CHAPTER 6

SOME INVESTIGATIONS ON THERMOELASTICITY THEORY WITH DUAL PHASE-LAGS

6.1 Stochastic Thermoelastic Interactions under Dual-Phase-Lag Model due to Deterministic and Random Temperature Distribution at the Boundary of a Half-Space[1]

6.1.1 Introduction

The present chapter is devoted to the study of dual-phase-lag thermoelasticity theory which is developed by Chandrasekharaiah (1998) on the basis of dual-phase-lag heat conduction law given by Tzou (1995b; 1995c). It has been realized by experimental observations that the fast-transient process of heat transfer in the gold film like structures involves micro-structural interaction effects (Brorson et al. (1987), Tzou (1995a)). In order to take into account these microscopic effects in heat transport mechanism, some models have been developed such as phonon-scattering model (Joseph and Preziosi (1989; 1990), Guyer and Krumhansl (1964)), phonon-electron interaction model (Bror-

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son et al. (1987), Anisimov et al. (1974), and Fujimoto et al. (1984)), and microscopic two-step model (Qiu and Tien (1992; 1993)). In 1995, Tzou (1995b; 1995c) has incorporated the effects of microstructural interactions in the fast transient process of heat transport phenomenon in a macroscopic formulation and proposed a more generalized law of heat conduction, known as dual-phase-lag heat conduction model. This dual-phase-lag heat conduction theory includes two different phase-lags associated with the heat-flux and the temperature gradient. The phase-lag associated to heat-flux (τ_q) emphasizes the fast-transient effects of thermal inertia while the phase-lag related to the temperature gradient (τ_θ) highlights the micro-structural interactions. This heat conduction model is further extended by Chandrasekharaiah (1998) to develop the generalized thermoelasticity theory that successfully overcomes the paradox of infinite speed of thermal waves. This theory is referred to as thermoelasticity with two phase-lags or dual phase-lags.

Dual-phase-lag thermoelastic model has been the area of interest to many researchers with respect to the study of the stability of the solution, various approximations, well-posedness, quality aspects, etc. Quintanilla (2003) has analyzed the exponential stability and well-posedness of dual-phase-lag thermoelasticity theory. Quintanilla and Racke (2006) have inspected the qualitative aspects of the dual-phase-lag theory. Roychoudhuri (2007b) has studied the problem of one-dimensional disturbances in an elastic half-space under two different boundary conditions. Mukhopadhyay et al. (2011a; 2014) have presented the domain of influence theorem, uniqueness theorem, variational theorem, and Danilovskaya's problem for this dual-phase-lag theory. In the present chapter, some aspects of dual-phase-lag thermoelasticity theory are analyzed by solving three different unsolved problems.

The main motive of the present subchapter is to investigate coupled thermoelastic interactions due to two distinct types of boundary conditions in the context of dual-phase-lag thermoelasticity theory. It is also aimed at investigating the effects of stochastic temperature distributions at the boundary of a half-space in the present context. A stochastic process is the mathematical tool that helps to deal with random nature of the system. It is basically a collection of random variables. Often, it makes the mathematical problem more realistic as compared to deterministic case due to the involvement of more possibilities by adding randomness to the problem. It facilitates the use of different samples along which the process can run despite the availability of initial conditions. However, some paths may be more probable than the others (Hoel et al. (1986), Platen and Kloeden (2006), and Lawler (2006)). There are many reasons that require replacing the rigorous deterministic model by stochastic model, out of which two reasons are worth mentioning (Bellomo and Flandoli (1989)). Firstly, incomplete isolation of the system that includes the interaction with a background field which gives rise to an additional noise. Secondly, the number of variables taken into account may not be sufficiently large in order to include all the variables representing the real physical system thus the remaining variables may give rise to some additional noise. Due to these reasons, there has been a popular trend to involve stochastic simulations to analyze many physical problems. Ahmadi (1978), Chen and Tien (1967), Kellar et al. (1978), and Tzou (1988) studied various thermal problems with random conductivity and problems in random medium. Ahmadi (1974), Chiba and Sugano (2007), and Gaikovich (1996) further studied the problems using random initial and boundary conditions. Val'kovskaya and Lenyuk (1996) considered the problems involving stochastic internal heat generation. Sherief et al. (2013; 2017) discussed the stochastic thermal shock problems in generalized thermoelasticity and generalized thermoelastic diffusion, respectively. Subsequently, Kant and Mukhopadhyay (2017; 2018) discussed the effects of the stochastic thermomechanical loading effects in the contexts of the theory of thermoelasticity without energy dissipation and the theory of thermoelasticity with two relaxation parameters.

This subchapter is arranged in the following way. In Subsection 6.1.2 and Subsection 6.1.3, respectively the basic equations are discussed and the problem under dual-phase-lag model for an isotropic homogeneous one-dimensional elastic half-space is formulated. Subsection 6.1.4 demonstrates the method of solving the problem and derives the solution of field variables in Laplace transform domain. In Subsection 6.1.5, the solution for temperature in physical domain is obtained. For deterministic case, two types of boundary conditions are considered. Case-I considers the application of thermal shock (Danilovskaya's Problem) at the stress free boundary of the half space, whereas Case-II considers ramp-type heating at the boundary. The solution in physical domain is obtained by using short time approximation and inverse Laplace transform. Analytical solutions of the field variables are analyzed. Next, in Sub-subsection 6.1.5.2, the stochastic type boundary conditions are introduced by incorporating the white noise to the deterministic ones and then it is described how to get the solution of the problem using the concept of Wiener process. Further, similar method is applied to find the solution for stress and displacement in both deterministic and stochastic case in Subsection 6.1.6 and Subsection 6.1.7, respectively. Subsection 6.1.8 involves the numerical computation of the problem for copper material to illustrate the problem as the special case. The solutions for the deterministic cases are potted and comparison with stochastic solution for different sample paths and the mean path is made. Lastly, in Subsection 6.1.9, the results are analyzed to mark the comparison between stochastic and deterministic type solutions of all the physical fields. It has been shown that prediction of finite wave speed by the present model is indicated in both the deterministic as well as stochastic boundary conditions.

6.1.2 Basic Equations

The dual-phase-lag heat conduction theory was proposed by Tzou (1995b; 1995c) in order to take into account the microscopic effects during heat transport process in a medium and this theory was extended to generalized thermoelasticity theory by Chandrasekharaiah (1998). Hence, following Tzou (1995b; 1995c) and Chandrasekharaiah (1998), the basic governing equations in the context of dual-phase-lag thermoelasticity in the absence of body forces and heat sources can be written as follows:

Modified heat conduction law with dual phase-lags:

$$\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)q_i=-K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\theta_{,i}. \tag{6.1.1}$$

Energy equation:

$$-q_{i,i}=\rho T_0\frac{\partial S}{\partial t}. \tag{6.1.2}$$

Entropy equation:

$$T_0\rho S=\rho c_E\theta+\beta T_0 e_{kk}. \tag{6.1.3}$$

Equation of motion:

$$\sigma_{ij,j}=\rho\ddot{u}_i. \tag{6.1.4}$$

Stress-strain-temperature relation:

$$\sigma_{ij}=\lambda e_{kk}\delta_{ij}+2\mu e_{ij}-\beta\theta\delta_{ij}. \tag{6.1.5}$$

Strain-displacement relation:

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}). \tag{6.1.6}$$

6.1.3 Problem Formulation

A problem of an isotropic and homogeneous elastic medium is taken. The formulation of problem is done for one-dimensional half-space, i.e, $x\geq 0$ in such a way that the boundary of the medium is experiencing zero stress and is subjected to a time-dependent temperature distribution. Each physical field variable is assumed to be bounded and vanishes as $x \to \infty$. Therefore, for one-dimension with displacement vector as $\vec{u} = (u(x,t), 0, 0)$, the governing equations can be obtained in the following way:

Using Eqs. (6.1.1-6.1.3) and Eq. (6.1.6), heat conduction equation is obtained as

$$K\left(1+\tau_\theta \frac{\partial}{\partial t}\right)\frac{\partial^2 \theta}{\partial x^2} = \left(1+\tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho c_E \frac{\partial \theta}{\partial t} + \beta T_0 \frac{\partial^2 u}{\partial x \partial t}\right). \tag{6.1.7}$$

Displacement equation of motion obtained by using Eqs. (6.1.4-6.1.6) takes the form

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda+2\mu)\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial \theta}{\partial x}. \tag{6.1.8}$$

Stress-displacement-temperature relation is derived using Eqs. (6.1.5-6.1.6) as

$$\sigma_{xx} = (\lambda+2\mu)\frac{\partial u}{\partial x} - \beta\theta. \tag{6.1.9}$$

For simplification of the problem, the following non-dimensional variables and parameters are involved :

$$x' = c_1\xi x,\ t' = c_1^2\xi t,\ \theta' = \frac{\theta}{T_0},\ u' = \frac{c_1(\lambda+2\mu)\xi u}{\beta T_0},\ \tau_q' = c_1^2\xi\tau_q,\ \tau_\theta' = c_1^2\xi\tau_\theta,$$
$$\varepsilon = \frac{\beta^2 T_0}{\rho^2 c_E c_1^2},\ \text{and}\ \sigma_{xx}' = \frac{\sigma_{xx}}{\beta T_0}.$$

where, $c_1 = \sqrt{\frac{(\lambda+2\mu)}{\rho}}$ is the speed of propagation of isothermal elastic waves, $\xi = \frac{\rho c_E}{K}$, and ε is the thermoelasticity coupling constant.

Therefore, using above non-dimensional variables and parameters, Eqs. (6.1.7-6.1.9) are transformed to the following forms:

$$\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\frac{\partial^2\theta}{\partial x^2}=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial\theta}{\partial t}+\varepsilon\frac{\partial^2 u}{\partial x\partial t}\right), \tag{6.1.10}$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}-\frac{\partial\theta}{\partial x}, \tag{6.1.11}$$

$$\sigma_{xx}=\frac{\partial u}{\partial x}-\theta. \tag{6.1.12}$$

Here, the primes are dropped for the convenience.

Initial and boundary conditions:

The boundary conditions are assumed in the following form:

$$\left.\begin{array}{l}\sigma_{xx}(x,t)|_{x=0}=0,\ u(x,t)|_{x=\infty}=0\ \text{ for } t>0\\ \theta(x,t)|_{x=0}=\theta_0(t),\ \theta(x,t)|_{x=\infty}=0\ \text{ for } t>0\end{array}\right\}, \tag{6.1.13}$$

where, $\theta_0(t)$ is any function of t. On the other hand, all the initial conditions are considered as homogeneous.

6.1.4 Solution of the Problem in the Laplace Transform Domain

After applying the Laplace transform on time, t to Eqs. (6.1.10-6.1.12), the following equations are obtained:

$$(nD^2-m)\bar{\theta}(s)=\varepsilon mD\bar{u}(s), \tag{6.1.14}$$

$$(D^2-s^2)\bar{u}(s)=D\bar{\theta}(s), \tag{6.1.15}$$

$$\bar{\sigma}_{xx}(s)=D\bar{u}(s)-\bar{\theta}(s), \tag{6.1.16}$$

where,

$$D=\frac{\mathrm{d}}{\mathrm{d}x},\ \ n=(1+\tau_\theta s),\ \ m=s\left(1+\tau_q s+\frac{1}{2}(\tau_q s)^2\right), \tag{6.1.17}$$

and $\bar{u}(s)$, $\bar{\theta}(s)$, and $\bar{\sigma}_{xx}(s)$ represent the Laplace transform of $u(t)$, $\theta(t)$, and $\sigma_{xx}(t)$, respectively.

Again, applying Laplace transform to boundary conditions (6.1.13) gives

$$\left.\begin{array}{c}\bar{\sigma}_{xx}(x,s)|_{x=0}=0,\ \bar{u}(x,s)|_{x=\infty}=0\\ \bar{\theta}(x,s)|_{x=0}=\bar{\theta}_0(s),\ \bar{\theta}(x,s)|_{x=\infty}=0\end{array}\right\}, \tag{6.1.18}$$

where, $\bar{\theta}_0(s)$ is the Laplace transform of $\theta_0(t)$.

Now, simplification of Eqs. (6.1.14-6.1.15) gives decoupled equations in terms of $\bar{u}$ and $\bar{\theta}$ as

$$[nD^4-(ns^2+m+m\varepsilon)D^2+ms^2](\bar{u},\bar{\theta})=0. \tag{6.1.19}$$

The corresponding auxiliary equation will be

$$nk^4-(ns^2+m+m\varepsilon)k^2+ms^2=0. \tag{6.1.20}$$

Since all the variables are vanishing as $x\to\infty$, only the roots with negative real parts of Eq. (6.1.20) are considered to avoid the positive powers of exponential while expressing the solution of differential equation (6.1.19).

Therefore, the solution of Eqs. (6.1.14-6.1.16) is acquired using the boundary condition (6.1.18) as

$$\bar{\theta}(x,\,s)=\frac{\bar{\theta}_0(s)}{(k_2^2-k_1^2)}[(s^2-k_1^2)\,e^{-k_1x}-(s^2-k_2^2)\,e^{-k_2x}], \tag{6.1.21}$$

$$\bar{u}(x,\,s)=\frac{\bar{\theta}_0(s)}{(k_2^2-k_1^2)}[k_1\,e^{-k_1x}-k_2\,e^{-k_2x}], \tag{6.1.22}$$

$$\bar{\sigma}_{xx}(x,\,s)=\frac{s^2\bar{\theta}_0(s)}{(k_2^2-k_1^2)}[e^{-k_2x}-\,e^{-k_1x}], \tag{6.1.23}$$

where, $-k_1$ and $-k_2$ are the roots of Eq. (6.1.20) such that $\mathrm{Re}(k_i)>0\,(i=1,2)$ and are functions of s alone.

6.1.5 Temperature Distribution in Physical (Space-Time) Domain

6.1.5.1 Deterministic Temperature

In order to obtain the solution of the problem in space-time domain, it is required to take the inverse Laplace transform of the solutions obtained in the previous subsection. Firstly, the deterministic case is considered and in order to evaluate field variables with deterministic type boundary conditions, two different types of such conditions are assumed. Hence, the boundary temperature $\theta_0(t)$ in Eq. (6.1.13) is defined in the following two ways:

Case-I: Thermal shock (Danilovskaya's Problem):

$$\theta_1(t) = \theta_0(t) = \theta^* \mathscr{H}(t) \quad \text{for } t > 0, \tag{6.1.24}$$

where, $\mathscr{H}(t)$ is Heaviside unit step function and θ^* is a constant temperature.

Case-II: Ramp-type heating at the boundary:

$$\theta_2(t) = \theta_0(t) = \theta^* \begin{cases} 0, & t \leq 0 \\ \frac{t}{t_c}, & 0 < t \leq t_c \\ 1, & t > t_c \end{cases}, \tag{6.1.25}$$

where, t_c is a constant time that is the fixed time of rise of ramp-type heating.

Applying Laplace transform to both the cases yields

Case-I:

$$\bar{\theta}_1(s) = \frac{\theta^*}{s} \tag{6.1.26}$$

Case-II:

$$\bar{\theta}_2(s) = \frac{\theta^*}{t_c s^2}(1 - e^{-t_c s}) \tag{6.1.27}$$

Taking Laplace inverse of Eqs. (6.1.21-6.1.23) to get a closed form solution of field variables is a formidable task. Therefore, an attempt to obtain the short time approximation is made for the large value of s to obtain the approximate solution of the problem. For this, the roots of Eq. (6.1.20) are considered as follows:

$$k_1 \approx \frac{s}{\nu_1} + \frac{1}{2}\frac{\lambda_2}{\lambda_1 \nu_1}, \tag{6.1.28}$$
$$k_2 \approx \frac{s}{\nu_2} + \frac{1}{2}\frac{\mu_2}{\mu_1 \nu_2}, \tag{6.1.29}$$

where,

$$\begin{aligned}
\lambda_1 &= a + \sqrt{a^2 - 4f},\ \mu_1 = a - \sqrt{a^2 - 4f}, \\
\nu_1 &= \sqrt{\frac{2\tau_\theta}{\lambda_1}},\ \nu_2 = \sqrt{\frac{2\tau_\theta}{\mu_1}}, \\
\lambda_2 &= b - \frac{a}{\tau_\theta} + \frac{\sqrt{a^2 - 4f}}{\tau_\theta}\left\{r\tau_\theta - 1\right\}, \\
\mu_2 &= b - \frac{a}{\tau_\theta} - \frac{\sqrt{a^2 - 4f}}{\tau_\theta}\left\{r\tau_\theta - 1\right\} \\
a &= \tau_\theta + \frac{1}{2}(1+\varepsilon)\tau_q^2,\ b = 1 + (1+\varepsilon)\tau_q, \\
d &= \tau_\theta \tau_q + \frac{1}{2}\tau_q^2,\ f = \frac{\tau_\theta \tau_q^2}{2}, \\
r &= \left(\frac{ab - 2d}{a^2 - 4f}\right).
\end{aligned} \tag{6.1.30}$$

It is noted that $a > 0$ and $a^2 - 4f = \frac{1}{4}\left[(2\tau_\theta - \tau_q)^2 + \varepsilon^2\tau_q^4 + 4\tau_\theta\tau_q^2\varepsilon\right] > 0$, which implies that $\sqrt{a^2 - 4f} > 0$.

Hence, substituting boundary conditions (6.1.26-6.1.27) and expressions from Eqs. (6.1.28-6.1.29) in Eq. (6.1.21) and then taking inverse, yield the following expression for temperature in case of deterministic distribution :

Case-I:

$$\theta^1(x,t) = \theta(x,t)$$
$$= \frac{\theta^*}{l_0}\left[e^{-\frac{\lambda_2 x}{2\lambda_1\nu_1}}\left\{\frac{(\nu_1^2-1)}{\nu_1^2} - \left(\frac{m_0}{l_0}\frac{(\nu_1^2-1)}{\nu_1^2} + \frac{\lambda_2}{\lambda_1\nu_1^2}\right)\left(t-\frac{x}{\nu_1}\right)\right\} h\left(t-\frac{x}{\nu_1}\right)\right.$$
$$\left. - e^{-\frac{\mu_2 x}{2\mu_1\nu_2}}\left\{\frac{(\nu_2^2-1)}{\nu_2^2} - \left(\frac{m_0}{l_0}\frac{(\nu_2^2-1)}{\nu_2^2} + \frac{\mu_2}{\mu_1\nu_2^2}\right)\left(t-\frac{x}{\nu_2}\right)\right\} h\left(t-\frac{x}{\nu_2}\right)\right], \quad (6.1.31)$$

Case-II:

$$\theta^2(x,t) = \theta(x,t)$$
$$= \frac{\theta^*}{l_0 t_c}\left[e^{-\frac{\lambda_2 x}{2\lambda_1\nu_1}}\left\{\frac{(\nu_1^2-1)}{\nu_1^2}\left(t-\frac{x}{\nu_1}\right) - \frac{1}{2}\left(\frac{m_0}{l_0}\frac{(\nu_1^2-1)}{\nu_1^2} + \frac{\lambda_2}{\lambda_1\nu_1^2}\right)\left(t-\frac{x}{\nu_1}\right)^2\right\} h\left(t-\frac{x}{\nu_1}\right)\right.$$
$$- e^{-\frac{\mu_2 x}{2\mu_1\nu_2}}\left\{\frac{(\nu_2^2-1)}{\nu_2^2}\left(t-\frac{x}{\nu_2}\right) - \left(\frac{m_0}{l_0}\frac{(\nu_2^2-1)}{\nu_2^2} + \frac{\mu_2}{\mu_1\nu_2^2}\right)\left(t-\frac{x}{\nu_2}\right)^2\right\} h\left(t-\frac{x}{\nu_2}\right)$$
$$- e^{-\frac{\lambda_2 x}{2\lambda_1\nu_1}}\left\{\frac{(\nu_1^2-1)}{\nu_1^2}\left(t-\frac{x}{\nu_1}-t_c\right) - \frac{1}{2}\left(\frac{m_0}{l_0}\frac{(\nu_1^2-1)}{\nu_1^2}\right.\right.$$
$$\left.\left. + \frac{\lambda_2}{\lambda_1\nu_1^2}\right)\left(t-\frac{x}{\nu_1}-t_c\right)^2\right\} h\left(t-\frac{x}{\nu_1}-t_c\right) + e^{-\frac{\mu_2 x}{2\mu_1\nu_2}}\left\{\frac{(\nu_2^2-1)}{\nu_2^2}\left(t-\frac{x}{\nu_2}-t_c\right)\right.$$
$$\left.\left. - \left(\frac{m_0}{l_0}\frac{(\nu_2^2-1)}{\nu_2^2} + \frac{\mu_2}{\mu_1\nu_2^2}\right)\left(t-\frac{x}{\nu_2}-t_c\right)^2\right\} h\left(t-\frac{x}{\nu_2}-t_c\right)\right], \quad (6.1.32)$$

where,

$$m_0 = \frac{\mu_2}{\nu_2^2\mu_1} - \frac{\lambda_2}{\nu_1^2\lambda_1},$$
$$l_0 = \frac{1}{\nu_2^2} - \frac{1}{\nu_1^2}.$$

It must be mentioned here that the case-I was studied by Roychoudhuri (2007a) and Mukhopadhyay et al. (2014) and similar results were reported.

