram of Closed loop Transfer Functions

Based on the steps outlined in section 3.2, an Internal Model Control (IMC) based PID controlled is designed for stable and unstable processes, as shown below:

3.2.1 IMC based PID Controller Design for First Order Stable Process

The transfer function of a first order stable process is represented as:

$$\bar{g}_p(s) = \frac{K_p}{(\tau_p s + 1)}$$ Eq. 3. 24

The invertible and non-invertible parts of the process transfer function are obtained by factorization (using equation 3.17) as:

$$\bar{g}_{p+}(\mathrm{s}) = 1$$ Eq. 3. 25

$$\bar{g}_{p-}(s) = \bar{g}_p(s) = \frac{K_p}{(\tau_p s + 1)}$$ Eq. 3. 26

The IMC filter is selected (based on equation 3.15) as:

$$f(s) = \frac{1}{(\lambda s + 1)} \qquad \text{Eq. 3. 27}$$

The model based controller, $q(s)$ is obtained (using Equation 3.14) as:

$$q(s) = \frac{1}{K_p} \frac{(\tau_p s + 1)}{(\lambda s + 1)} \qquad \text{Eq. 3. 28}$$

The IMC based PID controller, $g_c(s)$ is obtained (using Equation 3.18) as:

$$g_c(s) = \frac{\tau_p}{K_p \lambda} \frac{(\tau_p s + 1)}{\tau_p s} \qquad \text{Eq. 3. 29}$$

Comparing equation 3.29 with that of a standard Proportional Integral (PI) controller (equation 3.19), the PID controller parameters are obtained as:

$$K_c = \frac{1}{K_p} \frac{\tau_p}{\lambda} \qquad \text{Eq. 3. 30}$$

$$\tau_I = \tau_p \qquad \text{Eq. 3. 31}$$

The IMC based PID controller for a first order stable process results in a standard PI controller whose parameters are evaluated using equations 3.30 and 3.31, with λ as the tuning parameter.

3.2.2 IMC based PID Controller Deign for Second Order Stable Process

The transfer function of a second order stable process is represented as:

$$\bar{g}_p(s) = \frac{K_p}{(\tau_{p_1} s + 1)(\tau_{p_2} s + 1)} \qquad \text{Eq. 3. 32}$$

The invertible and non-invertible parts of the process transfer function are obtained by factorization (using equation 3.17) as:

$$\bar{g}_{p+}(\mathrm{s}) = 1 \qquad \text{Eq. 3. 33}$$

$$\bar{g}_{p-}(s) = \bar{g}_p(s) = \frac{K_p}{(\tau_{p_1}s+1)(\tau_{p_2}s+1)} \qquad \text{Eq. 3. 34}$$

The IMC filter is selected (based on equation 3.15) as:

$$f(s) = \frac{1}{(\lambda s+1)} \qquad \text{Eq. 3. 35}$$

The model based controller, $q(s)$ is obtained (using Equation 3.14) as:

$$q(s) = \frac{(\tau_{p_1}s+1)(\tau_{p_2}s+1)}{K_p(\lambda s+1)} \qquad \text{Eq. 3. 36}$$

The IMC based PID controller, $g_c(s)$ is obtained (using equation 3.18) as:

$$g_c(s) = \frac{(\tau_{p_1}+\tau_{p_2})}{K_p\lambda}\frac{\left[\tau_{p_1}\tau_{p_2}s^2+(\tau_{p_1}+\tau_{p_2})s+1\right]}{(\tau_{p_1}+\tau_{p_2})s} \qquad \text{Eq. 3. 37}$$

Comparing equation 3.37 with that of a standard Proportional Integral Derivative (PID) controller (equation 3.19), the PID controller parameters are obtained as:

$$K_c = \frac{(\tau_{p_1}+\tau_{p_2})}{K_p\lambda} \qquad \text{Eq. 3. 38}$$

$$\tau_I = (\tau_{p_1}+\tau_{p_2}) \qquad \text{Eq. 3. 39}$$

$$\tau_D = \frac{\tau_{p_1}\tau_{p_2}}{(\tau_{p_1}+\tau_{p_2})} \qquad \text{Eq. 3. 40}$$

The IMC based PID controller for a first order stable process results in a standard PID controller whose parameters are evaluated using equations 3.38-3.40, with λ as the tuning parameter.