6.1.5.2 Stochastic Temperature

Now, the temperature at the boundary of the half space is considered as stochastic distribution in the form

$$\theta_0(t) = \theta_i(t) + \psi_0(t), \tag{6.1.33}$$

where, $\theta_i(t)\,(i = 1, 2)$ is defined as in Eqs. (6.1.24-6.1.25) and $\psi_0(t)$ is a stochastic process based on the parameter t, satisfying

$$E[\psi_0(t)] = 0. \tag{6.1.34}$$

The stochastic process, ψ_0, is taken to be of white noise type as it is the most common type.

Recalling the property (Nowinski (1978) and Sherief et al. (2013)) satisfied by stochastic process $x(t)$ as

$$E[L\{x(t)\}] = L[E\{x(t)\}], \tag{6.1.35}$$

and since $\psi_0(t)$ is a stochastic process and each physical field involves boundary condition, therefore, each of them also becomes stochastic process mainly because of the random function $\psi_0(t)$. Therefore, Eq. (6.1.21) and Eq. (6.1.34) give

$$E[\bar{\theta}(x,\, s)] = L\,\{E[\theta(x,t)]\} = \frac{\bar{\theta}_0(s)}{(k_2^2 - k_1^2)}[(s^2 - k_1^2)\,e^{-k_1 x} - (s^2 - k_2^2)\,e^{-k_2 x}]. \tag{6.1.36}$$

It is observed that the mean of all the sample paths of the temperature field, $E[\theta(x,\, t)]$, is similar to the solution for both cases given in Eqs. (6.1.31-6.1.32) for the deterministic case.

Next, it is considered that

$$\overline{\Theta}(x, s) = \frac{1}{(k_2^2 - k_1^2)}[(s^2 - k_1^2)\,e^{-k_1 x} - (s^2 - k_2^2)\,e^{-k_2 x}]. \tag{6.1.37}$$

Therefore, Eq. (6.1.21) and Eq. (6.1.37) imply

$$\bar{\theta}(x,\, s) = \bar{\theta}_0(s)\bar{\Theta}(x,\, s). \tag{6.1.38}$$

Now, using Eq. (6.1.33), it is obtained that

$$\bar{\theta}(x,\, s) = \left(\bar{\theta}_i(s) + \bar{\psi}_0(s)\right)\bar{\Theta}(x,\, s). \tag{6.1.39}$$

Inverting the Laplace transform in above equation by using convolution property of Laplace inverse and the results from Eq. (6.1.21) and Eqs. (6.1.31-6.1.32) give the following:

$$\theta(x,t) = \theta^i(x,t) + \int\limits_0^t \psi(u)\Theta(x,\, t-u)\mathrm{d}u, \tag{6.1.40}$$

where, $\theta^i(x,t)(i = 1,\, 2)$ represents the deterministic temperature for two cases expressed as in Eqs. (6.1.31-6.1.32) and $\Theta(x,\, t)$ is the Laplace inverse of Eq. (6.1.37) obtained in the similar way as in Eqs. (6.1.31-6.1.32). Therefore,

$$\begin{aligned}\Theta(x,t) = \frac{1}{l_0}\Bigg[& e^{-\frac{\lambda_2 x}{2\lambda_1\nu_1}}\left\{\frac{(\nu_1^2-1)}{\nu_1^2}\delta\left(t-\frac{x}{\nu_1}\right) - \left(\frac{m_0}{l_0}\frac{(\nu_1^2-1)}{\nu_1^2} + \frac{\lambda_2}{\lambda_1\nu_1^2}\right)\right\} h\left(t-\frac{x}{\nu_1}\right)\\ & - e^{-\frac{\mu_2 x}{2\mu_1\nu_2}}\left\{\frac{(\nu_2^2-1)}{\nu_2^2}\delta\left(t-\frac{x}{\nu_2}\right) - \left(\frac{m_0}{l_0}\frac{(\nu_2^2-1)}{\nu_2^2} + \frac{\mu_2}{\mu_1\nu_2^2}\right)\right\} h\left(t-\frac{x}{\nu_2}\right)\Bigg],\end{aligned} \tag{6.1.41}$$

where, $\delta(t)$ represents the Dirac delta function.

Further, Eq. (6.1.40) can be written as

$$\theta(x,\, t) = \theta^i(x,\, t) + \int\limits_0^t \Theta(x,\, t-u)\,\mathrm{d}W(u), \tag{6.1.42}$$

where, $W(u)$ represents the Wiener process.

Therefore, stochastic temperature in different cases will be of the forms as given below:

Case-I:

$$\theta(x,\, t) = \theta^1(x,\, t) + \int\limits_0^t \Theta(x,\, t-u)\, \mathrm{d}W(u). \tag{6.1.43}$$

Case-II:

$$\theta(x,\, t) = \theta^2(x,\, t) + \int\limits_0^t \Theta(x,\, t-u)\, \mathrm{d}W(u). \tag{6.1.44}$$

Now, recalling one more property of Laplace Transform w.r.t. auto-correlation as (Nowinski (1978) and Sherief et al. (2013))

$$\overline{\overline{R}}_{xx}(s_1, s_2) = E[\overline{x}(s_1)\overline{x}(s_2)], \tag{6.1.45}$$

where, $R_{xx}(t_1, t_2) = E[x(t_1)x(t_2)]$ represents the auto-correlation function of a stochastic process $x(t)$ and

$$\overline{\overline{R}}_{xx}(s_1, s_2) = L_2\left\{R_{xx}(t_1, t_2)\right\} = \int\limits_0^\infty e^{-s_1 t_1}\left(\int\limits_0^\infty e^{-s_2 t_2} f(t_1, t_2)\mathrm{d}t_2\right)\mathrm{d}t_1, \tag{6.1.46}$$

where, $L_2(.)$ represents the the double Laplace transform.

Here, the process is assumed to be probabilistic stationary. A random process, $X(t)$, is called stationary if for all n and for every set of time instants $(t_i \,\epsilon\, T, i = 1, 2, ..n)$,

$$f_X(x_1, ..., x_n; t_1, ..., t_n) = f_X(x_1, ..., x_n; t_1 + \tau, ..., t_n + \tau),$$

where, f_X is joint probability density function. Therefore, the auto-correlation of a stationary function will be function of the time interval τ alone i.e.

$$E[X(t_1)X(t_2)] \equiv R_{XX}(\tau),\, \tau = t_1 - t_2. \tag{6.1.47}$$

From Eq. (6.1.21), Eq. (6.1.33), Eq. (6.1.34), Eq. (6.1.36), Eq. (6.1.37), and Eq.

(6.1.45), the following is acquired

$$
\begin{aligned}
&\overline{\overline{R}}_{\theta\theta}(x,\, s_1, s_2)\\
&= E[\bar{\theta}(x,\, s_1)\bar{\theta}(x,\, s_2)]\\
&= E\left[\frac{\bar{\theta}_0(s_1)}{(k_2^2(s_1)-k_1^2(s_1))}\left[(s_1^2-k_1^2(s_1))\, e^{-k_1(s_1)x}\right.\right.\\
&\left.-(s_1^2-k_2^2(s_1))\, e^{-k_2(s_1)x}\right]\cdot\frac{\bar{\theta}_0(s_2)}{(k_2^2(s_2)-k_1^2(s_2))}\left.[(s_2^2-k_1^2(s_2))\, e^{-k_1(s_2)x}-(s_2^2-k_2^2(s_2))\, e^{-k_2(s_2)x}]\right]\\
&= \overline{\Theta}(x, s_1)\overline{\Theta}(x, s_2)E[\bar{\theta}_0(s_1)\bar{\theta}_0(s_2)]\\
&= \overline{\Theta}(x, s_1)\overline{\Theta}(x, s_2)E\left[\left(\bar{\theta}_i(s_1)+\bar{\psi}_0(s_1)\right)\left(\bar{\theta}_i(s_2)+\bar{\psi}_0(s_2)\right)\right]\\
&= \overline{\Theta}(x, s_1)\overline{\Theta}(x, s_2)\left(E\left[\bar{\psi}_0(s_1)\bar{\psi}_0(s_2)\right]+\bar{\theta}_i(s_1)\bar{\theta}_i(s_1)\right)\\
&= \overline{\overline{R}}_{\psi_0\psi_0}(s_1, s_2)\overline{\Theta}(x, s_1)\overline{\Theta}(x, s_2)+E[\bar{\theta}(x,\, s_1)]E[\bar{\theta}(x,\, s_2)],
\end{aligned}
\tag{6.1.48}
$$

where, $\overline{\overline{R}}_{\psi_0\psi_0}(s_1, s_2) = E\left[\bar{\psi}_0(s_1)\bar{\psi}_0(s_2)\right]$.

In the present chapter, ψ_0 is assumed to be a white noise. Therefore, (Nowinski (1978)) yields

$$
R_{\psi_0\psi_0}(t_1,\, t_2) = \delta(\tau). \tag{6.1.49}
$$

From Eq. (6.1.46), the double Laplace of Eq. (6.1.49) can be obtained as

$$
\overline{\overline{R}}_{\psi_0\psi_0}(s_1, s_2) = \frac{1}{s_1+s_2}. \tag{6.1.50}
$$

Some properties of double Laplace transform can be stated as (Nowinski (1978)):

- $L_2\left\{f_1(x)f_2(x)\right\} = L\left\{f_1(x)\right\}L\left\{f_2(x)\right\}$,
- $\frac{1}{s_1+s_2}L_2\left\{f(x-\xi, y-\xi)\right\} = \begin{cases} L_2\left\{\int_0^x f(x-\xi, y-\xi)\mathrm{d}\xi\right\}, \text{ for } y>x \\ L_2\left\{\int_0^y f(x-\xi, y-\xi)\mathrm{d}\xi\right\}, \text{ for } y<x \end{cases}.$

Using above mentioned properties of double Laplace transform and Laplace transform, Eq. (6.1.48) and Eq. (6.1.50) give

$$\begin{aligned}
&\overline{\overline{R}}_{\theta\theta}(x,\, s_1, s_2)\\
&= \frac{1}{s_1+s_2} L_2\left\{\Theta(x,t_1)\Theta(x,t_2)\right\} + E[\bar{\theta}(x,\, s_1)]E[\bar{\theta}(x,\, s_2)]\\
&= L_2\left\{\int\limits_0^{t_1} \Theta(x,t_1-\xi)\Theta(x,t_2-\xi)\mathrm{d}\xi\right\} + L_2\left\{E[\theta(x,\, t_1)]E[\theta(x,\, t_2)]\right\},\ t_2 > t_1,
\end{aligned}$$

which implies

$$R_{\theta\theta}(x,\, t_1, t_2) = \int\limits_0^{t_1} \Theta(x,t_1-\xi)\Theta(x,t_2-\xi)\mathrm{d}\xi + E[\theta(x,\, t_1)]E[\theta(x,\, t_2)].$$

Taking $t_1 = t_2 = t$ above, implies

$$R_{\theta\theta}(x,\, t) = E[\theta^2(x,t)] = \int\limits_0^{t} \Theta^2(x,t-\xi)\mathrm{d}\xi + \left(E[\theta(x,\, t)]\right)^2.$$

Plugging $t-\xi=\vartheta$, gives

$$E[\theta^2(x,t)] = \int\limits_0^{t} \Theta^2(x,\vartheta)\mathrm{d}\vartheta + \left(E[\theta(x,\, t)]\right)^2.$$

Therefore, variance for the temperature distribution can be obtained as

$$Var[\theta(x,t)] = E[\theta^2(x,t)] - \left(E[\theta(x,\, t)]\right)^2 = \int\limits_0^{t} \Theta^2(x,\vartheta)\mathrm{d}\vartheta. \tag{6.1.51}$$

6.1.6 Stress Distribution

6.1.6.1 Deterministic Stress

In view of the deterministic boundary conditions (6.1.26-6.1.27) and following the same way as in subsection 6.1.5, the solution of stress field are obtained for both the cases in the following forms:

Case I:

$$\begin{aligned}\sigma^1_{xx}(x,t) &= \sigma_{xx}(x,t)\\ &= \frac{\theta^*}{l_0}\left[-e^{-\frac{\lambda_2 x}{2\lambda_1\nu_1}}\left\{1-\frac{m_0}{l_0}\left(t-\frac{x}{\nu_1}\right)\right\}h\left(t-\frac{x}{\nu_1}\right)\right.\\ &\left.+e^{-\frac{\mu_2 x}{2\mu_1\nu_2}}\left\{1-\frac{m_0}{l_0}\left(t-\frac{x}{\nu_2}\right)\right\}h\left(t-\frac{x}{\nu_2}\right)\right].\end{aligned} \quad (6.1.52)$$

Case II:

$$\begin{aligned}\sigma^2_{xx}(x,t) &= \sigma_{xx}(x,t)\\ &= \frac{\theta^*}{l_0 t_c}\left[-e^{-\frac{\lambda_2 x}{2\lambda_1\nu_1}}\left\{\left(t-\frac{x}{\nu_1}\right)-\frac{m_0}{2l_0}\left(t-\frac{x}{\nu_1}\right)^2\right\}h\left(t-\frac{x}{\nu_1}\right)\right.\\ &+e^{-\frac{\mu_2 x}{2\mu_1\nu_2}}\left\{\left(t-\frac{x}{\nu_2}\right)-\frac{m_0}{2l_0}\left(t-\frac{x}{\nu_2}\right)^2\right\}h\left(t-\frac{x}{\nu_2}\right)\\ &+e^{-\frac{\lambda_2 x}{2\lambda_1\nu_1}}\left\{\left(t-\frac{x}{\nu_1}-t_c\right)-\frac{m_0}{2l_0}\left(t-\frac{x}{\nu_1}-t_c\right)^2\right\}h\left(t-\frac{x}{\nu_1}-t_c\right)\\ &\left.-e^{-\frac{\mu_2 x}{2\mu_1\nu_2}}\left\{\left(t-\frac{x}{\nu_2}-t_c\right)-\frac{m_0}{2l_0}\left(t-\frac{x}{\nu_2}-t_c\right)^2\right\}h\left(t-\frac{x}{\nu_2}-t_c\right)\right].\end{aligned} \quad (6.1.53)$$

6.1.6.2 Stochastic Stress

Using the stochastic boundary condition as mentioned in Eq. (6.1.33) and proceeding in the same manner as in section 6.1.5, the following is obtained:

$$E[\bar{\sigma}_{xx}(x,\ s)] = L\left\{E[\sigma_{xx}(x,t)]\right\} = \frac{s^2\bar{\theta}_0(s)}{(k_2^2-k_1^2)}[e^{-k_2 x} -\ e^{-k_1 x}]. \quad (6.1.54)$$

Therefore, again it is observed that the mean of all the sample paths of the stress field, $E[\sigma_{xx}(x,\,t)]$, is similar to the solution for both cases given in Eqs. (6.1.52-6.1.53) for the deterministic case.

Now, considering

$$\bar{\Gamma}(x,s) = \frac{s^2}{(k_2^2 - k_1^2)}[e^{-k_2 x} - e^{-k_1 x}]. \tag{6.1.55}$$

Therefore, from Eq. (6.1.23) and Eq. (6.1.55), it is obtained that

$$\bar{\sigma}_{xx}(x,\,s) = \bar{\theta}_0(s)\bar{\Gamma}(x,\,s). \tag{6.1.56}$$

Further, Eq. (6.1.33) gives

$$\bar{\sigma}_{xx}(x,\,s) = \left(\bar{\theta}_i(s) + \bar{\psi}_0(t)\right)\bar{\Gamma}(x,\,s). \tag{6.1.57}$$

Taking Laplace inversion by using convolution and using the results of Eq. (6.1.23) and Eqs. (6.1.52-6.1.53) yield

$$\sigma_{xx}(x,t) = \sigma^i_{xx}(x,t) + \int_0^t \psi(u)\Gamma(x,\,t-u)\mathrm{d}u, \tag{6.1.58}$$

where, $\sigma^i_{xx}(x,t)\,(i=1,\,2)$ are given by Eqs. (6.1.52-6.1.53) and $\Gamma(x,\,t)$ is the Laplace inverse of Eq. (6.1.55) which can be expressed as

$$\begin{aligned}\Gamma(x,\,t) = \frac{1}{l_0}\Bigg[&-e^{-\frac{\lambda_2 x}{2\lambda_1\nu_1}}\left\{\delta\left(t-\frac{x}{\nu_1}\right)-\frac{m_0}{l_0}\right\}h\left(t-\frac{x}{\nu_1}\right)\\ &+e^{-\frac{\mu_2 x}{2\mu_1\nu_2}}\left\{\delta\left(t-\frac{x}{\nu_2}\right)-\frac{m_0}{l_0}\right\}h\left(t-\frac{x}{\nu_2}\right)\Bigg].\end{aligned} \tag{6.1.59}$$

Now, Eq. (6.1.58) can be rewritten as

$$\sigma_{xx}(x,t) = \sigma^i_{xx}(x,t) + \int_0^t \Gamma(x,\,t-u)\mathrm{d}W(u). \tag{6.1.60}$$

Therefore, the solution for stochastic stress in space-time domain for two different cases can be written as

Case-I:

$$\sigma_{xx}(x,t) = \sigma^1_{xx}(x,t) + \int_0^t \Gamma(x,\, t-u)\mathrm{d}W(u). \tag{6.1.61}$$

Case-II:

$$\sigma_{xx}(x,t) = \sigma^2_{xx}(x,t) + \int_0^t \Gamma(x,\, t-u)\mathrm{d}W(u). \tag{6.1.62}$$

Also, using the same argument as in section 6.1.5, variance for the stress distribution can be defined as

$$Var[\sigma_{xx}(x,t)] = \int_0^t \Gamma^2(x,\vartheta)\mathrm{d}\vartheta.$$

6.1.7 Displacement Distribution

6.1.7.1 Deterministic Displacement

Similar to above subsections, displacement in cases of two different deterministic boundary conditions can be expressed as follows:

Case-I:

$$\begin{aligned} &u^1(x,t) \\ &= \frac{\theta^*}{l_0}\left[e^{-\frac{\lambda_2 x}{2\lambda_1\nu_1}}\left\{\frac{1}{\nu_1}\left(t-\frac{x}{\nu_1}\right)-\frac{1}{2\nu_1}\left(\frac{m_0}{l_0}+\frac{\lambda_2}{2\lambda_1}\right)\left(t-\frac{x}{\nu_1}\right)^2\right\}h\left(t-\frac{x}{\nu_1}\right)\right. \\ &\left.-e^{-\frac{\mu_2 x}{2\mu_1\nu_2}}\left\{\frac{1}{\nu_2}\left(t-\frac{x}{\nu_2}\right)-\frac{1}{2\nu_2}\left(\frac{m_0}{l_0}+\frac{\mu_2}{2\mu_1}\right)\left(t-\frac{x}{\nu_2}\right)^2\right\}h\left(t-\frac{x}{\nu_2}\right)\right]. \end{aligned} \tag{6.1.63}$$

Case-II:

$$u^2(x,t)$$
$$= \frac{\theta^*}{l_0 t_c}\left[e^{-\frac{\lambda_2 x}{2\lambda_1\nu_1}}\left\{\frac{1}{2\nu_1}\left(t-\frac{x}{\nu_1}\right)^2-\frac{1}{6\nu_1}\left(\frac{m_0}{l_0}+\frac{\lambda_2}{2\lambda_1}\right)\left(t-\frac{x}{\nu_1}\right)^3\right\}h\left(t-\frac{x}{\nu_1}\right)\right.$$
$$-e^{-\frac{\mu_2 x}{2\mu_1\nu_2}}\left\{\frac{1}{2\nu_2}\left(t-\frac{x}{\nu_2}\right)^2-\frac{1}{6\nu_2}\left(\frac{m_0}{l_0}+\frac{\mu_2}{2\mu_1}\right)\left(t-\frac{x}{\nu_2}\right)^3\right\}h\left(t-\frac{x}{\nu_2}\right)$$
$$-e^{-\frac{\lambda_2 x}{2\lambda_1\nu_1}}\left\{\frac{1}{2\nu_1}\left(t-\frac{x}{\nu_1}-t_c\right)^2-\frac{1}{6\nu_1}\left(\frac{m_0}{l_0}+\frac{\lambda_2}{2\lambda_1}\right)\left(t-\frac{x}{\nu_1}-t_c\right)^3\right\}h\left(t-\frac{x}{\nu_1}-t_c\right)$$
$$\left.+e^{-\frac{\mu_2 x}{2\mu_1\nu_2}}\left\{\frac{1}{2\nu_2}\left(t-\frac{x}{\nu_2}-t_c\right)^2-\frac{1}{6\nu_2}\left(\frac{m_0}{l_0}+\frac{\mu_2}{2\mu_1}\right)\left(t-\frac{x}{\nu_2}-t_c\right)^3\right\}h\left(t-\frac{x}{\nu_2}-t_c\right)\right]. \quad (6.1.64)$$

6.1.7.2 Stochastic Displacement

Again, using the stochastic boundary condition as mentioned in Eq. (6.1.33) and proceeding in the same manner as in above subsections, the following is obtained:

$$E[\bar{u}(x,\, s)] = L\left\{E[u(x,t)]\right\} = \frac{\overline{\theta}_0(s)}{(k_2^2-k_1^2)}[k_1\, e^{-k_1 x} - k_2\, e^{-k_2 x}]. \quad (6.1.65)$$

Therefore, the mean of all the sample paths of the displacement field, $E[u(x,\, t)]$, is similar to the solution for both cases given in Eqs. (6.1.63-6.1.64) for the deterministic case.

Now, consider

$$\bar{U}(x,s) = \frac{1}{(k_2^2-k_1^2)}[e^{-k_2 x} - \, e^{-k_1 x}]. \quad (6.1.66)$$

Therefore, Eq. (6.1.22) and Eq. (6.1.66) return

$$\bar{u}(x,\, s) = \bar{\theta}_0(s)\bar{U}(x,\, s). \quad (6.1.67)$$

Further, from Eq. (6.1.33), it is obtained that

$$\bar{u}(x,\, s) = \left(\bar{\theta}_i(s) + \bar{\psi}_0(t)\right) \bar{U}(x,\, s). \tag{6.1.68}$$

Applying Laplace inversion by using convolution property and using the results of Eq. (6.1.22) and Eqs. (6.1.63-6.1.64) give

$$u(x,t) = u^i(x,t) + \int\limits_0^t \psi(u) U(x,\, t-u)\mathrm{d}u, \tag{6.1.69}$$

where, $u^i(x,t)\,(i=1,\,2)$ are given by Eqs. (6.1.63-6.1.64) and $U(x,\,t)$ is Laplace inverse of Eq. (6.1.66) which can be expressed as

$$\begin{aligned} U(x,\,t) = \frac{1}{l_0}\Bigg[& e^{-\frac{\lambda_2 x}{2\lambda_1\nu_1}}\left\{\frac{1}{\nu_1} - \frac{1}{\nu_1}\left(\frac{m_0}{l_0} + \frac{\lambda_2}{2\lambda_1}\right)\left(t - \frac{x}{\nu_1}\right)\right\} h\left(t - \frac{x}{\nu_1}\right) \\ & - e^{-\frac{\mu_2 x}{2\mu_1\nu_2}}\left\{\frac{1}{\nu_2} - \frac{1}{\nu_2}\left(\frac{m_0}{l_0} + \frac{\mu_2}{2\mu_1}\right)\left(t - \frac{x}{\nu_2}\right)\right\} h\left(t - \frac{x}{\nu_2}\right)\Bigg]. \end{aligned} \tag{6.1.70}$$

Now, Eq. (6.1.69) can be modified as

$$u(x,t) = u^i(x,t) + \int\limits_0^t U(x,\, t-u)\mathrm{d}W(u). \tag{6.1.71}$$

Therefore, stochastic displacement for two different cases can be written as

Case-I:

$$u(x,t) = u^1(x,t) + \int\limits_0^t U(x,\, t-u)\mathrm{d}W(u). \tag{6.1.72}$$

Case-II:

$$u(x,t) = u^2(x,t) + \int\limits_0^t U(x,\, t-u)\mathrm{d}W(u). \tag{6.1.73}$$

With the previous subsection's discussion, variance for the displacement distribution can be given as

$$Var[u(x,t)] = \int\limits_0^t U^2(x,\vartheta)\mathrm{d}\vartheta.$$

6.1.8 Numerical Results

In this subsection, the present problem is illustrated by finding numerical solution by considering copper material with the following materialistic data (Sherief et al. (2013)):

$$\lambda = 7.76{\times}10^{10}\ \text{kg}\ m^{-1}s^{-2},\ \mu = 3.86 \times 10^{10}\ \text{kg}\ m^{-1}s^{-2},\ \rho = 8954\ \text{kg}\ m^{-3},$$
$$c_E = 383.1\ \text{J kg}^{-1}\text{K},\ \beta_\theta = 1.78 \times 10^{-5} K^{-1},\ K = 386\ \text{W}m^{-1}K^{-1},\ T_0 = 293\,\text{K}.$$

The value of non-dimensional entities are taken as follows:

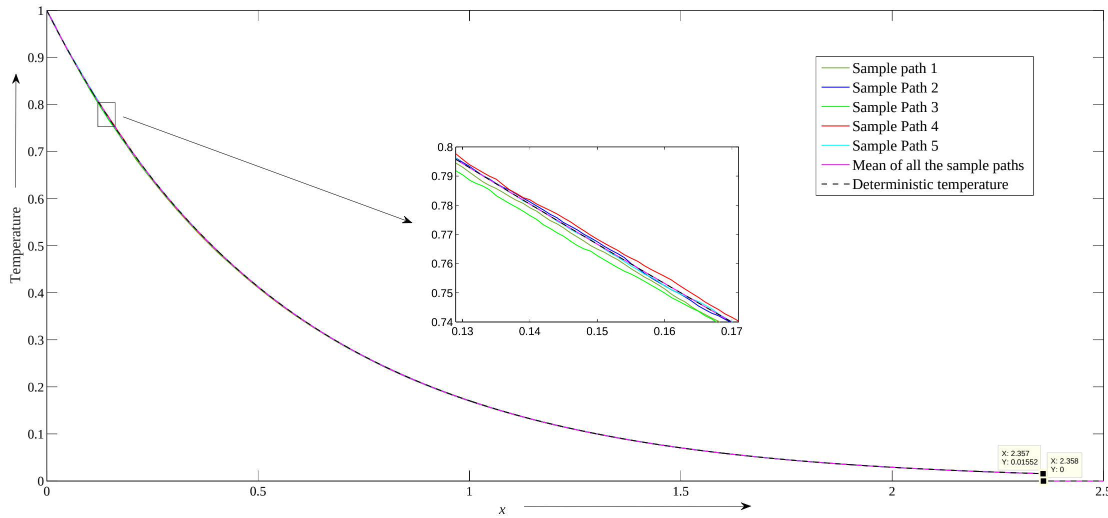
$$\tau_\theta = 0.01,\ \tau_q = 0.015,\ t = 0.25,\ t_c = 0.1,\ \theta^* = 1.$$

The solution of the physical fields in the space-time domain is obtained for both the cases by carrying out numerical computation of the solution obtained in Subsections (6.1.5-6.1.7). The theory of Brownian motion or standard Wiener process is used to compute the stochastic integration following Higham (2001). Five thousand sample paths have been taken during the computation of all physical variables for stochastic distribution among which five random paths have been displayed along with the deterministic and the mean of all sample paths to mark the difference between deterministic and stochastic distributions. The mean is computed numerically by considering the mean of the values in five thousand sample paths.

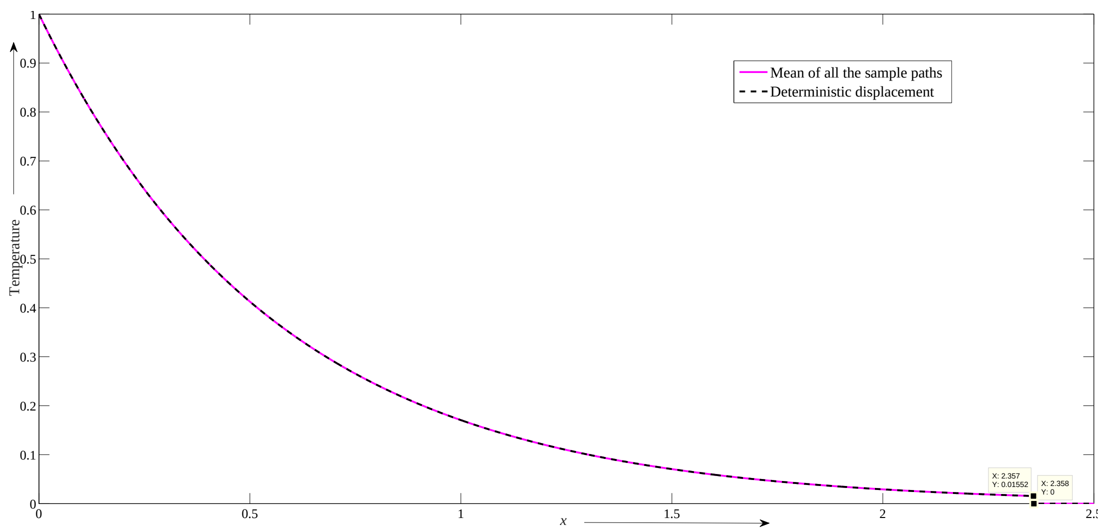
Results for Case-I are shown in Fig. 6.1.1 (a,b), Fig. 6.1.2 (a,b), and Fig. 6.1.3 (a,b), whereas results for Case-II are shown in Fig. 6.1.4 (a,b), Fig. 6.1.5 (a,b), and Fig. 6.1.6 (a,b). Fig. 6.1.1 (a) and Fig. 6.1.4 (a) show the comparison between deterministic temperature distribution and stochastic temperature distribution for five sample paths along with mean of stochastic temperature distribution for five thousand sample paths for Case-I and Case-II, respectively. It is noted that the stochastic temperature distribution along different sample paths show variations with deterministic temperature distribution for small magnitude which later coincides completely. In order to show the comparison between deterministic temperature distribution and mean of the stochastic temperature distribution for all the sample paths, Fig. 6.1.1 (b) and Fig. 6.1.4 (b) are plotted separately for Case-I and Case-II, respectively. From Fig. 6.1.1 (b) and Fig 6.1.4 (b), it is clear that the mean of all the sample paths coincides with deterministic temperature distribution which further verifies our analytical result in this respect for both the cases. For the present values, $\nu_1 = 0.9999$ and $\nu_2 = 9.4290$ where, ν_1 represent the speed of elastic wave and ν_2 represents the speed of thermal wave. For Case-I, temperature distribution experiences two finite jumps, one at $x = 0.2499$ and other at $x = 2.357$ while for case-II, temperature distribution is continuous overall.

Combined plot of the deterministic stress distribution along with stochastic stress distribution for five sample paths and mean of all the sample paths is shown in Fig. 6.1.2 (a) and Fig. 6.1.5 (a) for Case-I and Case-II, respectively. Prominent variation between stochastic stress distribution and deterministic stress distribution along different sample paths is observed near the boundary which later coincides with deterministic distribution. A separate plot to compare deterministic stress distribution and mean of stochastic stress distribution is shown in Fig. 6.1.2 (b) and Fig. 6.1.5 (b) for the two different cases. From Fig. 6.1.2 (b) and Fig. 6.1.5 (b), the analytic result for stress for both the cases is verified as the curve of deterministic stress distribution coincides with the mean of stochastic stress distribution for all sample paths. Finite jumps, one at $x = 0.2499$ and other at $x = 2.357$ is seen for Case-I while for Case-II, stress distribution is found to be continuous.

Deterministic displacement distribution with stochastic displacement distribution for five random sample paths and mean of all the paths are shown in Fig. 6.1.3 (a) and Fig. 6.1.6 (a) for Case-I and Case-II, respectively. Similar to stress distribution, significant variation between stochastic displacement distribution and deterministic displacement distribution is observed near boundary. Unlike, temperature and stress, displacement distribution is continuous for both the cases.

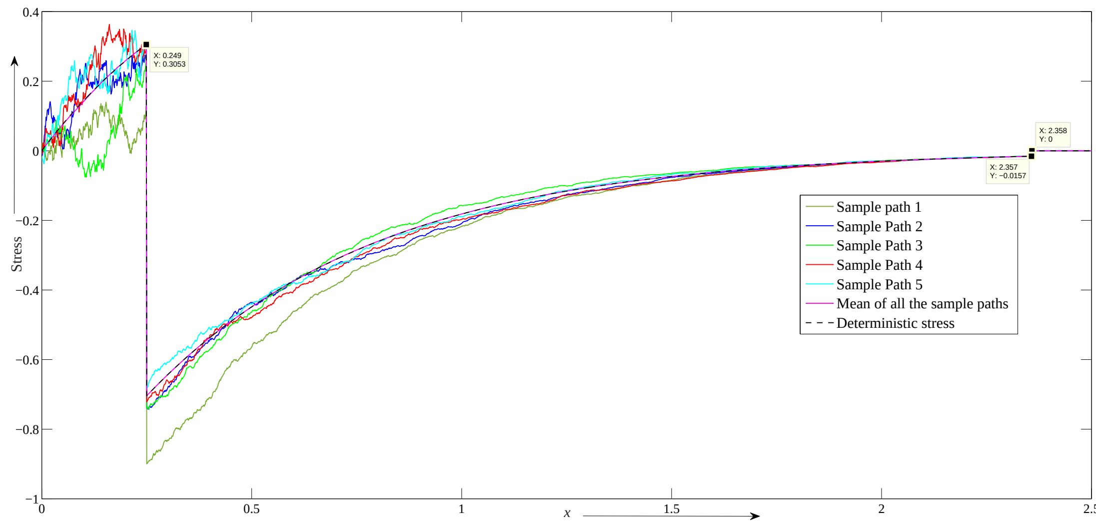


6.1.1 (a)

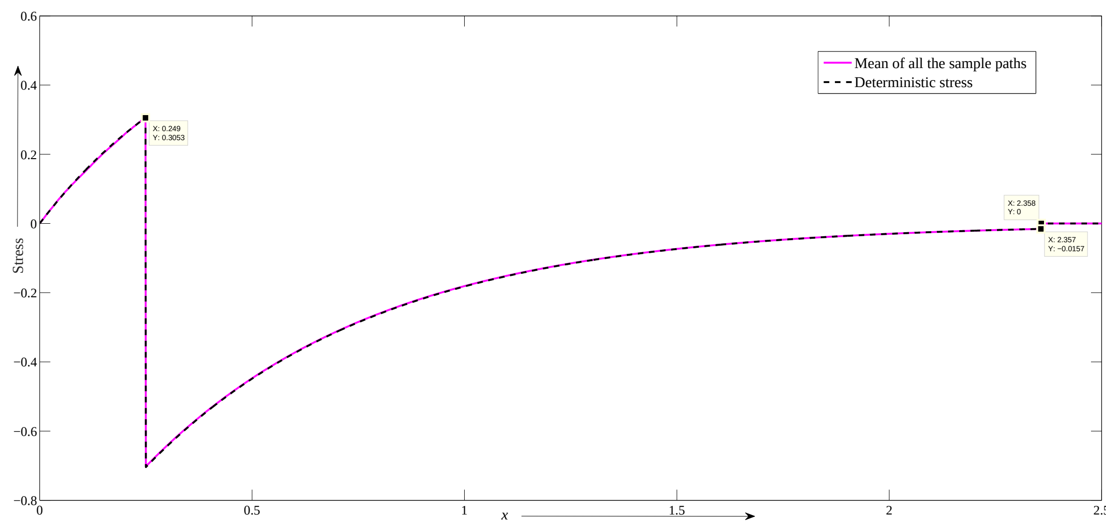


6.1.1 (b)

Figure 6.1.1: **(a).** Deterministic temperature distribution and stochastic temperature distribution along five sample paths and its mean for Case-I **(b).** Deterministic temperature distribution and mean of stochastic temperature distribution for Case-I

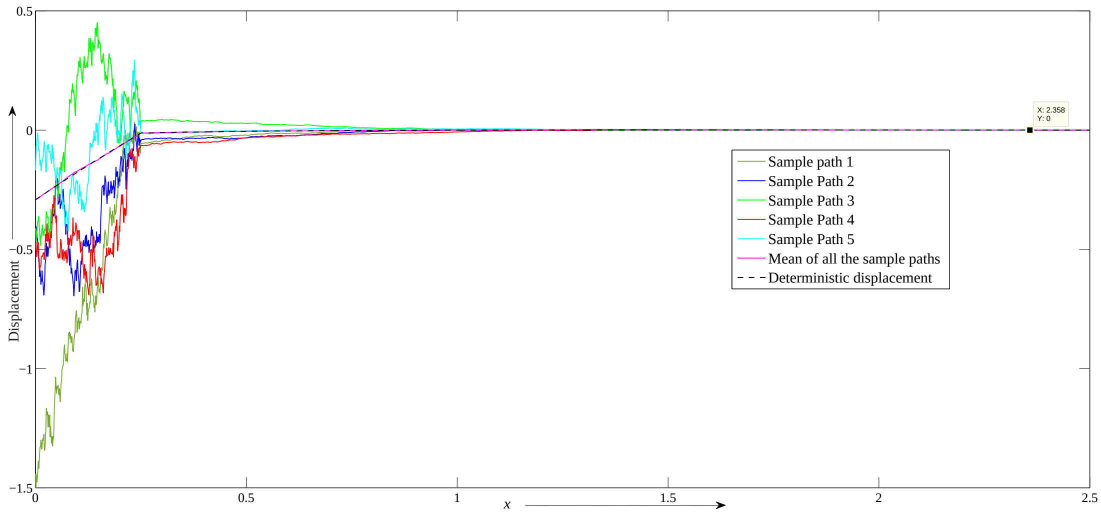


6.1.2 (a)

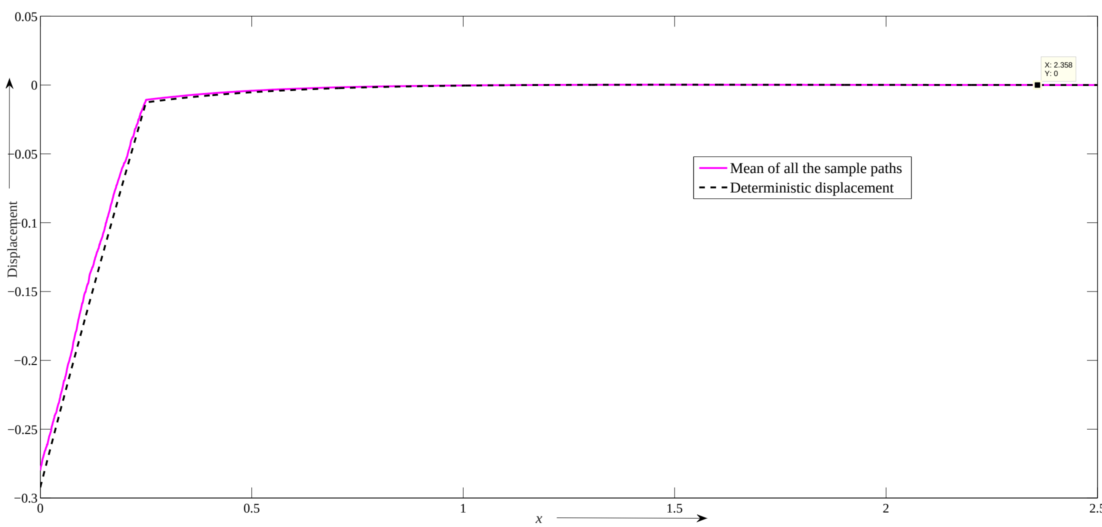


6.1.2 (b)