3.2.3 IMC based PID Controller Deign for First Order Plus Dead Time Stable Process

The transfer function of a First order Plus Dead Time (FOPDT) Process is represented as:

$$\bar{g}_p(s) = \frac{K_p}{(\tau_p s + 1)} e^{-\theta s} \qquad \text{Eq. 3. 41}$$

Using first order Pade' approximation for dead time (equation 3.1), the process transfer function is represented as:

$$\bar{g}_p(s) = \frac{K_p}{(\tau_p s + 1)} \frac{(-0.5\theta s + 1)}{(0.5\theta s + 1)} \qquad \text{Eq. 3. 42}$$

The invertible and non-invertible parts of the process transfer function are obtained by factorization (using equation 3.17) as:

$$\bar{g}_{p+}(s) = (-0.5\theta s + 1) \qquad \text{Eq. 3. 43}$$

$$\bar{g}_{p-}(s) = \frac{K_p}{(\tau_p s + 1)(0.5\theta s + 1)} \qquad \text{Eq. 3. 44}$$

The IMC filter is selected (based on equation 3.15) as:

$$f(s) = \frac{1}{(\lambda s + 1)} \qquad \text{Eq. 3. 45}$$

The model based controller, $q(s)$ is obtained (using equation 3.14) as:

$$q(s)=\frac{1}{K_p}\frac{(\tau_p s+1)(0.5\theta s+1)}{(\lambda s+1)} \qquad \text{Eq. 3. 46}$$

The IMC based PID controller, $g_c(s)$ is obtained (using equation 3.18) as:

$$g_c(s)=\frac{(\tau_p+0.5\theta)}{K_p(\lambda+0.5\theta)}\frac{\left[0.5\theta\tau_p s^2+(\tau_p+0.5\theta)s+1\right]}{(\tau_p+0.5\theta)s} \qquad \text{Eq. 3. 47}$$

Comparing equation 3.47 with that of a standard Proportional Integral (PI) controller (equation 3.19), the PID controller parameters are obtained as:

$$K_c=\frac{(\tau_p+0.5\theta)}{K_p(\lambda+0.5\theta)} \qquad \text{Eq. 3. 48}$$

$$\tau_I=(\tau_p+0.5\theta) \qquad \text{Eq. 3. 49}$$

$$\tau_D=\frac{0.5\theta\tau_p}{(\tau_p+0.5\theta)} \qquad \text{Eq. 3. 50}$$

The IMC based PID controller for a first order stable process results in a standard PID controller whose parameters are evaluated using equations 3.48-50, with λ as the tuning parameter.

3.2.4 IMC based PID Controller Deign for First Order Unstable Process

Let the transfer function of a first order unstable process (having one RHP pole) be represented as:

$$\bar{g}_p(s)=\frac{K_p}{(-\tau_u s+1)} \qquad \text{Eq. 3. 51}$$

Where, the unstable pole is represented as:

$$p_u = \frac{1}{\tau_u}$$ Eq. 3. 52

The invertible and non-invertible parts of the process transfer function are obtained by factorization (using equation 3.17) as:

$$\bar{g}_{p+}(\mathrm{s}) = 1$$ Eq. 3. 53

$$\bar{g}_{p-}(s) = \bar{g}_p(s) = \frac{K_p}{(-\tau_u s + 1)}$$ Eq. 3. 54

The IMC filter is selected (based on equation 3.16) as:

Let $f(s) = \frac{(\gamma s + 1)}{(\lambda s + 1)^2}$ Eq. 3. 55

Since the additional parameter γ in the IMC filter defined above introduces an additional degree of freedom, the following additional condition is therefore used:

$$f(s)\big|_{s=p_u} = \frac{\left[\gamma(\frac{1}{\tau_u}) + 1\right]}{\left[\lambda(\frac{1}{\tau_u}) + 1\right]^2} = 1$$ Eq. 3. 56

A relationship between the two parameters (λ and γ) of the IMC filter is obtained from the solution of equation 3.56, as shown:

$$\gamma = \lambda\left(\frac{\lambda}{\tau_u} + 2\right)$$ Eq. 3. 57

The model based controller, $q(s)$ is obtained (using equation 3.14) as:

$$q(s) = \frac{(-\tau_u s + 1)}{K_p}\frac{(\gamma s + 1)}{(\lambda s + 1)^2}$$ Eq. 3. 58

The IMC based PID controller, $g_c(s)$ is obtained (using equations 3.18 and 3.58) as:

$$g_c(s) = \frac{(\gamma s + 1)}{K_p(2\lambda - \gamma)s} \qquad \text{Eq. 3. 59}$$

Comparing equation 3.59 with that of a standard Proportional Integral (PI) controller (equation 3.19), the PI controller parameters are obtained as:

$$K_c = \frac{1}{K_p}\frac{\gamma}{(2\lambda - \gamma)} \qquad \text{Eq. 3. 60}$$

$$\tau_I = \gamma = \lambda\left(\frac{\lambda}{\tau_u} + 2\right) \qquad \text{Eq. 3. 61}$$

The IMC based PID controller for a first order unstable process results in a standard PI controller whose parameters are evaluated using equations 3.60 and 3.61. λ as the tuning parameter.