Figure 6.1.2: **(a).** Deterministic stress distribution and stochastic stress distribution along five sample paths and its mean for Case-I **(b).** Deterministic stress distribution and mean of stochastic stress distribution for Case-I

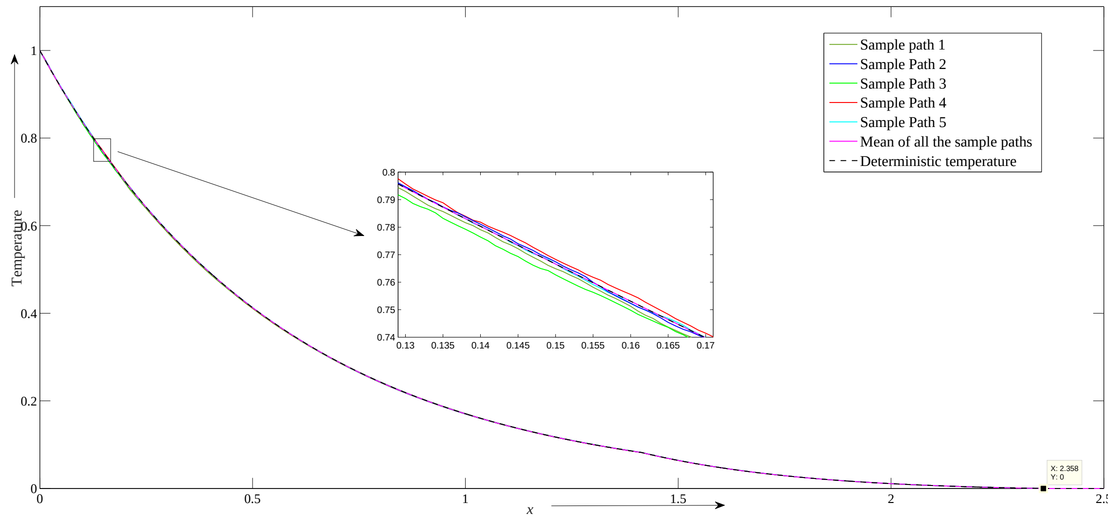


6.1.3 (a)

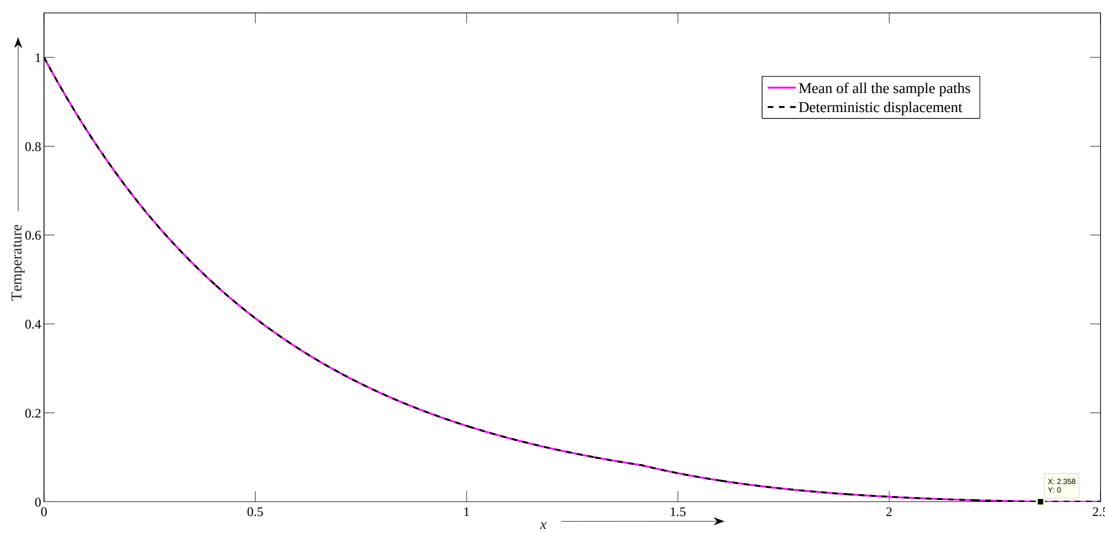


6.1.3 (b)

Figure 6.1.3: **(a).** Deterministic displacement distribution and stochastic displacement distribution along five sample paths and its mean for Case-I **(b).** Deterministic displacement distribution and mean of stochastic displacement distribution for Case-I

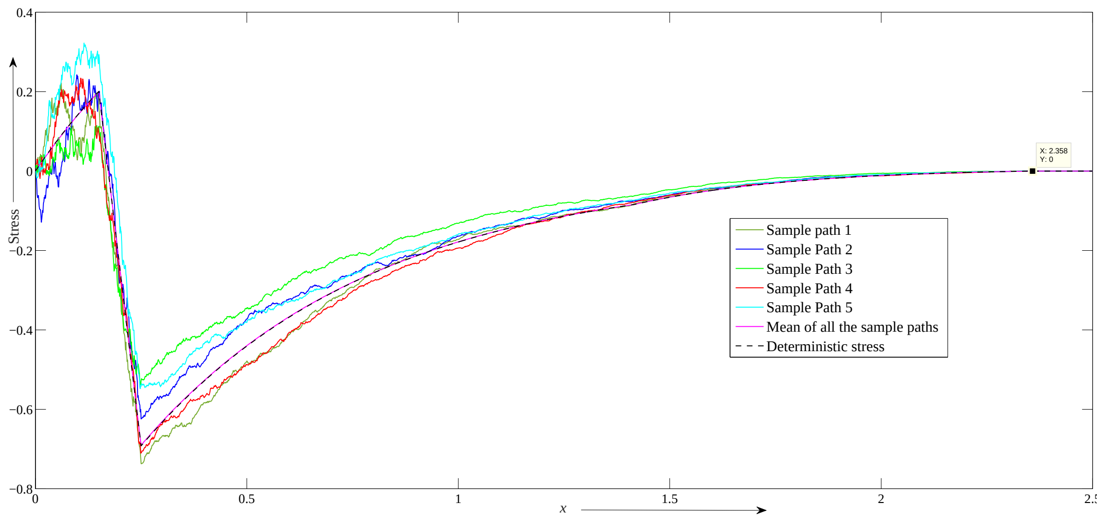


6.1.4 (a)

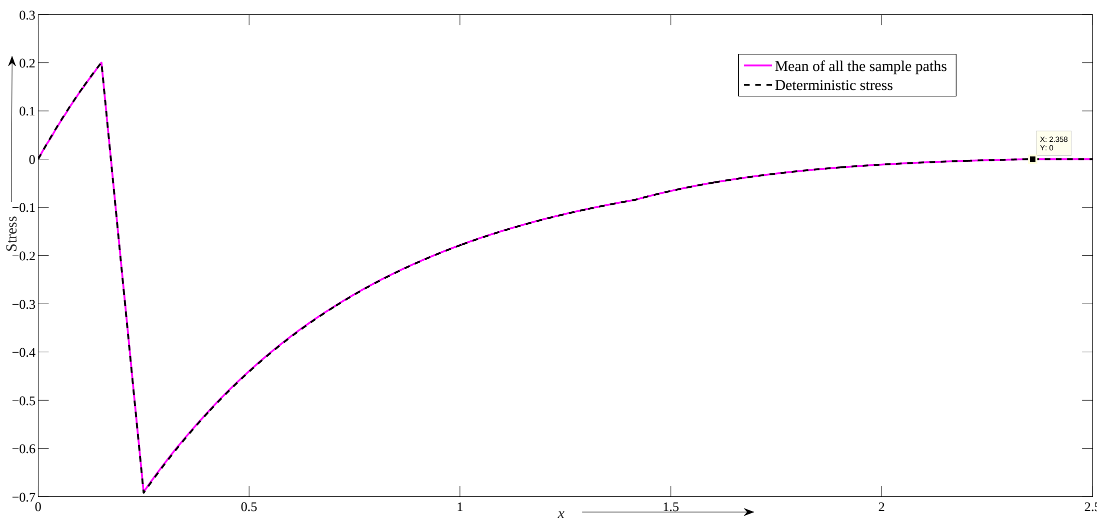


6.1.4 (b)

Figure 6.1.4: **(a).** Deterministic temperature distribution and stochastic temperature distribution along five sample paths and its mean for Case-II **(b).** Deterministic temperature distribution and mean of stochastic temperature distribution for Case-II

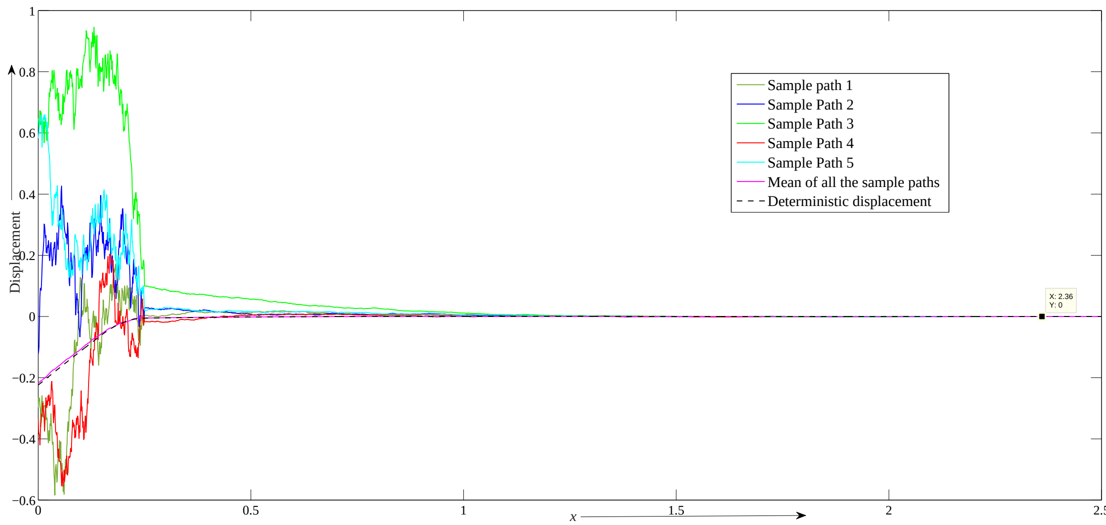


6.1.5 (a)

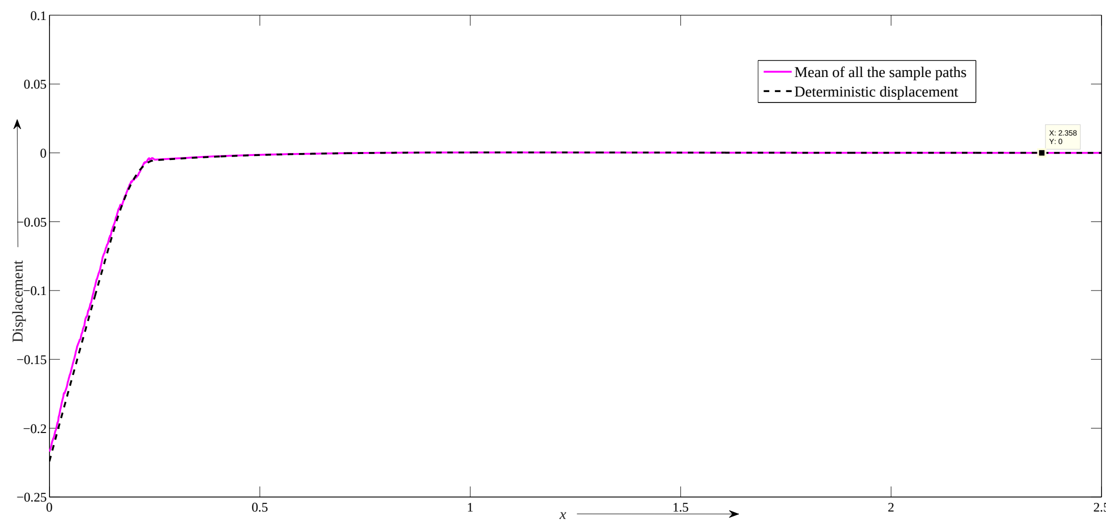


6.1.5 (b)

Figure 6.1.5: **(a).** Deterministic stress distribution and stochastic stress distribution along five sample paths and its mean for Case-II **(b).** Deterministic stress distribution and mean of stochastic stress distribution for Case-II



6.1.6 (a)



6.1.6 (b)

Figure 6.1.6: **(a).** Deterministic displacement distribution and stochastic displacement distribution along five sample paths and its mean for Case-II **(b).** Deterministic displacement distribution and mean of stochastic displacement distribution for Case-II

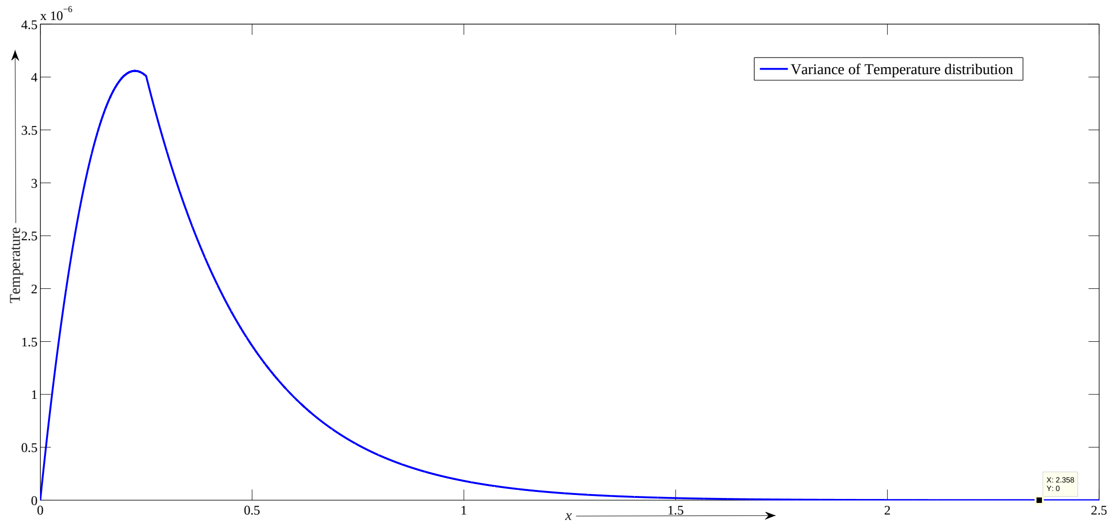


Figure 6.1.7: Variance of temperature distribution

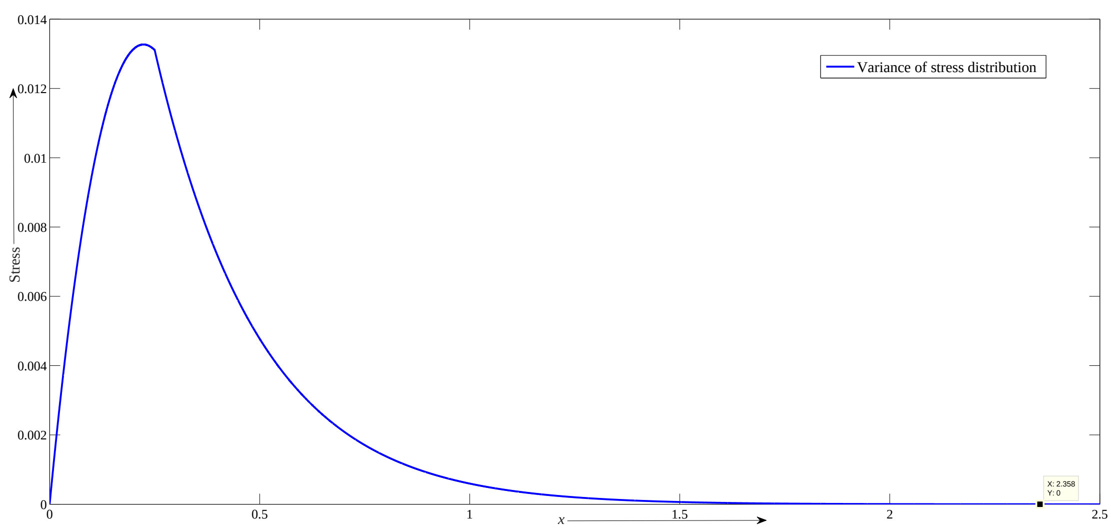


Figure 6.1.8: Variance of stress distribution

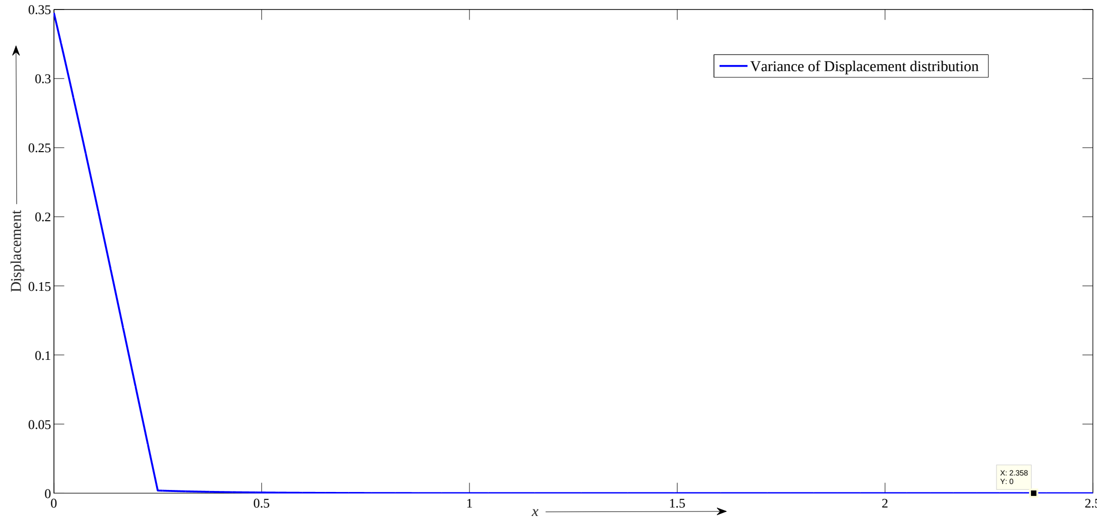


Figure 6.1.9: Variance of displacement distribution

Comparison between deterministic displacement distribution and mean of stochastic displacement distribution for all sample paths for two cases is shown in Fig. 6.1.3 (b) and Fig. 6.1.6 (b), respectively which further verify the analytical result for displacement as the result for deterministic displacement distribution and mean of stochastic displacement distribution almost coincide.

Figs. (6.1.7-6.1.9) represent the variation of temperature distribution, stress distribution, and displacement distribution, respectively. All the field variables vanish after finite distance ($x = 2.358$) for stochastic as well as deterministic distribution which confirms the existence of a finite domain of influence and the same region of influence.

6.1.9 Conclusion

In this subchapter, a one-dimensional half-space problem for an isotropic elastic homogeneous medium is solved and discussed for two types of thermal boundary conditions. Two distinct stochastic type thermal boundary conditions are defined by adding the white noise in the deterministic boundary conditions. The problem is analyzed by finding both analytical as well as numerical solutions. The comparison is made between deterministic type field variables and their corresponding stochastic distributions using five thousand sample paths and their mean for numerical results. In case of deterministic distribution as well as stochastic distribution, all the field variables are observed to vanish after a finite distance at any time which shows the presence of finite domain of influence in both the cases. This fact is also verified from our analytical solutions which show that the distributions of the field variables consist of two coupled waves (predominantly elastic and predominantly thermal waves) and both the waves propagate with finite speed. Furthermore, the numerical results show that all field variables vanish at the same distance which confirms the same region of influence for deterministic and stochastic cases. Unlike stress and displacement, temperature field shows an insignificant difference between stochastic temperature distribution and deterministic temperature distribution. Both analytical and numerical arguments have been given to verify the coincidence of mean of the stochastic solution with their respective deterministic solution for all the physical fields. This coincidence obviously verifies the reliability of the obtained numerical results.