3.2.5 IMC based PID Controller Deign for First Order Unstable Process (RHP pole and Numerator dynamics)

The transfer function of a first order unstable process having RHP pole and numerator dynamics is represented as:

$$\bar{g}_p(s) = \frac{K_p(\tau_n s + 1)}{(-\tau_u s + 1)(\tau_p s + 1)} \qquad \text{Eq. 3. 62}$$

Where, the unstable pole is represented as:

$$p_u = \frac{1}{\tau_u} \qquad \text{Eq. 3. 63}$$

The invertible and non-invertible parts of the process transfer function are obtained by factorization (using equation 3.17) as:

$$\bar{g}_{p+}(s) = 1 \qquad \text{Eq. 3. 64}$$

$$\bar{g}_{p-}(s) = \bar{g}_p(s) = \frac{K_p(\tau_n s+1)}{(-\tau_u s+1)(\tau_p s+1)} \qquad \text{Eq. 3. 65}$$

The IMC filter is selected (based on equation 3.16) as:

Let $f(s) = \dfrac{(\gamma s+1)}{(\lambda s+1)^2}$ Eq. 3. 66

The model based controller, $q(s)$ is obtained (using equation 3.14) as:

$$q(s) = \frac{(-\tau_u s+1)(\tau_p s+1)}{K_p(\tau_n s+1)} \frac{(\gamma s+1)}{(\lambda s+1)^2} \qquad \text{Eq. 3. 67}$$

Using the IMC filters parameter relationships (equations 3.56 and 3.57), the IMC based PID controller, $g_c(s)$ is obtained (using equations 3.18) as:

$$g_c(s) = \left(\frac{1}{K_p}\frac{\gamma}{(2\lambda-\gamma)}\frac{(\gamma s+1)}{\gamma s}\right)\left(\frac{(\tau_p s+1)}{(\tau_n s+1)}\right) \qquad \text{Eq. 3. 68}$$

Comparing equation 3.68 with that of a standard Proportional Integral (PI) controller (equation 3.19), the PI controller parameters are obtained as:

$$K_c = \frac{1}{K_p}\frac{\gamma}{(2\lambda-\gamma)} \qquad \text{Eq. 3. 69}$$

$$\tau_I = \gamma = \lambda\left(\frac{\lambda}{\tau_u}+2\right) \qquad \text{Eq. 3. 70}$$

Lead lag filter:

$$g_f(s) = \frac{(\tau_p s+1)}{(\tau_n s+1)} \qquad \text{Eq. 3. 71}$$

The IMC based PID controller for a first order unstable process having RHP pole and numerator dynamics results in a standard PI controller cascaded with a lead lag filter (equation 3.71). The PI parameters are evaluated using equations 3.69 and 3.70. λ as the tuning parameter of the IMC based PI controller.

3.3 Process Models for Controller Design

Considering a general nonlinear multivariable dynamic process model (in time domain) having 'n' number of state variables, 'm' number of input variables and 'r' number of measured output variables ($r \le n$), as shown:

$$\frac{dx_1}{dt} = f_1(x_1, x_2, ..., x_i, ..., x_n, \mathrm{u}_1, u_2, ..., u_j, ..., \mathrm{u}_m) \qquad \text{Eq. 3. 72}$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2, ..., x_i, ..., x_n, \mathrm{u}_1, u_2, ..., u_j, ..., \mathrm{u}_m) \qquad \text{Eq. 3. 73}$$

$$\vdots$$

$$\frac{dx_i}{dt} = f_i(x_1, x_2, ..., x_i, ..., x_n, \mathrm{u}_1, u_2, ..., u_j, ..., \mathrm{u}_m) \qquad \text{Eq. 3. 74}$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n(x_1, x_2, ..., x_i, ..., x_n, \mathrm{u}_1, u_2, ..., u_j, ..., \mathrm{u}_m) \qquad \text{Eq. 3. 75}$$