6.2 Domain of Influence Results of Dual-Phase-Lag Thermoelasticity Theory for Natural Stress-Heat-Flux Problem [2]

6.2.1 Introduction

The domain of influence (DOI) theorem is very important to analyze the deformation of the elastic material when acted upon by thermomechanical loadings. It mathematically proves the presence of bounded domain at any finite time such that out of the domain, the solution of mixed initial-boundary value problem vanishes, which further affirms the hyperbolicity of the theory. In other words, domain of influence states that the disturbances caused due to thermomechanical loadings does not propagate in the entire medium at a finite time. One of the application of this theorem is that it can help in deciding the model to be used when a particular area of medium is undergoing laser treatment. The DOI results were initially conceptualized by Nunziato and Cowin (1979) for elastic materials with void. In 1978, Ignaczak (1978) also presented the DOI theorem for linear thermoelasticity. Further, Chandrasekharaiah (1987b) applied the results of DOI theorem to obtain the uniqueness of problem in elastic materials with void. Ignaczak and Bialy (1980) proved the domain of influence theorem for thermoelasticity model with one relaxation time (Lord and Shulman (1967)). The DOI results of the generalized thermoelastic models given by Lord and Shulman (1967) and Green and Lindsay (1972) were discussed by Ignaczak (1991) and Ignaczak and Ostoja-Starzewski (2010). Marin (1997; 2010) extended the concept of domain of influence by Nunziato and Cowin (1979), to thermoelastic bodies with voids and microstructural elastic materials, respectively. Cimmelli and Rogolino (2002) proved the DOI results for linear thermoelasticity with thermal relaxation and internal variable. Carbonaro and Ignaczak

[2] The content of this chapter is under review in an international journal.

(1987) considered anisotropic inhomogeneous unbounded body to establish work and energy theorem and reciprocity theorem along with domain of influence theorem under temperature-rate-dependent thermoelasticity theory. Moreover, Mukhopadhyay and co-researchers (2011a; 2017a; 2021b; 2021a) articulated domain of influence results on various thermoelastic theories and also established the comparison among them. Recently, Marin et al. (2020) presented the DOI results for Moore-Gibson-Thompson (Quintanilla (2019)) theory for dipolar medium.

In the previous subchapter, the finite speed of thermoelastic disturbances inside a one dimensional half-space was observed in the context of dual-phase-lag thermoelasticity theory for deterministic as well as stochastic type boundary conditions. Being motivated by this results, the present subchapter is devoted to establish the hyperbolicity of this theory. Hence, this section of the thesis focuses upon the domain of influence results for stress-heat-flux problem under dual-phase-lag thermoelasticity theory for a three dimensional medium of isotropic homogeneous thermoelastic material. The problem that describes a thermoelastic process in terms of stress and heat-flux is termed as natural stress-heat-flux problem (Ignaczak and Ostoja-Starzewski (2010)). The work is arranged in the following manner. The thermoelastic governing equations are formulated in terms of stress and heat-flux along with generalized mixed initial and boundary conditions in Subsection 6.2.2. The main objective is to extend the domain of influence notion to DPL thermoelastic model in the considered problem. In order to prove the theorem, an energy identity is first established in Subsection 6.2.4. Further, a detailed proof of domain of influence is given in Subsection 6.2.5 under which an upper bound is established for the speed of stress-heat-flux disturbances. Lastly, analogous results for other thermoelasticity theories like Lord-Shulman (1967) and classical theory (Biot (1956)) are recovered using the obtained theorem and it is shown that the present analysis in special cases matches with the corresponding DOI results reported in the literature.

6.2.2 Basic Equations and Problem Formulation

A homogeneous and isotropic material is considered which occupies a regular region A of three-dimensional Euclidean space. Let $\tilde{A}$ denotes the closure of a bounded, connected and open set with ∂A denoting the boundary of $\tilde{A}$ and A as the interior of $\tilde{A}$. The components of outward unit normal to ∂A is represented by n_i . The fundamental system governing the DPL thermoelasticity theory consists of the following equations:

Stress equation of motion:

$$\sigma_{ij,j} + \rho H_i = \rho \ddot{u}_i, \tag{6.2.1}$$

Energy equation:

$$-\, q_{i,i} + \rho R = C_\sigma \dot{\theta} + T_0 \beta \dot{\sigma}_{kk}, \tag{6.2.2}$$

Temperature-strain-stress relation:

$$e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right) + \beta\theta\delta_{ij}, \tag{6.2.3}$$

Heat conduction equation:

$$\left(1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right) q_i = -K\left(1 + \tau_\theta \frac{\partial}{\partial t}\right)\theta_{,i}, \tag{6.2.4}$$

Strain-displacement relation:

$$e_{ij} = u_{(i,j)} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{6.2.5}$$

where, C_σ denotes the specific heat at constant stress.

Now, the present thermoelastic process is considered in view of stress and heat-flux for an isotropic and homogeneous material. Hence, above Eqs. (6.2.1-6.2.5) for this natural problem can be rewritten such that a pair (σ_{ij}, q_i) satisfies the following field equations (Ignaczak (1991) and Ignaczak and Ostoja-Starzewski (2010)):

$$\rho^{-1}\sigma_{(ik,kj)} - \left[\frac{1}{2\mu}\left(\ddot{\sigma}_{ij} - \frac{\lambda}{3\lambda+2\mu}\ddot{\sigma}_{kk}\delta_{ij}\right) - \frac{T_0\beta^2}{C_\sigma}\ddot{\sigma}_{kk}\delta_{ij}\right] + \frac{\beta}{C_\sigma}\dot{q}_{k,k}\delta_{ij} = \frac{1}{C_\sigma}\rho\beta\dot{R}\delta_{ij} - H_{(i,j)}, \tag{6.2.6}$$

$$K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\frac{1}{C_\sigma}\left(q_{k,k}+\beta T_0\dot{\sigma}_{kk}\right)_{,i} - \left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\dot{q}_i = K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\frac{\rho}{C_\sigma}R_{,i}, \tag{6.2.7}$$

where, $H_{(i,j)} = \frac{1}{2}\left(H_{i,j}+H_{j,i}\right)$ and $\sigma_{(ik,kj)} = \frac{1}{2}\left(\sigma_{ik,kj}+\sigma_{jk,ki}\right)$.

The following assumptions are imposed on the material constants:

$$\begin{aligned} &\rho > 0,\ \ \lambda > 0,\ \ \mu > 0,\ \ K > 0,\ \ C_\sigma > 0,\ \ \beta > 0,\ \ T_0 > 0,\\ &\tau_q > 0,\ \ 3\lambda+2\mu > 0. \end{aligned} \tag{6.2.8}$$

Moreover, the following relation is considered:

$$2\tau_\theta > \tau_q. \tag{6.2.9}$$

Here, it is worth recalling the fact examined by Quintanilla (2002b; 2003) that the exponentially stability of the solutions in the context of DPL theory of thermoelasticity is satisfied if the phase-lag constants τ_q and τ_θ satisfy the relation (6.2.9).

Now, in view of the Eq. (6.2.6) and Eq. (6.2.7), the notations are further introduced as follows:

$$\mathscr{T}_{(ij)} = H_{(i,j)} - \frac{1}{C_\sigma}\rho\beta\dot{R}\delta_{ij}, \tag{6.2.10}$$

$$g_i = -K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\frac{\rho}{C_\sigma}R_{,i}. \tag{6.2.11}$$

Then, Eq. (6.2.6) and Eq. (6.2.7) yield

$$\rho^{-1}\sigma_{(ik,kj)} - \left[\frac{1}{2\mu}\left(\ddot{\sigma}_{ij} - \frac{\lambda}{3\lambda+2\mu}\ddot{\sigma}_{kk}\delta_{ij}\right) - \frac{T_0\beta^2}{C_\sigma}\ddot{\sigma}_{kk}\delta_{ij}\right] + \frac{\beta}{C_\sigma}\dot{q}_{k,k}\delta_{ij} = -\mathscr{T}_{(ij)}, \tag{6.2.12}$$

$$K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\frac{1}{C_\sigma}\left(q_{k,k}+\beta T_0\dot{\sigma}_{kk}\right)_{,i} - \left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\dot{q}_i = -g_i. \tag{6.2.13}$$

The initial conditions on $\boldsymbol{x} = (x_1, x_2, x_3) \in A$ for the above Eq. (6.2.12) and Eq. (6.2.13) are assumed in the following way:

$$\begin{aligned}
\sigma_{ij}(\boldsymbol{x},0) &= \sigma_{ij}^0(\boldsymbol{x}),\\
\dot{\sigma}_{ij}(\boldsymbol{x},0) &= \dot{\sigma}_{ij}^0(\boldsymbol{x}),\\
q_i(\boldsymbol{x},0) &= q_i^0(\boldsymbol{x}),\\
\dot{q}_i(\boldsymbol{x},0) &= \dot{q}_i^0(\boldsymbol{x}),\\
\ddot{q}_i(\boldsymbol{x},0) &= \ddot{q}_i^0(\boldsymbol{x}),
\end{aligned} \tag{6.2.14}$$

and boundary conditions on $\partial A \times [0,\infty[$ are considered as:

$$\begin{aligned}
\sigma_{ij}n_j &= \sigma_i',\\
q_i n_i &= q'.
\end{aligned} \tag{6.2.15}$$

6.2.3 Preliminaries

Now, in order to obtain the domain of influence results for the present problem, the following important definitions (Ignaczak and Ostoja-Starzewski (2010)) are recalled.

Definition 1. For a fixed time $t \in (0, \infty)$, the set

$$\mathfrak{D}_0(t) = \begin{cases} & (1)\,\text{for}\, \boldsymbol{x} \in A,\; \sigma_{ij}^0 \neq 0 \;\text{ or }\; \dot{\sigma}_{ij}^0 \neq 0 \;\text{ or }\; q_i^0 \neq 0 \;\text{ or }\; \dot{q}_i^0 \neq 0 \;\text{ or }\; \ddot{q}_i^0 \neq 0 \\ \boldsymbol{x} \in \tilde{A}: & (2)\,\text{for}\; (\boldsymbol{x}, \tau) \in \partial A \times [0, t]\,,\; \sigma_i' \neq 0 \;\text{ or }\; q' \neq 0 \\ & (3)\,\text{for}\; (\boldsymbol{x}, \tau) \in \partial A \times [0, t]\,,\; \mathscr{T}_{(ij)} \neq 0 \;\text{ or }\; g_i \neq 0 \end{cases} \tag{6.2.16}$$

is known as the thermomechanical load support at time t in the context of the present system (6.2.12-6.2.15).

Definition 2. Let V is any real parameter and an open ball $\Omega(\boldsymbol{x}, Vt)$ with radius Vt and center at $\boldsymbol{x}$, then the set

$$\mathfrak{D}(t) = \left\{ \boldsymbol{x} \in \tilde{A} : \overline{\Omega(\boldsymbol{x}, Vt)} \cap \mathfrak{D}_0(t) \neq \emptyset \right\} \tag{6.2.17}$$

represents the set of all the points of $\tilde{A}$ which can be affected by the thermomechanical disturbances propagating from $\mathfrak{D}_0(t)$ with a finite speed less than or equal to V (see Ignaczak and Ostoja-Starzewski (2010)). Hence, the set $\mathfrak{D}(t)$ defines the domain of influence corresponding to the thermomechanical load, $\mathfrak{D}_0(t)$.

6.2.4 Energy Identity

In this section, the following theorem presents an energy identity for the present context which is a counterpart of the identity acquired by Ignaczak and Ostoja-Starzewski (2010).

Theorem 6.2.4.1.

Statement: Let a smooth solution to the mixed problem (6.2.12-6.2.15) is defined by (σ_{ij}, q_i) and a scalar field is represented by $m(\boldsymbol{x}) \in C^1(\tilde{A})$ in a such manner that the set

$$E_0 = \left\{ \boldsymbol{x} \in \tilde{A} : m(\boldsymbol{x}) > 0 \right\} \tag{6.2.18}$$

is bounded. Then

$$
\begin{aligned}
&\frac{1}{2}\int_{A}\left\{M_0\left(\boldsymbol{x}, m\left(\boldsymbol{x}\right)\right)-\left[m\left(\boldsymbol{x}\right)\dot{M}_0\left(\boldsymbol{x},0\right)+M_0\left(\boldsymbol{x},0\right)\right]\right\}\mathrm{d}A\\
&+\frac{1}{2}\int_{A}\left\{\int_{0}^{m(\boldsymbol{x})}M_1\left(\boldsymbol{x},t\right)dt-m\left(\boldsymbol{x}\right)M_1\left(\boldsymbol{x},0\right)\right\}\mathrm{d}A\\
&+\int_{A}\left\{\int_{0}^{m(\boldsymbol{x})}\left[m\left(\boldsymbol{x}\right)-t\right]N\left(\boldsymbol{x},t\right)\mathrm{d}t\right\}\mathrm{d}A+\int_{A}\left\{\int_{0}^{m(\boldsymbol{x})}P_i\left(\boldsymbol{x},t\right)m_{,i}\left(\boldsymbol{x}\right)\mathrm{d}t\right\}\mathrm{d}A\\
&=\int_{\partial A}\left\{\int_{0}^{m(\boldsymbol{x})}\left[m\left(\boldsymbol{x}\right)-t\right]P_i\left(\boldsymbol{x},t\right)n_i\left(\boldsymbol{x}\right)\mathrm{d}t\right\}dB+\int_{A}\left\{\int_{0}^{m(\boldsymbol{x})}\left[m\left(\boldsymbol{x}\right)-t\right]Q\left(\boldsymbol{x},t\right)\mathrm{d}t\right\}\mathrm{d}A,
\end{aligned}
\tag{6.2.19}
$$

where,

$$
M_0\left(\boldsymbol{x},t\right)=\frac{K\tau_q^2}{2T_0}\left(\dot{q}_i\right)^2, \tag{6.2.20}
$$

$$
\begin{aligned}
M_1\left(\boldsymbol{x},t\right)=&\rho^{-1}\hat{\sigma}_{ik,k}\hat{\sigma}_{ij,j}+\frac{1}{2\mu}\left(\dot{\hat{\sigma}}_{ij}\dot{\hat{\sigma}}_{ij}-\frac{\lambda}{3\lambda+2\mu}\left(\dot{\hat{\sigma}}_{kk}\right)^2\right)-\frac{\beta^2T_0}{C_\sigma}\left(\dot{\hat{\sigma}}_{kk}\right)^2+\\
&+\frac{1}{C_\sigma T_0}\left(\hat{q}_{k,k}\right)^2+\frac{K\tau_q+K\tau_\theta}{T_0}\left(\dot{q}_i\right)^2+\frac{K\tau_\theta\tau_q^2}{2T_0}\left(\ddot{q}_i\right)^2,
\end{aligned}
\tag{6.2.21}
$$

$$
N\left(\boldsymbol{x},t\right)=\frac{K}{T_0}\left[\left(\dot{q}_i\right)^2+\left(\frac{2\tau_q\tau_\theta-\tau_q^2}{2}\right)\left(\ddot{q}_i\right)^2\right], \tag{6.2.22}
$$

$$
P_i\left(\boldsymbol{x},t\right)=\rho^{-1}\dot{\hat{\sigma}}_{ij}\hat{\sigma}_{jk,k}+\frac{1}{C_\sigma}\left(\beta\dot{\hat{\sigma}}_{kk}+\frac{1}{T_0}\hat{q}_{k,k}\right)K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\dot{q}_i, \tag{6.2.23}
$$

$$
Q\left(\boldsymbol{x},t\right)=\hat{\mathscr{T}}_{(ij)}\dot{\hat{\sigma}}_{ij}+\frac{1}{T_0}g_i\dot{\hat{q}}_i, \tag{6.2.24}
$$

where, for a function $h=h\left(\boldsymbol{x},t\right)$ defined on $\boldsymbol{x}\in\tilde{A}\times[0,\infty[$, $\hat{h}()$ is denoted as follows:

$$
\hat{h}=K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)h. \tag{6.2.25}
$$

Proof. The proof is initiated by applying the hat operator as given by Eq. (6.2.25), on the Eq. (6.2.12). Then, after multiplying $\dot{\hat{\sigma}}_{ij}$ to both sides of Eq. (6.2.12), the following equation is obtained:

$$\rho^{-1}\hat{\sigma}_{(ik,kj)}\dot{\hat{\sigma}}_{ij} - \left[\frac{1}{2\mu}\left(\ddot{\hat{\sigma}}_{ij}\dot{\hat{\sigma}}_{ij} - \frac{\lambda}{3\lambda+2\mu}\ddot{\hat{\sigma}}_{kk}\dot{\hat{\sigma}}_{kk}\right) - \frac{T_0\beta^2}{C_\sigma}\ddot{\hat{\sigma}}_{kk}\dot{\hat{\sigma}}_{kk}\right] + \frac{\beta}{C_\sigma}\dot{\hat{q}}_{k,k}\dot{\hat{\sigma}}_{kk} = -\hat{\mathscr{T}}_{(ij)}\dot{\hat{\sigma}}_{ij}. \tag{6.2.26}$$

Now, on multiplying both sides by $T_0^{-1}\dot{\hat{q}}_i$, Eq. (6.2.13) transforms to

$$\left\{K\left(1+\tau_q\frac{\partial}{\partial t}\right)\frac{1}{C_\sigma}\left(q_{k,k}+\beta T_0\dot{\sigma}_{kk}\right)_{,i}\right\}T_0^{-1}\dot{\hat{q}}_i - \left\{\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\dot{q}_i\right\}T_0^{-1}\dot{\hat{q}}_i = -g_i T_0^{-1}\dot{\hat{q}}_i. \tag{6.2.27}$$

Adding Eq. (6.2.26) and Eq. (6.2.27) and then after detailed manipulations, the following equation is acquired:

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}M_0(\boldsymbol{x},t) + M_1(\boldsymbol{x},t)\right\} + N(\boldsymbol{x},t) = P_{i,i}(\boldsymbol{x},t) + Q(\boldsymbol{x},t), \tag{6.2.28}$$

where, M_0, M_1, N, P_i, and Q are given by Eqs. (6.2.20-6.2.24), respectively.

Recalling the following relation (Ignaczak and Ostoja-Starzewski (2010)):

$$\int_0^{m(\boldsymbol{x})} P_{i,i}\left[m(\boldsymbol{x})-t\right]\mathrm{d}t = \int_0^{m(\boldsymbol{x})}\left\{\left[P_i(\boldsymbol{x},t)\left[m(\boldsymbol{x})-t\right]\right]_{,i} - P_i(\boldsymbol{x},t)\,m_{,i}(\boldsymbol{x})\right\}\mathrm{d}t = \left[\int_0^{m(\boldsymbol{x})} P_i(\boldsymbol{x},t)\left[m(\boldsymbol{x})-t\right]\mathrm{d}t\right]_{,i} - \int_0^{m(\boldsymbol{x})} P_i(\boldsymbol{x},t)\,m_{,i}(\boldsymbol{x})\,\mathrm{d}t. \tag{6.2.29}$$

Therefore, taking double integration of Eq. (6.2.28) over t from $t=0$ to $t=m(\boldsymbol{x})$ and using Eq. (6.2.29), gives as follows:

$$\frac{1}{2}\left[M_0(\boldsymbol{x},m(\boldsymbol{x})) - M_0(\boldsymbol{x},0) - m(\boldsymbol{x})\dot{M}_0(\boldsymbol{x},0)\right] + \frac{1}{2}\left[\int_0^{m(\boldsymbol{x})} M_1(\boldsymbol{x},t)\,\mathrm{d}t - m(\boldsymbol{x})\,M_1(\boldsymbol{x},0)\right] + \int_0^{m(\boldsymbol{x})}\left[m(\boldsymbol{x})-t\right]N(\boldsymbol{x},t)\,\mathrm{d}t + \int_0^{m(\boldsymbol{x})} P_i(\boldsymbol{x},t)m_{,i}(\boldsymbol{x})\,\mathrm{d}t = \left[\int_0^{m(\boldsymbol{x})} P_i(\boldsymbol{x},t)\left[m(\boldsymbol{x})-t\right]\mathrm{d}t\right]_{,i} + \int_0^{m(\boldsymbol{x})}\left[m(\boldsymbol{x})-t\right]Q(\boldsymbol{x},t)\,\mathrm{d}t. \tag{6.2.30}$$

Each term in Eq. (6.2.30) is bounded since the set E_0 is bounded from the definition of Eq. (6.2.18). Therefore, integrating Eq. (6.2.30) over A and applying the divergence theorem in the R.H.S., the Eq. (6.2.19) is obtained. This completes the proof of Theorem 6.2.4.1.