In vector-matrix notation,

the vector of state variables having dimensions (nx1) is defined as:

$$\mathbf{x} = (x_1, x_2, ..., x_i, ..., x_n)^T \qquad \text{Eq. 3. 76}$$

the vector of input variables having dimensions (mx1) is defined as:

$$\mathbf{u} = (\mathrm{u}_1, u_2, ..., u_j, ..., \mathrm{u}_m)^T \qquad \text{Eq. 3. 77}$$

the vector of nonlinear functions having dimensions (nx1) is defined as:

$$\mathbf{f} = (f_1, f_2, ..., f_i, ..., f_n)^T \qquad \text{Eq. 3. 78}$$

the vector of time derivatives of state variables having dimensions (nx1) is defined as:

$$\dot{\mathbf{x}} = (\dot{x}_1, \dot{x}_2, ..., \dot{x}_i, \cdots \dot{x}_n)^T = (\frac{dx_1}{dt}, \frac{dx_2}{dt}, ..., \frac{dx_i}{dt}, ... \frac{dx_n}{dt})^T \qquad \text{Eq. 3. 79}$$

In vector-matrix compact notation, the equations 3.72-3.75 are written as

$$\dot{\mathbf{x}} = \mathbf{f(x,u)} \qquad \text{Eq. 3. 80}$$

Under steady state conditions, equations 3.76-3.80 are correspondingly written as:

$$\mathbf{x_s} = (x_{1s}, x_{2s}, ..., x_{is}, ..., x_{ns})^T \qquad \text{Eq. 3. 81}$$

$$\mathbf{u_s} = (\mathrm{u}_{1s}, u_{2s}, ..., u_{js}, ..., \mathrm{u}_{ms})^T \qquad \text{Eq. 3. 82}$$

$$\mathbf{f_s} = (f_{1s}, f_{2s}, ..., f_{is}, ..., f_{ns})^T = \mathbf{0} \qquad \text{Eq. 3. 83}$$

$$\dot{\mathbf{x}}_\mathbf{s} = (\dot{x}_{1s}, \dot{x}_{2s}, ..., \dot{x}_{is}, \cdots \dot{x}_{ns})^T = (\frac{dx_{1s}}{dt}, \frac{dx_{2s}}{dt}, ..., \frac{dx_{is}}{dt}, ... \frac{dx_{ns}}{dt})^T = \mathbf{0} \qquad \text{Eq. 3. 84}$$

$$\dot{\mathbf{x}}_\mathbf{s} = \mathbf{f(x_s, u_s) = 0} \qquad \text{Eq. 3. 85}$$

Here the subscript, ‘s’ is used to denote the steady state condition.

Under steady state conditions, the vector equation 3.85 represents a set of nonlinear algebraic equations in ‘n’ unknown variables, whose solution is numerically obtained by multivariable Newton Raphson Method (Scarborough, 1958) (Appendix A). The tridiagonal matrix algorithm (Appendix C) is very useful in solving a set of linear (or linearized nonlinear) algebraic equations that are most frequently encountered in the steady state solution of all counter current flow equilibrium staged processes such as distillation columns, tray column absorbers etc.(Tomich, 1970) (Broyden, 1965) (Friday & Smith, 1964) (Tierney & Bruno, 1967)

3.3.1 Linearized State Space Model (Time domain)

Linearization of the nonlinear functions (equation 3.80) around their steady state operating point, is performed using the Taylor series expansion (up to linear terms only):

$$f_1 \approx f_{1s} + \left[(\frac{\partial f_1}{\partial x_1})|_s \, (x_1 - x_{1s}) + (\frac{\partial f_1}{\partial x_2})|_s \, (x_2 - x_{2s}) + ... + (\frac{\partial f_1}{\partial x_n})|_s \, (x_n - x_{ns}) \right] + \left[(\frac{\partial f_1}{\partial u_1})|_s \, (\mathrm{u}_1 - u_{1s}) + (\frac{\partial f_1}{\partial u_2})|_s \, (\mathrm{u}_2 - u_{2s}) + ... + (\frac{\partial f_1}{\partial u_m})|_s \, (\mathrm{u}_m - u_{ms}) \right] \quad \text{Eq. 3. 86}$$