Next, the domain of influence results is established under DPL thermoelastic model in view of a natural stress-heat-flux problem in the following theorem.

6.2.5 Domain of Influence Theorem

Theorem 6.2.5.1.

Statement: Let V denotes a real number which satisfies the following inequality:

$$V \geq \max(V_1, V_2, V_3, V_4), \tag{6.2.31}$$

where,

$$V_1 = \left(\frac{2\mu}{\rho}\right)^{\frac{1}{2}}, \tag{6.2.32}$$

$$V_2 = \left\{\frac{2K\tau_\theta}{\tau_q^2 C_\sigma}\left[1 + \frac{C_\sigma}{C_E}\left(1 - \frac{C_E}{C_\sigma}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}, \tag{6.2.33}$$

$$V_3 = \left\{\frac{K}{(\tau_q + \tau_\theta)C_\sigma}\left[1 + \frac{C_\sigma}{C_E}\left(1 - \frac{C_E}{C_\sigma}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}, \tag{6.2.34}$$

$$V_4 = \left\{\frac{(3\lambda + 2\mu)}{\rho}\frac{C_\sigma}{C_E}\left[1 - \left(1 - \frac{C_E}{C_\sigma}\right)^{\frac{1}{2}}\right]^{-1}\right\}^{\frac{1}{2}}, \tag{6.2.35}$$

and, C_E denotes the specific heat at constant strain which satisfies the relation with C_σ as follows:

$$C_\sigma = C_E + 3\beta^2 (3\lambda + 2\mu) T_0. \tag{6.2.36}$$

Then, for the set $\mathfrak{D}(t)$ defining the domain of influence for thermomechanical load $\mathfrak{D}_0(t)$ at time t and the pair (σ_{ij}, q_i) defining a smooth solution of mixed problem (6.2.12-6.2.15), the following results hold:

$$\sigma_{ij} = 0, \quad q_i = 0, \quad \text{on } \left\{\tilde{A} - \mathfrak{D}(t)\right\} \times [0, t]. \tag{6.2.37}$$

Proof. Let

$$\Lambda = \tilde{A} \cap \overline{\Omega(\boldsymbol{w}, V\tau)}, \tag{6.2.38}$$

where, $(\boldsymbol{w}, \tau) \in \{A - \mathfrak{D}(t)\} \times (0, t)$ is a fixed point.

Then, consider the following:

$$m_\tau(\boldsymbol{x}) = \begin{cases} \tau - \frac{1}{V}|\boldsymbol{x} - \boldsymbol{w}|, & \text{for } \boldsymbol{x} \in \Lambda \\ 0, & \text{for } \boldsymbol{x} \notin \Lambda \end{cases}, \tag{6.2.39}$$

where, V represents a parameter as given in inequality (6.2.31).

Since $\tau < t$ and employing the definitions of domain $\mathfrak{D}(t)$ and Λ as given by the Eq. (6.2.17) and Eq. (6.2.38), respectively, it is found that

$$\Lambda \cap \mathfrak{D}_0(t) = \emptyset. \tag{6.2.40}$$

Therefore,

$$\sigma_{ij} n_j = 0, \;\; q_i n_i = 0, \;\; \text{on } \; (\Lambda \cap \partial A) \times [0, t], \tag{6.2.41}$$

and

$$\dot{\sigma}_{ij} n_j = 0, \;\; \dot{q}_i n_i = 0, \;\; \ddot{q}_i n_i = 0, \;\; \text{on } \; (\Lambda \cap \partial A) \times [0, t], \tag{6.2.42}$$

$$\mathscr{T}_{(ij)} = 0, \;\; g_i = 0, \;\; \text{on } \Lambda \times (0, t). \tag{6.2.43}$$

Furthermore,

$$\sigma_{ij}(\boldsymbol{x}, 0) = \dot{\sigma}_{ij}(\boldsymbol{x}, 0) = q_i(\boldsymbol{x}, 0) = \dot{q}_i(\boldsymbol{x}, 0) = \ddot{q}_i(\boldsymbol{x}, 0), \;\; \text{on } \Lambda. \tag{6.2.44}$$

Now, using the Eq. (6.2.23), Eq. (6.2.39), and Eq. (6.2.42), give

$$\int_{\partial A} \left\{ \int_0^{m_\tau(\boldsymbol{x})} [m_\tau(\boldsymbol{x}) - t] \, P_i(\boldsymbol{x}, t) \, n_i(\boldsymbol{x}) \, \mathrm{d}t \right\} \mathrm{d}B = 0. \tag{6.2.45}$$

Similarly, in view of the Eq. (6.2.24), Eq. (6.2.39), and Eq. (6.2.43), it is obtained that

$$\int_{A} \left\{ \int_0^{m_\tau(\boldsymbol{x})} [m_\tau(\boldsymbol{x}) - t] \, Q(\boldsymbol{x}, t) \, \mathrm{d}t \right\} \mathrm{d}A = 0. \tag{6.2.46}$$

Further, using the definitions of $M_0(\boldsymbol{x},t)$, $M_1(\boldsymbol{x},t)$, and $m_\tau(\boldsymbol{x})$, it is acquired that

$$M_0(\boldsymbol{x}, m_\tau(\boldsymbol{x})) - M_0(\boldsymbol{x},0) - m_\tau(\boldsymbol{x})\dot{M}_0(\boldsymbol{x},0) = \begin{cases} M_0(\boldsymbol{x}, m_\tau(\boldsymbol{x})), & \text{for } \boldsymbol{x} \in \Lambda \\ 0, & \text{for } \boldsymbol{x} \notin \Lambda \end{cases}, \tag{6.2.47}$$

and

$$\int_0^{m_\tau(\boldsymbol{x})} M_1(\boldsymbol{x},t)\,\mathrm{d}t - m_\tau(\boldsymbol{x}) M_1(\boldsymbol{x},0) = \begin{cases} \int_0^{m_\tau(\boldsymbol{x})} M_1(\boldsymbol{x},t)\,\mathrm{d}t, & \text{for } \boldsymbol{x} \in \Lambda \\ 0, & \text{for } \boldsymbol{x} \notin \Lambda \end{cases}. \tag{6.2.48}$$

Now, on substituting $m_\tau(\boldsymbol{x})$ into the Eq. (6.2.19) and making use of the Eqs. (6.2.45-6.2.48), yield

$$\begin{aligned} &\frac{1}{2}\int_\Lambda M_0(\boldsymbol{x}, m_\tau(\boldsymbol{x}))\,\mathrm{d}A + \frac{1}{2}\int_\Lambda \int_0^{m_\tau(\boldsymbol{x})} M_1(\boldsymbol{x},t)\,\mathrm{d}t\mathrm{d}A \\ &+ \int_\Lambda \left\{ \int_0^{m_\tau(\boldsymbol{x})} [m_\tau(\boldsymbol{x}) - t]\, N(\boldsymbol{x},t)\,\mathrm{d}t \right\} \mathrm{d}A = -\int_\Lambda \left\{ \int_0^{m_\tau(\boldsymbol{x})} P_i(\boldsymbol{x},t)\, m_{\tau,i}(\boldsymbol{x})\,\mathrm{d}t \right\} \mathrm{d}A. \end{aligned} \tag{6.2.49}$$

Since, Eq. (6.2.9) imply that $N \geq 0$ on Λ. Therefore, from the Eq. (6.2.39) and Eq. (6.2.49), the following inequality is found:

$$\frac{1}{2}\int_\Lambda M_0(\boldsymbol{x}, m_\tau(\boldsymbol{x}))\,\mathrm{d}A + \frac{1}{2}\int_\Lambda \int_0^{m_\tau(\boldsymbol{x})} M_1(\boldsymbol{x},t)\,\mathrm{d}t\mathrm{d}A \leq \frac{1}{V}\int_\Lambda \int_0^{m_\tau(\boldsymbol{x})} |P_i(\boldsymbol{x},t)\,\mathrm{d}t|\,\mathrm{d}A. \tag{6.2.50}$$

Now,

$$\begin{aligned} \frac{1}{V}|P_i| \leq & \rho^{-1}\left|\frac{\dot{\hat{\sigma}}_{ij}}{V}\hat{\sigma}_{jk,k}\right| + \frac{1}{C_\sigma}\left|\beta\dot{\hat{\sigma}}_{kk} + \frac{1}{T_0}\hat{q}_{k,k}\right| \left|K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\frac{\dot{q}_i}{V}\right| \\ \leq & \rho^{-1}\left|\frac{\dot{\hat{\sigma}}_{ij}}{V}\right| |\hat{\sigma}_{jk,k}| + \frac{|\beta|}{C_\sigma}\left|\dot{\hat{\sigma}}_{kk}\right| \left|\frac{K\dot{q}_i + K\tau_\theta\ddot{q}_i}{V}\right| + \frac{1}{C_\sigma T_0}|\hat{q}_{k,k}| \left|\frac{K\dot{q}_i + K\tau_\theta\ddot{q}_i}{V}\right|. \end{aligned} \tag{6.2.51}$$

To calculate each term of the R.H.S. of Eq. (6.2.51) and simplify the Eq. (6.2.51), the following relation of real numbers is used:

$$\sqrt{ab} \leq \frac{1}{2}\left(\epsilon a + \epsilon^{-1} b\right), \tag{6.2.52}$$

ϵ represents a dimensionless positive parameter and a and b are non-negative physical fields of same dimension.

To calculate the first term of Eq. (6.2.51), $a = \left(\hat{\sigma}_{jk,k}\right)^2$, $b = \left(\frac{\dot{\hat{\sigma}}_{ij}}{V}\right)^2$, and $\epsilon = 1$ are used in Eq. (6.2.52) and then it is obtained that

$$\left|\hat{\sigma}_{jk,k}\right| \left|\frac{\dot{\hat{\sigma}}_{ij}}{V}\right| \leq \frac{1}{2}\left(\hat{\sigma}_{ij,j}\hat{\sigma}_{ik,k} + \frac{1}{V^2}\dot{\hat{\sigma}}_{ij}\dot{\hat{\sigma}}_{ij}\right). \tag{6.2.53}$$

Further to calculate the second term of Eq. (6.2.51), the following is employed:

$$a = \left(\dot{\hat{\sigma}}_{kk}\right)^2, \quad b = \frac{1}{V^2\left(\beta T_0\right)^2}\left(K\dot{q}_i + K\tau_\theta \ddot{q}_i\right)^2, \quad \epsilon = \frac{C_E}{C_\sigma}\left(1 - \frac{C_E}{C_\sigma}\right)^{-\frac{1}{2}}. \tag{6.2.54}$$

Therefore, using Eq. (6.2.54) in Eq. (6.2.52) gives

$$\begin{aligned}
&\left|\dot{\hat{\sigma}}_{kk}\right| \frac{1}{V\left|\beta\right| T_0}\left|K\dot{q}_i + K\tau_\theta \ddot{q}_i\right| \\
&\leq \frac{1}{2}\left\{\frac{C_E}{C_\sigma}\left(1 - \frac{C_E}{C_\sigma}\right)^{-\frac{1}{2}}\left(\dot{\hat{\sigma}}_{kk}\right)^2 + \frac{C_\sigma}{C_E}\left(1 - \frac{C_E}{C_\sigma}\right)^{\frac{1}{2}}\frac{1}{V^2\beta^2 T_0^2}\left(K\dot{q}_i + K\tau_\theta \ddot{q}_i\right)^2\right\} \\
&\leq \frac{1}{2}\left\{\frac{C_E}{C_\sigma}\left(1 - \frac{C_E}{C_\sigma}\right)^{-\frac{1}{2}}\left(\dot{\hat{\sigma}}_{kk}\right)^2 + \frac{C_\sigma}{C_E}\left(1 - \frac{C_E}{C_\sigma}\right)^{\frac{1}{2}}\frac{K^2}{V^2\beta^2 T_0^2}\left[\left(\dot{q}_i\right)^2 + \tau_\theta^2\left(\ddot{q}_i\right)^2\right]\right\} \\
&+ \frac{K^2\tau_\theta}{2V^2\beta^2 T_0^2}\frac{C_\sigma}{C_E}\left(1 - \frac{C_E}{C_\sigma}\right)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}_i\right)^2.
\end{aligned} \tag{6.2.55}$$

Now, by fixing $a = \left(\hat{q}_{k,k}\right)^2$, $b = \left(\frac{K\dot{q}_i + K\tau_\theta \ddot{q}_i}{V}\right)^2$, and $\epsilon = 1$ in Eq. (6.2.52), the last term of Eq. (6.2.51) is obtained as

$$\begin{aligned}
\left|\hat{q}_{k,k}\right|\left|\frac{K\dot{q}_i + K\tau_\theta \ddot{q}_i}{V}\right| &\leq \frac{1}{2}\left\{\left(\hat{q}_{k,k}\right)^2 + \frac{1}{V^2}\left(K\dot{q}_i + K\tau_\theta \ddot{q}_i\right)^2\right\} \\
&\leq \frac{1}{2}\left\{\left(\hat{q}_{k,k}\right)^2 + \frac{K^2}{V^2}\left[\left(\dot{q}_i\right)^2 + \tau_\theta^2\left(\ddot{q}_i\right)^2\right]\right\} + \frac{K^2\tau_\theta}{2V^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}_i\right)^2.
\end{aligned} \tag{6.2.56}$$

Thus, Eq. (6.2.51), Eq. (6.2.53), Eq. (6.2.55), and Eq. (6.2.56) yield

$$\frac{1}{V}\left|P_i\right| \leq \frac{\rho^{-1}}{2}\left(\hat{\sigma}_{ij,j}\hat{\sigma}_{ik,k}+\frac{1}{V^2}\dot{\hat{\sigma}}_{ij}\dot{\hat{\sigma}}_{ij}\right)$$
$$+\frac{\beta^2 T_0}{2C_\sigma}\left\{\frac{C_E}{C_\sigma}\left(1-\frac{C_E}{C_\sigma}\right)^{-\frac{1}{2}}\left(\dot{\hat{\sigma}}_{kk}\right)^2+\frac{C_\sigma}{C_E}\left(1-\frac{C_E}{C_\sigma}\right)^{\frac{1}{2}}\frac{K^2}{V^2\beta^2T_0^2}\left[\left(\dot{q}_i\right)^2+\tau_\theta^2\left(\ddot{q}_i\right)^2\right]\right\}$$
$$+\frac{1}{2C_\sigma T_0}\left\{\left(\hat{q}_{k,k}\right)^2+\frac{K^2}{V^2}\left[\left(\dot{q}_i\right)^2+\tau_\theta^2\left(\ddot{q}_i\right)^2\right]\right\}$$
$$+\frac{K^2\tau_\theta}{2V^2C_\sigma T_0}\left\{1+\frac{C_\sigma}{C_E}\left(1-\frac{C_E}{C_\sigma}\right)^{\frac{1}{2}}\right\}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}_i\right)^2. \tag{6.2.57}$$

Further, the relation (6.2.36) implies that

$$\frac{\beta^2 T_0}{C_\sigma}=\frac{1}{3\left(3\lambda+2\mu\right)}\left(1-\frac{C_E}{C_\sigma}\right). \tag{6.2.58}$$

Therefore, from Eq (6.2.50), Eq. (6.2.57), and the relation (6.2.58), the following is obtained:

$$\left(\frac{1}{2\mu}-\frac{1}{\rho V^2}\right)\int_\Lambda\int_0^{m_\tau(x)}\left(\dot{\sigma}_{ij}-\frac{1}{3}\dot{\sigma}_{kk}\delta_{ij}\right)\left(\dot{\sigma}_{ij}-\frac{1}{3}\dot{\sigma}_{kk}\delta_{ij}\right)\mathrm{d}t\mathrm{d}A$$
$$+\frac{K}{T_0}\left[\tau_q+\tau_\theta-\frac{K}{C_\sigma V^2}\left\{1+\frac{C_\sigma}{C_E}\left(1-\frac{C_E}{C_\sigma}\right)^{\frac{1}{2}}\right\}\right]\int_\Lambda\int_0^{m_\tau(x)}\left(\dot{q}_i\right)^2\mathrm{d}t\mathrm{d}A$$
$$+\frac{K\tau_\theta}{T_0}\left[\frac{\tau_q^2}{2}-\frac{K\tau_\theta}{C_\sigma V^2}\left\{1+\frac{C_\sigma}{C_E}\left(1-\frac{C_E}{C_\sigma}\right)^{\frac{1}{2}}\right\}\right]\int_\Lambda\int_0^{m_\tau(x)}\left(\ddot{q}_i\right)^2\mathrm{d}t\mathrm{d}A$$
$$+\frac{1}{3}\left[\frac{1}{3\lambda+2\mu}\frac{C_E}{C_\sigma}\left\{1-\left(1-\frac{C_E}{C_\sigma}\right)^{\frac{1}{2}}\right\}-\frac{1}{\rho V^2}\right]\int_\Lambda\int_0^{m_\tau(x)}\left(\dot{\sigma}_{kk}\right)^2\mathrm{d}t\mathrm{d}A$$
$$+\frac{K}{T_0}\left[\frac{\tau_q^2}{2}-\frac{K\tau_\theta}{C_\sigma V^2}\left\{1+\frac{C_\sigma}{C_E}\left(1-\frac{C_E}{C_\sigma}\right)^{\frac{1}{2}}\right\}\right]\int_\Lambda\left(\dot{q}_i\right)^2\mathrm{d}A\leq 0. \tag{6.2.59}$$

Now, with respect to Eq. (6.2.31), the non-negativity of the coefficients of each integrals in Eq. (6.2.59) is acquired. Thus, equality sign must hold in Eq. (6.2.59) which therefore, implies that each term of Eq. (6.2.59) vanishes on Λ.

In particular, it can be stated that

$$\dot{\sigma}_{ij}\left(\boldsymbol{x},m_\tau(\boldsymbol{x})\right)=0,\;\; \dot{q}_i\left(\boldsymbol{x},m_\tau(\boldsymbol{x})\right)=0\;\text{ on }\Lambda. \tag{6.2.60}$$

Now, smoothness property of (σ_{ij}, q_i) and definition of $m_\tau(\boldsymbol{x})$ imply that

$$\left.\begin{aligned} \dot{\hat{\sigma}}_{ij}\left(\boldsymbol{x}, m_\tau(\boldsymbol{x})\right) \to \dot{\hat{\sigma}}_{ij}\left(\boldsymbol{w}, \tau\right) \\ \dot{q}_i\left(\boldsymbol{x}, m_\tau(\boldsymbol{x})\right) \to \dot{q}_i\left(\boldsymbol{w}, \tau\right) \end{aligned}\right\} \text{as } \boldsymbol{x} \to \boldsymbol{w}. \tag{6.2.61}$$

Consequently, taking the limit $\boldsymbol{x} \to \boldsymbol{w}$ in Eq. (6.2.60), Eq. (6.2.39) yields

$$\dot{\hat{\sigma}}_{ij}\left(\boldsymbol{w}, \tau\right) = 0, \;\; \dot{q}_i\left(\boldsymbol{w}, \tau\right) = 0 \;\text{ on }\; \left\{\tilde{A} - \mathfrak{D}(t)\right\} \times [0, t]\,. \tag{6.2.62}$$

Since (σ_{ij}, q_i) is sufficiently smooth in $\tilde{A} \times [0, \infty[$ and since $(\boldsymbol{w}, \tau)$ is an arbitrary point of $\left\{\tilde{A} - \mathfrak{D}(t)\right\} \times (0, t)$, therefore, it is found that

$$\dot{\hat{\sigma}}_{ij} = 0, \;\; \dot{q}_i = 0 \;\text{ on }\; \left\{\tilde{A} - \mathfrak{D}(t)\right\} \times [0, t]\,. \tag{6.2.63}$$

Now, in view of $(\boldsymbol{x}, \tau) \in \left\{\tilde{A} - \mathfrak{D}\left(t\right)\right\} \times [0, t]$, Eq. (6.2.63) yields

$$\sigma_{ij}\left(\boldsymbol{x}, \tau\right) = \sigma_{ij}\left(\boldsymbol{x}, 0\right) + \left\{1 - e^{-\frac{1}{\tau_\theta}\tau}\right\} \tau_\theta \dot{\sigma}_{ij}\left(\boldsymbol{x}, 0\right), \tag{6.2.64}$$

and

$$q_i\left(\boldsymbol{x}, \tau\right) = q_i\left(\boldsymbol{x}, 0\right). \tag{6.2.65}$$

From the definition of $\mathfrak{D}(t)$, it is concluded that

$$\sigma_{ij}\left(\boldsymbol{x}, 0\right) = \dot{\sigma}_{ij}\left(\boldsymbol{x}, 0\right) = q_i\left(\boldsymbol{x}, 0\right) = 0 \;\text{ on }\; \left\{\tilde{A} - \mathfrak{D}\left(t\right)\right\}. \tag{6.2.66}$$

Hence, Eq. (6.2.64) and Eq. (6.2.65) combining with Eq. (6.2.66) finally give

$$\sigma_{ij} = 0, \;\; q_i = 0 \;\text{ on }\; \left\{\tilde{A} - \mathfrak{D}(t)\right\} \times [0, t]\,. \tag{6.2.67}$$

Therefore, the required result is proved.