$$f_2 \approx f_{2s} + \left[(\frac{\partial f_2}{\partial x_1})|_s \, (x_1 - x_{1s}) + (\frac{\partial f_2}{\partial x_2})|_s \, (x_2 - x_{2s}) + ... + (\frac{\partial f_2}{\partial x_n})|_s \, (x_n - x_{ns}) \right] + \left[(\frac{\partial f_2}{\partial u_1})|_s \, (\mathrm{u}_1 - u_{1s}) + (\frac{\partial f_2}{\partial u_2})|_s \, (\mathrm{u}_2 - u_{2s}) + ... + (\frac{\partial f_2}{\partial u_m})|_s \, (\mathrm{u}_m - u_{ms}) \right] \quad \text{Eq. 3. 87}$$

$$\vdots$$

$$f_n \approx f_{ns} + \left[(\frac{\partial f_n}{\partial x_1})|_s \, (x_1 - x_{1s}) + (\frac{\partial f_n}{\partial x_2})|_s \, (x_2 - x_{2s}) + ... + (\frac{\partial f_n}{\partial x_n})|_s \, (x_n - x_{ns}) \right] + \left[(\frac{\partial f_n}{\partial u_1})|_s \, (\mathrm{u}_1 - u_{1s}) + (\frac{\partial f_n}{\partial u_2})|_s \, (\mathrm{u}_2 - u_{2s}) + ... + (\frac{\partial f_n}{\partial u_m})|_s \, (\mathrm{u}_m - u_{ms}) \right] \quad \text{Eq. 3. 88}$$

The vector of state variables in deviation form, having dimensions (nx1), is defined as:

$$\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_i, ..., \hat{x}_n)^T = (x_1 - x_{1s}, x_2 - x_{2s}, ..., x_i - x_{is}, ..., x_n - x_{ns})^T \quad \text{Eq. 3. 89}$$

The vector of time derivatives of state variables in deviation form, having dimensions (nx1), is defined as:

$$\dot{\hat{\mathbf{x}}} = \left[\frac{d(x_1 - x_{1s})}{dt}, \frac{d(x_2 - x_{2s})}{dt}, ..., \frac{d(x_i - x_{is})}{dt}, ... \frac{d(x_n - x_{ns})}{dt} \right]^T \qquad \text{Eq. 3. 90}$$

The vector of input variables in deviation form, having dimensions (mx1), is defined as:

$$\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, ..., \hat{u}_j, ..., \hat{u}_m)^T = (\mathrm{u}_1 - u_{1s}, u_2 - u_{2s}, ..., u_j - u_{js}, ..., \mathrm{u}_m - u_{ms})^T \qquad \text{Eq. 3. 91}$$

vector of nonlinear functions in deviation form, having dimensions (nx1), is defined as:

$$\hat{\mathbf{f}} = (\hat{f}_1, \hat{f}_2, ..., \hat{f}_i, ..., \hat{f}_n)^T = (f_1 - f_{1s}, f_2 - f_{2s}, ..., f_i - f_{is}, ..., f_n - f_{ns})^T \qquad \text{Eq. 3. 92}$$

The square matrix of partial derivatives of nonlinear functions with respect to state variables, evaluated at steady state and having dimensions (nxn) is defined as:

$$\mathbf{A} = \begin{pmatrix} (\frac{\partial f_1}{\partial x_1})|_s & (\frac{\partial f_1}{\partial x_2})|_s \cdots & (\frac{\partial f_1}{\partial x_i})|_s \cdots & (\frac{\partial f_1}{\partial x_n})|_s \\ (\frac{\partial f_2}{\partial x_1})|_s & (\frac{\partial f_2}{\partial x_2})|_s \cdots & (\frac{\partial f_2}{\partial x_i})|_s \cdots & (\frac{\partial f_2}{\partial x_n})|_s \\ \vdots & \vdots & \vdots & \vdots \\ (\frac{\partial f_n}{\partial x_1})|_s & (\frac{\partial f_n}{\partial x_2})|_s \cdots & (\frac{\partial f_n}{\partial x_i})|_s \cdots & (\frac{\partial f_n}{\partial x_n})|_s \end{pmatrix} \qquad \text{Eq. 3. 93}$$