6.2.6 Conclusion

In the present subchapter, a theorem is established that states that if the condition (6.2.9) holds then the pair (σ_{ij}, q_i) satisfying the system (6.2.12-6.2.15) in view of the DPL theory, generates no stress-heat-flux disturbance outside the bounded set $\mathfrak{D}\left(t\right)$ for a finite time t and a prescribed bounded support of thermomechanical loading. Furthermore, this theorem also find that the stress-heat-flux disturbance propagates with finite speed less than or equal to V defined by inequality (6.2.31), when the relation given by (6.2.9) holds. Clearly, V depends on the material parameters including the two phase-lags, τ_q and τ_θ, and the other thermoelastic parameters. Further, it must be mentioned that in early twenty-first century, Quintanilla (2002b; 2003) stated a condition for the exponential stability of the solutions in the context of DPL theory. According to that condition, the exponential stability is attained if the phase-lag constants, τ_q and τ_θ satisfy the relation (6.2.9).

After further analyzing the results, it can be clearly noted that in the special case of only one thermal relaxation parameter (i.e., when $\tau_\theta = 0, \tau_q > 0$), the present inequality given by Eq. (6.2.9) reduces to the same inequality as reported by Ignaczak and Ostoja-Starzewski (2010) for the Lord-Shulman thermoelastic model. Moreover, when both the phase-lag parameters in the results are vanished, the propagation of thermoelastic wave generated by the pair (σ_{ij}, q_i) is obtained to be of infinite speed. This result corresponds to classical thermoelasticity theory as plugging both phase-lags as zero reduces the current thermoelastic model to the classical model. Hence, the domain of influence theorem proves the hyperbolicity of the DPL theory as described by the present system of governing equations and from this theorem, one can find the upper bound of the speed of stress-heat-flux variations.

6.3 A Study on Generalized Thermoelasticity Theory Based on Non-Local Heat Conduction Model with Dual Phase-Lags[3]

6.3.1 Introduction

Further to elaborate the discussion on dual-phase-lag thermoelasticity theory, this subchapter investigates the extension of the dual-phase-lag thermoelasticity theory considering the non-local effects in heat conduction. Non-local continuum theory helps to analyze the influence of all the points of the body at any particular material point (Eringen (2002)). Involvement of non-local factor, i.e., size effect enhances pronounce microscopic effects in heat transport process at a macroscopic level. The non-local response is same as the lagging response in space as the latter in time. As discussed by Tzou (1997), the phase-lag captures the ultrafast response in femtosecond domain. However, the non-local response enlightens the physical mechanism at nanoscale level.

In recent times, a heat conduction model is proposed by Tzou and Guo (2010) which brings out the notion of thermomass by introducing the non-local response in dual-phase-lag heat conduction model without actually using the concept of thermomass. Thermomass is defined according to Einstein's mass-energy relation as the equivalent mass of phonon gas in dielectrics. The concepts of thermal as well as mechanical fields are employed in the thermomass heat conduction theory. Subsequently, Wang et al. (2014) has developed a thermoelastic theory based on the heat conduction with thermomass concept. Moreover, Tzou and Guo (2010) have developed a new heat conduction law which considers non-local behavior with thermal lagging. Therefore, the main objective in this subchapter is to propose the general thermoelasticity theory based on the non-local heat conduction model given by Tzou and Guo (2010).

[3]The content of this subchapter is published in *Journal of Thermal Stresses*, 42(9), 2019.

This subchapter of the thesis starts by introducing the general basic governing equations of the proposed model. Then in order to analyze the prediction of the present non-local model, a one-dimensional problem is investigated for an isotropic half-space medium. The Danilovskaya's problem is considered and then the problem is formulated in Laplace transform domain. Further, the solution is obtained numerically by finding the Laplace inversion through Stehfest method (Stehfest (1970)). An attempt is made to analyze the specific predictions of the non-local model with previously developed models. However, special attention is paid to investigate the effects of the non-local length parameter and phase-lags on physical fields. The effects of different parameters is elaborated by plotting various graphs of field variables in the last subsection.

6.3.2 Basic Governing Equations

In this subsection, the governing equations are formulated in the context of generalized thermoelasticity theory based on the non-local heat conduction law with two phase-lags. The basic equations in the coordinate form in the same context for isotropic homogeneous medium can be constituted as follows:

Modified heat conduction law for non-local model (Tzou and Guo (2010))**:**

$$\left(1+(\lambda_q)_k\frac{\partial}{\partial x_k}+\tau_q\frac{\partial}{\partial t}\right)q_i=-K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\theta_{,i}. \tag{6.3.1}$$

Energy equation:

$$-q_{i,i}=\rho T_0\frac{\partial S}{\partial t}. \tag{6.3.2}$$

Entropy equation:

$$T_0\rho S=\rho c_E\theta+\beta T_0 e_{kk}. \tag{6.3.3}$$

Equation of motion:

$$\sigma_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2}. \tag{6.3.4}$$

Stress-strain-temperature relation:

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \beta\theta\delta_{ij}. \tag{6.3.5}$$

Strain-displacement relation:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{6.3.6}$$

Here, $(\lambda_q)_k$ is the component of non-local length vector.

The first governing equation is obtained by combining Eqs. (6.3.1-6.3.3) and Eq. (6.3.6) to get the heat conduction equation in the form

$$K\left(1 + \tau_\theta \frac{\partial}{\partial t}\right)\theta_{,ii} = \left(1 + (\lambda_q)_i \frac{\partial}{\partial x_i} + \tau_q \frac{\partial}{\partial t}\right)\left(\rho c_E \frac{\partial \theta}{\partial t} + \beta T_0 \frac{\partial}{\partial t}(u_{i,i})\right). \tag{6.3.7}$$

Next, by using Eqs. (6.3.4-6.3.6), the displacement equation of motion takes the form as

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu)u_{j,ji} + \mu u_{i,jj} - \beta\theta_{,i}. \tag{6.3.8}$$

Lastly, stress-displacement-temperature relation, using Eqs. (6.3.5-6.3.6), is obtained as follows:

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) - \beta\theta\delta_{ij}. \tag{6.3.9}$$

Hence, the Eq. (6.3.7) and Eq. (6.3.8) along with Eq. (6.3.9) represent the basic governing equations of the generalized thermoelasticity theory based on non-local heat conduction model given by Tzou and Guo (2010).

6.3.3 Problem Formulation

A problem of one-dimensional half-space ($x \geq 0$) is investigated for an isotropic elastic homogeneous medium in the respect of proposed non-local model. The formulation of problem for traction free boundary of the medium is considered that is subjected to a time-dependent temperature distribution. All the physical field variables are assumed to be bounded and vanish as $x \to \infty$. Therefore, $\vec{u} = (u(x,t), 0, 0)$ is considered as one-dimension displacement vector and the governing equations for one-dimension obtained from Eqs. (6.3.7-6.3.9) are as follows:

$$K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\frac{\partial^2\theta}{\partial x^2} = \left(1+\lambda_q\frac{\partial}{\partial x}+\tau_q\frac{\partial}{\partial t}\right)\left(\rho c_E\frac{\partial\theta}{\partial t}+\beta T_0\frac{\partial^2 u}{\partial x\partial t}\right), \tag{6.3.10}$$

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda+2\mu)\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial\theta}{\partial x}, \tag{6.3.11}$$

$$\sigma_{xx} = (\lambda+2\mu)\frac{\partial u}{\partial x} - \beta\theta. \tag{6.3.12}$$

For the simplification of the problem, the following non-dimensional variables and parameters are used :

$$x^{'} = c_1\xi x,\ t^{'} = c_1^2\xi t,\ \theta^{'} = \frac{\theta}{T_0},\ u^{'} = \frac{c_1(\lambda+2\mu)\xi u}{\beta T_0},\ \tau_q^{'} = c_1^2\xi\tau_q,$$
$$\tau_\theta^{'} = c_1^2\xi\tau_\theta,\ \lambda_q^{'} = c_1\xi\lambda_q,\ \text{and}\,\sigma_{xx}^{'} = \frac{\sigma_{xx}}{\beta T_0},$$

where, $c_1 = \sqrt{\frac{(\lambda+2\mu)}{\rho}}$ is the speed of propagation of isothermal elastic waves and $\xi = \frac{\rho c_E}{K}$.

By using above non-dimensional variables and parameters, Eqs. (6.3.10-6.3.12) are transformed into the following forms:

$$\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\frac{\partial^2\theta}{\partial x^2}=\left(1+\lambda_q\frac{\partial}{\partial x}+\tau_q\frac{\partial}{\partial t}\right)\left(\frac{\partial\theta}{\partial t}+\varepsilon\frac{\partial^2 u}{\partial x\partial t}\right), \tag{6.3.13}$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}-\frac{\partial\theta}{\partial x}, \tag{6.3.14}$$

$$\sigma_{xx}=\frac{\partial u}{\partial x}-\theta, \tag{6.3.15}$$

where, $\varepsilon=\frac{\beta^2 T_0}{\rho^2 c_E c_1^2}$ is the thermoelastic coupling constant. Here, the primes are dropped for the convenience.

Initial and boundary conditions:

All the initial conditions are considered as homogeneous and the boundary conditions are assumed as follows:

$$\left.\begin{aligned}\sigma_{xx}(x,t)|_{x=0}=0,\ u(x,t)|_{x=\infty}=0\ \text{ for } t>0\\ \theta(x,t)|_{x=0}=\theta_0\mathscr{H}(t),\ \theta(x,t)|_{x=\infty}=0\ \text{ for } t>0\end{aligned}\right\}, \tag{6.3.16}$$

where, $\mathscr{H}(t)$ is Heaviside unit step function and θ_0 is a constant temperature. The above initial and boundary conditions constitute the Danilovskaya's problem.

6.3.4 Solution of the Problem in the Laplace Transform Domain

After applying the Laplace transform on time, t to Eqs. (6.3.13-6.3.15), the following equations are obtained:

$$(nD^2-m-\lambda_q sD)\bar{\theta}=\varepsilon(mD+\lambda_q sD^2)\bar{u}, \tag{6.3.17}$$

$$(D^2-s^2)\bar{u}=D\bar{\theta}, \tag{6.3.18}$$

$$\bar{\sigma}_{xx}=D\bar{u}-\bar{\theta}, \tag{6.3.19}$$

where,

$$D=\frac{\mathrm{d}}{\mathrm{d}x},\ \ n=(1+\tau_\theta s),\ \ m=s\left(1+\tau_q s\right), \tag{6.3.20}$$

and $\bar{u}(s)$, $\bar{\theta}(s)$, and $\bar{\sigma}_{xx}(s)$ represents the Laplace transform of $u(t)$, $\theta(t)$, and $\sigma_{xx}(t)$, respectively with s as the Laplace Transform parameter.

Again, applying Laplace transform to boundary conditions (6.3.16) gives

$$\left.\begin{aligned} \bar{\sigma}_{xx}(x,s)|_{x=0} = 0,\ \bar{u}(x,s)|_{x=\infty} = 0, \\ \bar{\theta}(x,s)|_{x=0} = \tfrac{\theta_0}{s},\ \bar{\theta}(x,s)|_{x=\infty} = 0. \end{aligned}\right\} \tag{6.3.21}$$

Now, solving the Eqs. (6.3.17-6.3.18) gives decoupled equations in terms of $\bar{u}$ and $\bar{\theta}$ as

$$[nD^4 - (1+\varepsilon)\lambda_q sD^3 - (ns^2 + m + m\varepsilon)D^2 + \lambda_q s^3 D + ms^2](\bar{u}, \bar{\theta}) = 0. \tag{6.3.22}$$

The corresponding auxiliary equation will be

$$nk^4 - (1+\varepsilon)\lambda_q sk^3 - (ns^2 + m + m\varepsilon)k^2 + \lambda_q s^3 k + ms^2 = 0. \tag{6.3.23}$$

Since, all the variables are vanishing as $x \to \infty$, only the roots with negative real parts of Eq.(6.3.23) are considered to avoid the positive powers of exponential while expressing the solution of differential Eq. (6.3.22).

Therefore, the solution of Eqs. (6.3.17-6.3.19) is acquired using the boundary condition (6.3.21) as

$$\bar{\theta}(x,\, s) = \frac{\theta_0}{s\,(k_2^2 - k_1^2)}[(s^2 - k_1^2)\, e^{-k_1 x} - (s^2 - k_2^2)\, e^{-k_2 x}], \tag{6.3.24}$$

$$\bar{u}(x,\, s) = \frac{\theta_0}{s\,(k_2^2 - k_1^2)}[k_1\, e^{-k_1 x} - k_2\, e^{-k_2 x}], \tag{6.3.25}$$

$$\bar{\sigma}_{xx}(x,\, s) = \frac{s\,\theta_0}{(k_2^2 - k_1^2)}[e^{-k_2 x} - \, e^{-k_1 x}], \tag{6.3.26}$$

where, $-k_1$ and $-k_2$ are the roots of Eq. (6.3.23) such that $\mathrm{Re}(k_i) > 0\,(i = 1, 2)$.

Therefore, Eqs. (6.3.24-6.3.26) give the solution of physical fields in the Laplace transform domain.

6.3.5 Numerical Results and Discussion

In this subsection, the solution to the problem is found numerically. The aid of mathematical softwares, MATLAB and Mathematica is used to compute the roots of the auxiliary Eq. (6.3.23). The Stehfest method (Stehfest (1970)) is used to compute the Laplace inversion involved in the solutions obtained in previous subsection. The main aim of this subsection is to analyze the newly developed model on the basis of parameters involved and also by comparing it to the corresponding results of previously established generalized thermoelasticity models. The discussion is made by considering the numerical data for copper material as following (Sherief et al. (2013)):

$$\lambda = 7.76{\times}10^{10}\ \mathrm{kg\ m^{-1}s^{-2}},\ \mu = 3.86\times 10^{10}\ \mathrm{kg\ m^{-1}}s^{-2},\ \rho = 8954\ \mathrm{kg\ m^{-3}},$$
$$c_E = 383.1\ \mathrm{J\ k}g^{-1}\mathrm{K}^{-1}, \beta_\theta = 1.78\times 10^{-5}\mathrm{K}^{-1},\ K = 386\ \mathrm{Wm^{-1}K^{-1}},\ T_0 = 293\,\mathrm{K}.$$

The effects are analyzed at two non-dimensional time, one at $t = 0.1$ and other at $t = 0.5$. The dashed line in each figure represents the result for $t = 0.1$ while the solid line represents the result for $t = 0.5$.

6.3.5.1 Effect of λ_q

The effect of λ_q is analyzed by taking the five values : $0, 0.012,\ 0.12,\ 1.2,$ and 12 and by keeping $\tau_q = 0.015$ and $\tau_\theta = 0.01$ fixed. Here, the case of $\lambda_q = 0$ corresponds to DPL model. Further, the case of $\lambda_q = 0$ and $\tau_\theta = 0$ corresponds to the LS model. Figs. (6.3.1-6.3.3) show the distributions of non-dimensional temperature, stress and displacement fields, respectively for different values of λ_q. An attempt is made to analyze the effects of non-local model by comparing the results for $\lambda_q \neq 0$ with the corresponding results for LS model and DPL model. The main effect that is observed is the change in domain of influence. The domain of influence increases with increase in value of λ_q. However, at a particular time, the nature of graphs remains the same for different values of λ_q, i.e., behavior does not alter significantly with the change in value of λ_q. The graphs of non-local model show minute variation with that of LS model and DPL model for small values of λ_q whereas for higher values of λ_q, a significant difference can be observed in the results. Like, for $\lambda_q = 0.012$, it is noticed that the results predicted by DPL and LS models almost coincide with that of the non-local model but for higher values of λ_q, the significant variation in the predictions by non-local model and other generalized thermoelastic models is observed. However, it is noticed that as the time increases the effect of λ_q diminishes in case of temperature. A similar effect is observed in case of stress and displacement fields. Further, the increase in the domain of influence with increase of time is observed for each field. It can be concluded from these observations that the involvement of non-local length parameter, λ_q have significant effects that make the non-local thermoelastic model different from previous other models.

6.3.5.2 Effect of τ_q

The effect of τ_q on the behavior of physical fields is analyzed by taking the five values of τ_q as 0.0015, 0.015, 0.15, 1.5, and 15 and keeping $\lambda_q = 0.012$ and $\tau_\theta = 0.01$ fixed. Figs. (6.3.4-6.3.6) show the effect of τ_q on non-dimensional temperature, stress, and displacement, respectively. Effect on the domain of influence is significant. However, the effect of this parameter is lesser than the effect due to the non-local parameter, λ_q. The effect of time on domain of influence is clearly visible in this case too. The nature of graphs of each field is same for the first two values, but it changes for higher values. Further, for higher values of τ_q, the negative values for temperature is reflected in the graph. Such behavior in the temperature field is not observed for smaller values of τ_q. In case of displacement, unlike the effect of λ_q, the magnitude of displacement decreases as the value of τ_q increases for small values of x.