The matrix of partial derivatives of nonlinear functions with respect to input variables, evaluated at steady state and having dimensions (nxm) is defined as:

$$\mathbf{B} = \begin{pmatrix} (\frac{\partial f_1}{\partial u_1})|_s & (\frac{\partial f_1}{\partial u_2})|_s \cdots & (\frac{\partial f_1}{\partial u_j})|_s \cdots & (\frac{\partial f_1}{\partial u_m})|_s \\ (\frac{\partial f_2}{\partial u_1})|_s & (\frac{\partial f_2}{\partial u_2})|_s \cdots & (\frac{\partial f_2}{\partial u_j})|_s \cdots & (\frac{\partial f_2}{\partial u_m})|_s \\ \vdots & \vdots & \vdots & \vdots \\ (\frac{\partial f_n}{\partial u_1})|_s & (\frac{\partial f_n}{\partial u_2})|_s \cdots & (\frac{\partial f_n}{\partial u_j})|_s \cdots & (\frac{\partial f_n}{\partial u_m})|_s \end{pmatrix} \qquad \text{Eq. 3. 94}$$

Equations 3.86 to 3.88 are compactly written in vector matrix notation, in terms of deviation variables as:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\hat{\mathbf{u}}$$ Eq. 3. 95

In general notation, the superscript ^ in equation 3.95 is omitted and the linearized state space model (in time domain) is represented as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$ Eq. 3. 96

It is implied that all the variables in the vector-matrix equation 3.96 are expressed in deviation form.

If the state variables of a process are not measurable or if only a subset of state variables are require to be measured for the sake of controller design, the linearized state space model is accompanied by a measurement equation, as shown:

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$ Eq. 3. 97

Where, the measurement vector, $\mathbf{y}$ having dimensions (rx1), is represented as:

$$\mathbf{y} = (y_1, y_2, ..., y_k, ..., y_r)^T$$ Eq. 3. 98

The measurement vector, $\mathbf{y}$ is a subset of the state variable vector $\mathbf{x}$, such that $r \le n$.

The Matrices of constant coefficients $\mathbf{C}$ having dimensions (rxn) and $\mathbf{D}$ having dimensions (rxm) are appropriately chosen. In most cases,

$$\mathbf{D} = \mathbf{0}$$ Eq. 3. 99

3.3.2 Transfer Function Model (Laplace domain)

Equations 3.96 and 3.97 represent the linearized state space model of a process in time domain. Taking Laplace transform of the vector-matrix equations, 3.96 and 3.97:

$$L\{\dot{\mathbf{x}}\} = s\mathbf{x}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{x}(s) + \mathbf{B}\mathbf{u}(s) \qquad \text{Eq. 3. 100}$$

$$L\{\mathbf{y}\} = \mathbf{y}(s) = \mathbf{C}\mathbf{x}(s) \qquad \text{Eq. 3. 101}$$

Choosing the initial steady state as initial condition, results in:

$$\mathbf{x}(0) = \mathbf{0} \qquad \text{Eq. 3. 102}$$

Solution of equation 3.100 yields:

$$\mathbf{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\,\mathbf{u}(s) \qquad \text{Eq. 3. 103}$$

Substitution of equation 3.103 into 3.101 gives:

$$\mathbf{y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\,\mathbf{u}(s) \qquad \text{Eq. 3. 104}$$

Defining a multivariable process transfer function matrix, $\mathbf{G}(s)$, equation 3.104 is simplified as:

$$\mathbf{y}(s) = \mathbf{G}(s)\mathbf{u}(s) \qquad \text{Eq. 3. 105}$$

Where,

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \qquad \text{Eq. 3. 106}$$

Each element of the multivariable process transfer function matrix, $\mathbf{G}(s)$, relates the k^{th} output of a process to its j^{th} input. Scalar elements of the multivariable process transfer function matrix, $\mathbf{G}(s)$ are represented as:

$$\mathbf{G}(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s)\cdots & g_{1j}(s)\cdots & g_{1m}(s) \\ g_{21}(s) & g_{22}(s)\cdots & g_{2j}(s)\cdots & g_{2m}(s) \\ \vdots & \vdots & \vdots \quad \cdots & \vdots \\ g_{k1}(s) & g_{k2}(s)\cdots & g_{kj}(s)\cdots & g_{km}(s) \\ \vdots & \vdots & \vdots & \vdots \\ g_{r1}(s) & g_{r2}(s) & g_{rj}(s) & g_{rm}(s) \end{bmatrix}$$ Eq. 3. 107

A general element of the multivariable process transfer function matrix is represented as:

$$g_{kj}(s) = \frac{y_k(s)}{u_j(s)}$$ Eq. 3. 108

Based on equation 3.105, the transfer function of Single Input Single Output (SISO) process is represented as:

$$y_1(s) = g_{11}(s)u_1(s)$$ Eq. 3. 109

The block diagram of a Single Input Single Output (SISO) process is shown in Figure 3.6.

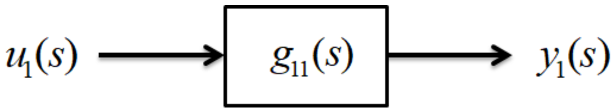

Figure 3. 6 Block diagram of Single Input Single Output (SISO) process

Similarly, based on equation 3.105, the transfer functions of a Two Input Two Output (TITO) process are represented as:

$$\begin{bmatrix} y_1(s) = g_{11}(s)u_1(s) + g_{12}(s)u_2(s) \\ y_2(s) = g_{21}(s)u_1(s) + g_{22}(s)u_2(s) \end{bmatrix}$$ Eq. 3. 110

The block diagram of a Two Input Two Output (TITO) process is shown in Figure 3.7.

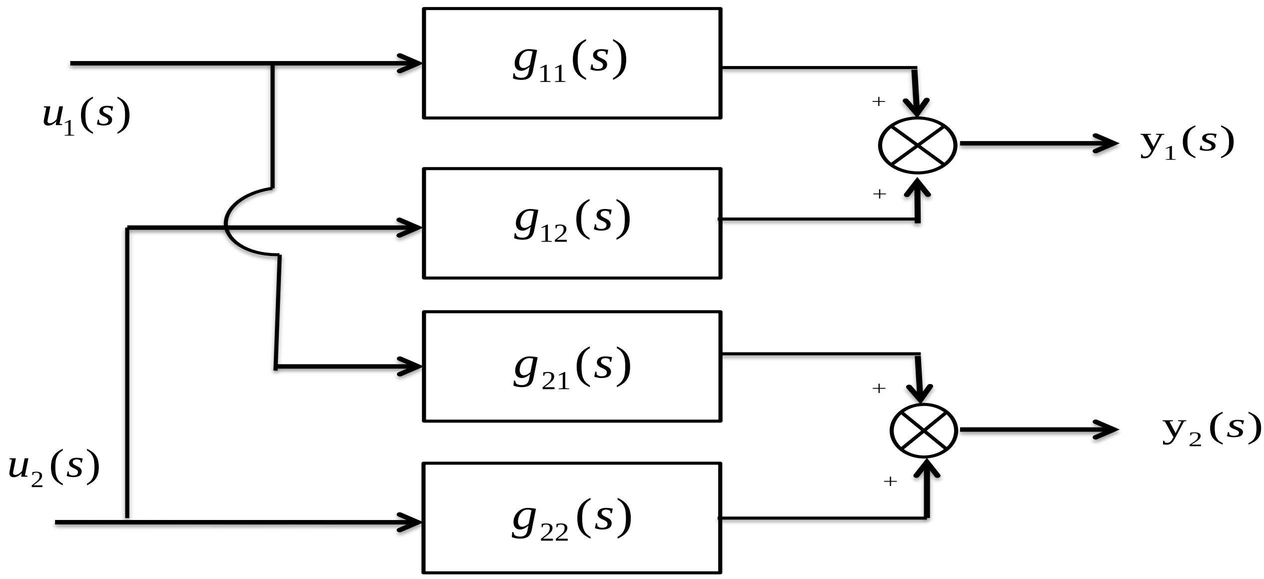

Figure 3. 7 Block diagram of Two Input Two Output (TITO) process

A description of the vectors and matrices used in the linearized state space and transfer function models, along with their dimensions, is given in Table 3.5

Table 3. 5 Dimensionality of State Space and Transfer Function Models

Notation	**Dimensions**	**Description**
x	(nx1)	Vector of state variables in deviation form
$\dot{\mathbf{x}}$	(nx1)	Vector of time derivatives of state variables in deviation form
u	(mx1)	Vector of input variables in deviation form
y	(rx1) $r \le n$	Vector of measured outputs in deviation form
A	(nxn)	Matrix of partial derivatives with respect to state variables evaluated at steady state
B	(nxm)	Matrix of partial derivatives with respect to input variables evaluated at steady state
C	(rxn)	Matrix of constant coefficients in measurement equation
D	(rxm)	Matrix of constant coefficients in measurement equation
G	(rxm)	Matrix of multivariable transfer functions