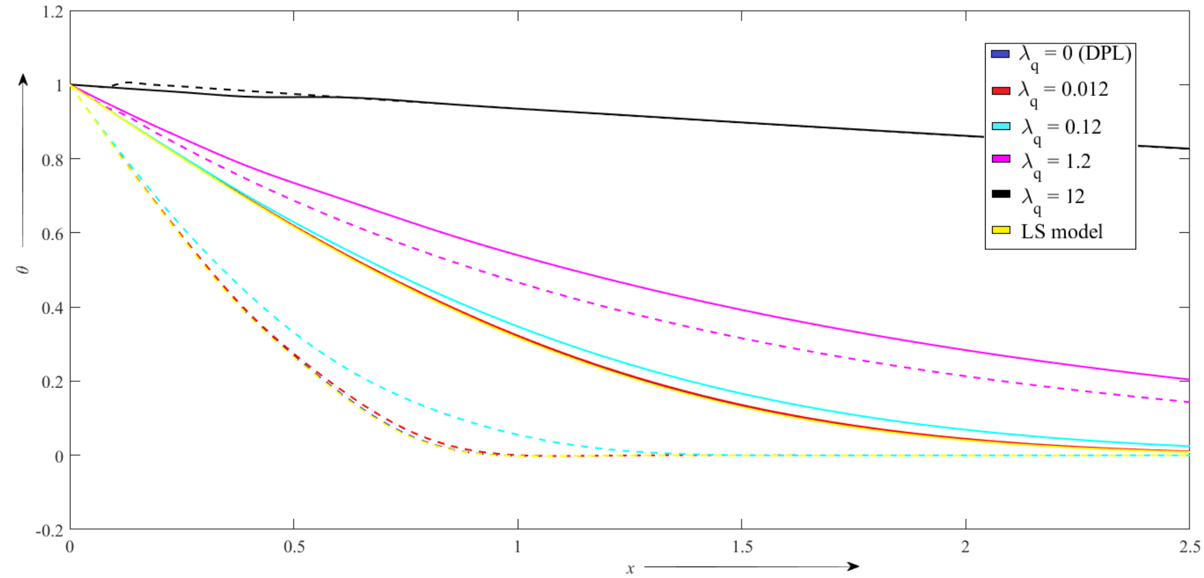


Figure 6.3.1: Effect of non-local parameter, λ_q on temperature distribution

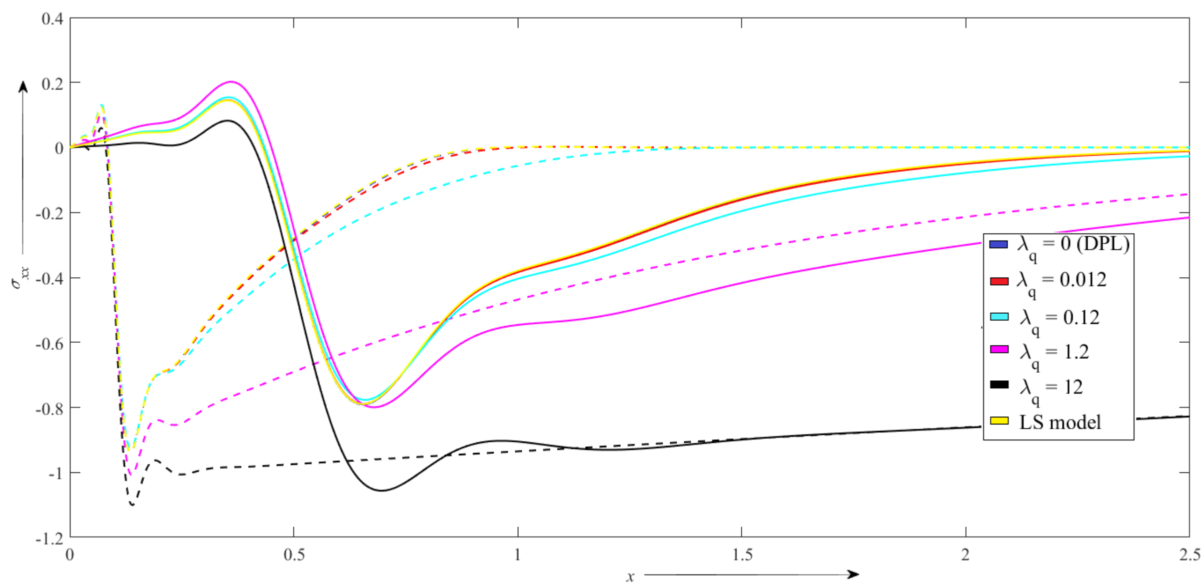


Figure 6.3.2: Effect of non-local parameter, λ_q on stress distribution

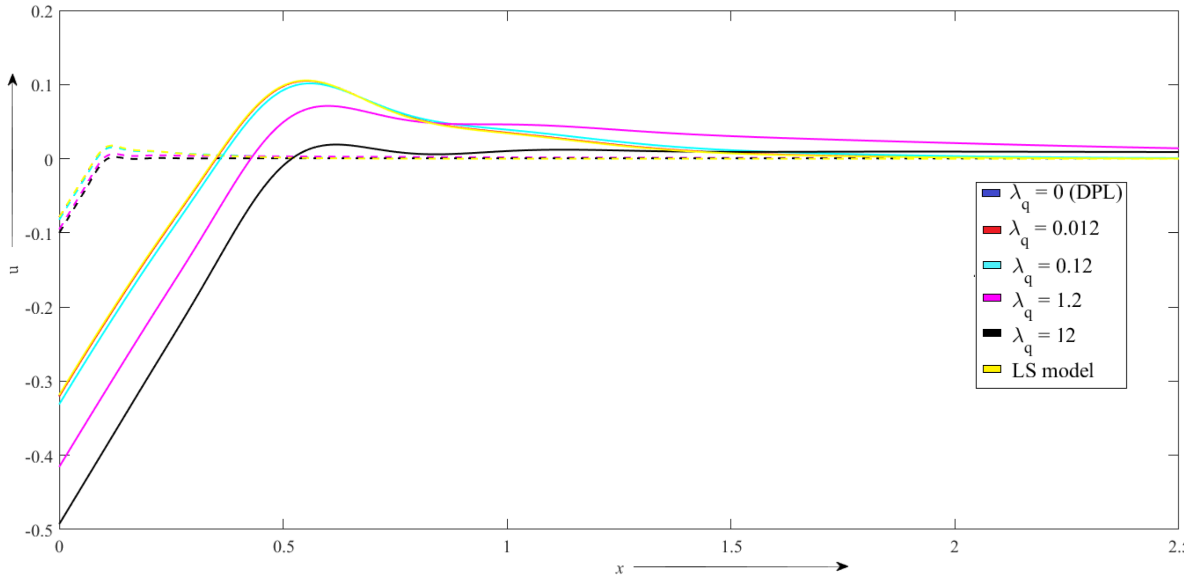


Figure 6.3.3: Effect of non-local parameter, λ_q on displacement distribution

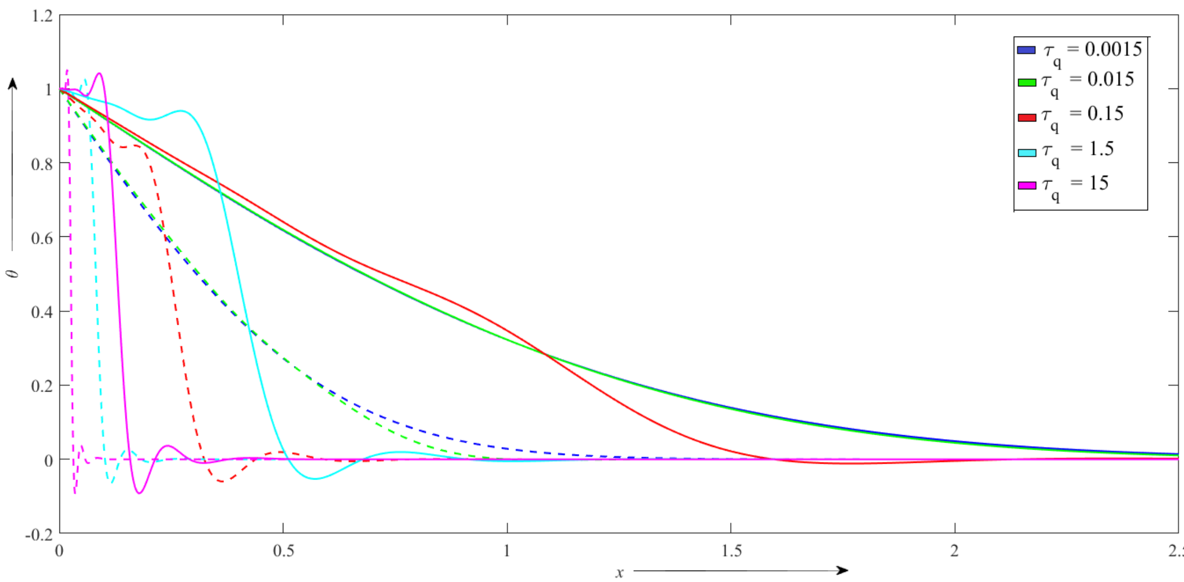


Figure 6.3.4: Effect of phase-lag, τ_q on temperature distribution

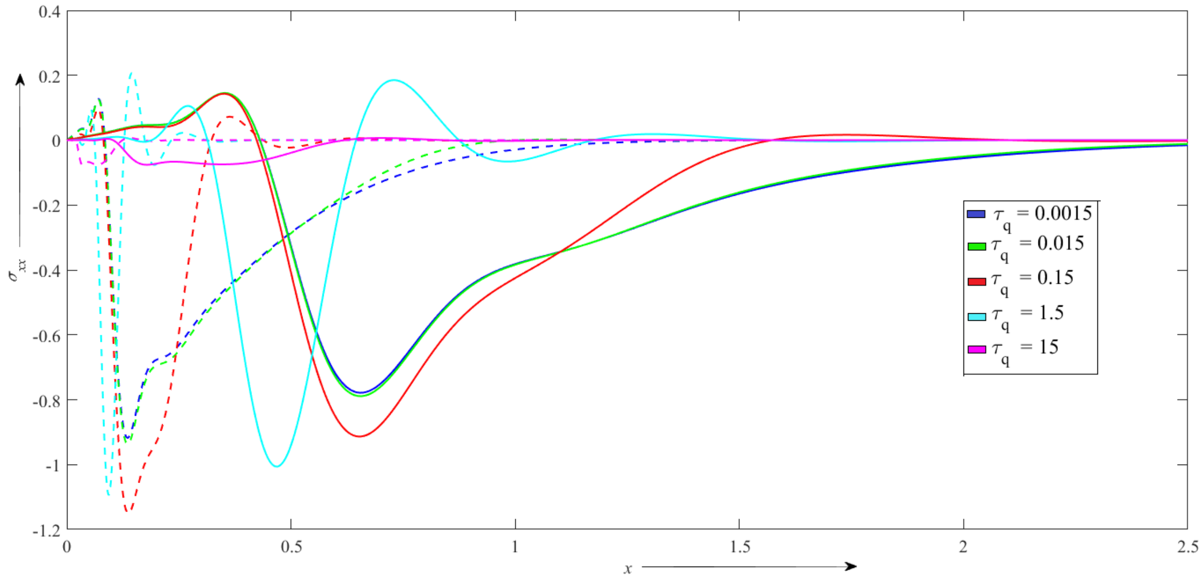


Figure 6.3.5: Effect of phase-lag, τ_q on stress distribution

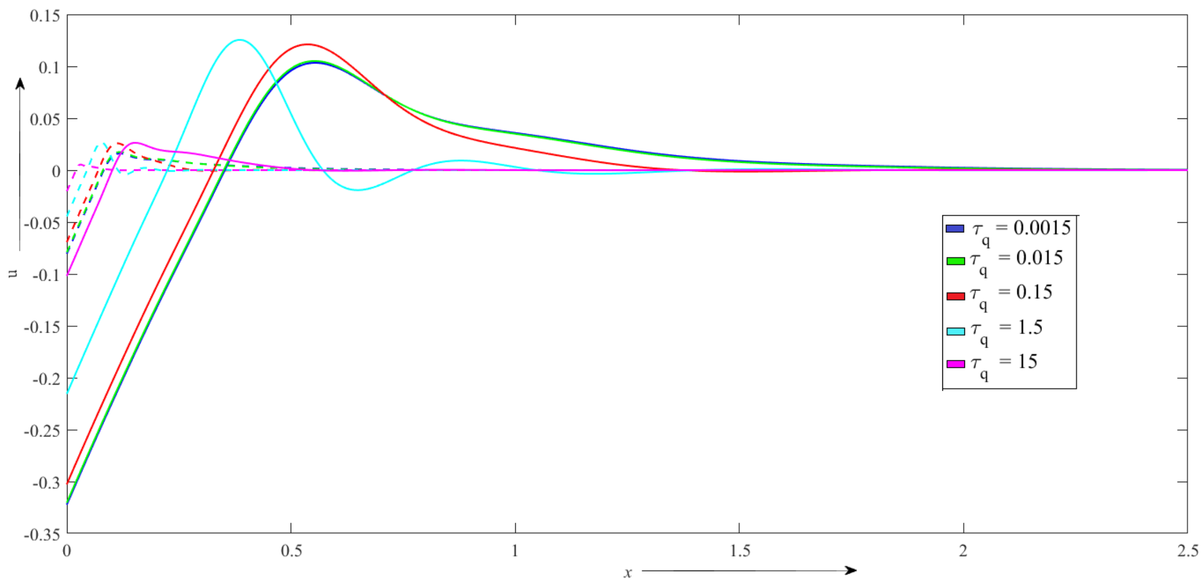


Figure 6.3.6: Effect of phase-lag, τ_q on displacement distribution

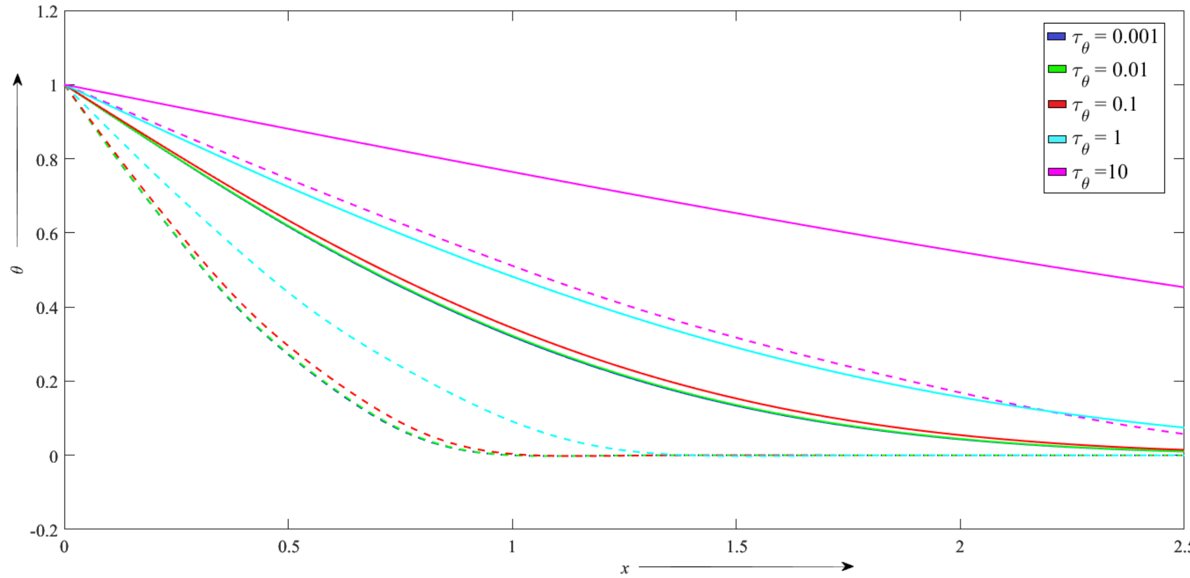


Figure 6.3.7: Effect of phase-lag, τ_θ on temperature distribution

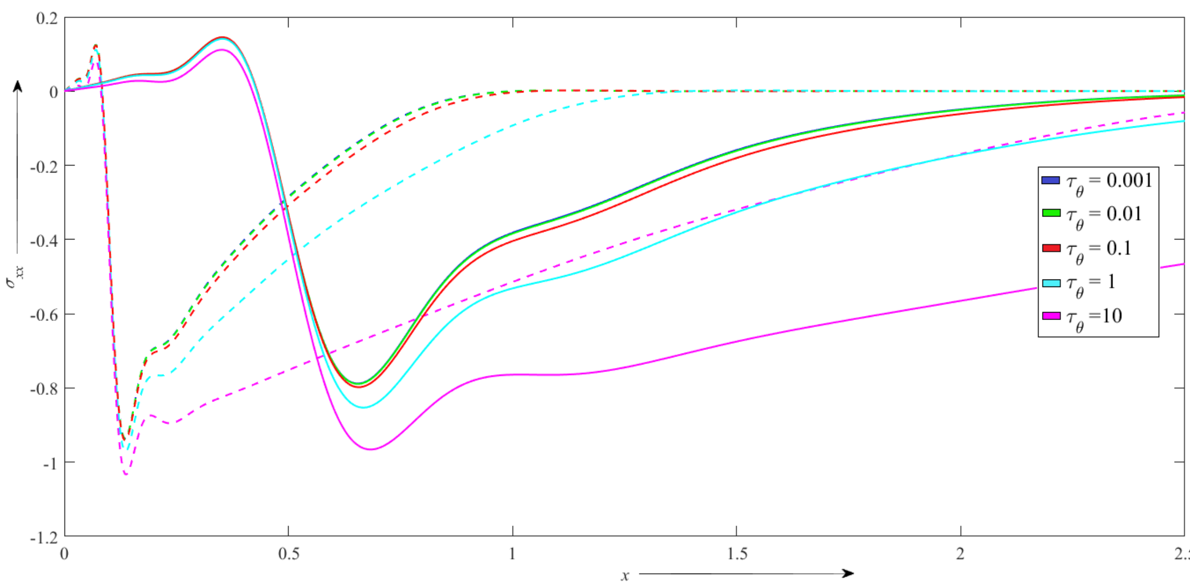


Figure 6.3.8: Effect of phase-lag, τ_θ on stress distribution

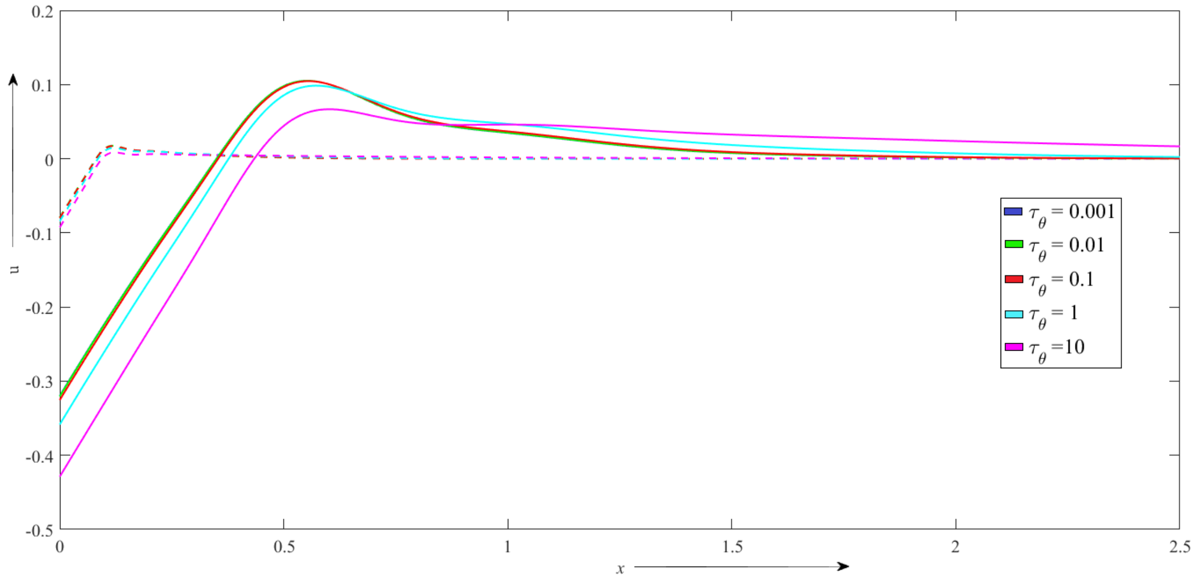


Figure 6.3.9: Effect of phase-lag, τ_θ on displacement distribution

6.3.5.3 Effect of τ_θ

The values of τ_θ are considered as 0.001, 0.01, 0.1, 1. and 10 with $\lambda_q = 0.012$ and $\tau_q = 0.015$. Figs. (6.3.7-6.3.9) show the effect of τ_θ on non-dimensional temperature, stress and displacement, respectively. A prominent effect of τ_θ on each field is observed. The effect increases with increase of time and it is more significant in temperature and stress as compared to displacement. The domain of influence for each field is significantly affected by τ_θ, and it is more visible at higher time.

6.3.6 Conclusion

The new non-local heat conduction model introduced by Tzou and Guo (2010) that involves thermal lagging in the form of dual phase-lags, has given the insight of thermomass effect. Further, the inclusion of non-local factor, i.e., size effect enhances the microscopic effects in heat transport process at a macroscopic level. However, the heat conduction model in this theory has been formulated without actually using the concept of thermomass. In this subchapter, an effort has been made to extend the concept of this new non-local heat conduction model to the generalized thermoelasticity theory. The basic governing equations for the generalized thermoelasticity theory in the context of this non-local heat conduction model have been formulated and a one-dimensional problem has been investigated. The predictions of the new thermoelastic model have been analyzed by specially paying attention to the effect of non-local length parameter, λ_q, which is the characteristic of this non-local heat conduction model. A significant effect of this parameter is observed on the behavior of physical fields like, displacement, temperature and stress. The results of the present context are further compared with the corresponding results of the other existing models, like LS model and DPL model. The variation of LS and DPL model from this non-local model increases with an increase in value of λ_q. The effect of λ_q is also observed to be prominent on the domain of influence of the field variables. The impacts of phase-lag parameters, τ_q and τ_θ, in presence of λ_q are also investigated. It is observed that the effect of phase-lag, τ_q is more prominent as compared to the phase-lag, τ_θ. It is believed that the present investigation will help in understanding the non-local effect of heat conduction in mutual interactions due to thermomechanical loading in a medium. Further study in this direction will result in developing better and more appropriate model of thermoelasticity.