3.4 Process Identification

3.4.1 Identification from Open loop Step Response Experimental Data

The open loop step response experimental data is used for identification of transfer function models (Chidambaram, 1998). Cohen and Coon suggested that a higher order system can be approximated as a First Order Plus Dead-Time (FOPDT) process (Cohen & Coon, 1953). The transfer function model of a First Order Plus Dead-Time (FOPDT) FOPDT process is represented as:

$$g(s) = \frac{y(s)}{u(s)} = \frac{K_p}{\tau_p s + 1} e^{-\theta s} \qquad \text{Eq. 3. 111}$$

Where,

K_p represents the steady state gain of the process

τ_p represents the time constant of the process

θ represents the time delay/dead-time

For a step input of magnitude Δu in the process input variable:

$$\mathrm{u}(s) = \frac{\Delta u}{s} \qquad \text{Eq. 3. 112}$$

Substituting equation 3.112 into 3.111:

$$y(s) = g(s)u(s) = \frac{K_p \Delta u}{s(\tau_p s + 1)} e^{-\theta s} \qquad \text{Eq. 3. 113}$$

Taking invers Laplace transform of equation 3.113, the transient response (in time domain) is obtained as:

$$y(\mathrm{t}) = \begin{Bmatrix} 0; \quad \forall\, t < \theta \\ K_p \Delta u \left\{1 - \exp\left[-(t-\theta)/\tau_p\right]\right\}; \quad \forall\, t \geq \theta \end{Bmatrix} \qquad \text{Eq. 3. 114}$$

In equation 3.114, the unknown process parameter K_p is estimated from the ultimate (final steady state) value of response:

$$y(\infty) = K_p \Delta u \qquad \text{Eq. 3. 115}$$

The other (unknown) process parameters τ_p and θ are estimated from the step response experimental data. At this stage, the equation of $y(\mathrm{t})$ (equation 3.114) is nonlinear with respect to process parameters. 3.114 is therefore combined with equation 3.115 and rearranged as shown:

$$\frac{y(\infty) - \mathrm{y(t)}}{y(\infty)} = \exp\left[-(t-\theta)/\tau_p\right] \qquad \text{Eq. 3. 116}$$

Taking logarithm on both sides, equation 3.116 is transformed as:

$$\ln\left(\frac{y(\infty) - \mathrm{y(t)}}{y(\infty)}\right) = \frac{\theta}{\tau_p} - \frac{t}{\tau_p} \qquad \text{Eq. 3. 117}$$

Defining the transformed dependent variable (ordinate) as:

$$Y = \ln\left(\frac{y(\infty) - \mathrm{y(t)}}{y(\infty)}\right) \qquad \text{Eq. 3. 118}$$

Defining the transformed (yet to be estimated) parameters as:

$$\beta_1 = \frac{\theta}{\tau_p} \qquad \text{Eq. 3. 119}$$

$$\beta_2 = -\frac{1}{\tau_p} \qquad \text{Eq. 3. 120}$$

Equation 3.117 is written in terms of transformed variables and parameters as:

$$Y = \beta_1 + \beta_2 t \qquad \text{Eq. 3. 121}$$

Equation 3.121 is linear with respect to the two parameters, β_1 and β_2. The unknown parameters are estimated by any of the following two methods:

i. Graphical method:

In the graphical method, a straight line is plotted between the transformed dependent variable (ordinate), Y against the abscissa, t. β_2 is obtained from the slope of straight line and β_1 is obtained from the intercept (on the ordinate) of straight line.

ii. Linear regression (Least squares parameter estimation)

More accurate values of the unknown parameter vector $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$ are estimated using the method of linear regression (Ross, 2020), (Himmelblau, 1970) (Least squares parameter estimation). The linear least squares parameter estimation methodology is elaborately described in Appendix B.

Finally, equations 3.115, 3.119 and 3.120 are used to identify the unknown process parameters.

3.4.2 Identification from Steady State Input-Output Experimental Data

The steady state input-output experimental data is of utmost importance in understanding the nonlinear behaviour of a process. The chemical engineering processes are mostly nonlinear and exhibit higher order (sluggish) response. As suggested by Cohen and Coon, the higher order systems can be approximated as FOPDT processes.

First order flow systems are characterized by (a) their capacity to store mass or thermal energy and (b) their resistance offered to flow of mass or energy (Svrcek et al., 2014).

The (graphical) plots of steady state input-output experimental data is therefore very useful in establishing the linear/nonlinear relationships between the flows and system states. Typically, in first order liquid level systems, the following process parameters are estimated based on the steady state experimental data:

1. Linear/nonlinear flow resistance
2. Process time constant (product of capacitance and resistance)
3. Steady state process gain