CHAPTER 4

THEORETICAL ANALYSIS OF MODIFIED GREEN-LINDSAY THERMOELASTICITY THEORY

4.1 Introduction[1]

The present chapter and the subsequent chapter of the thesis aim at discussing some aspects of newly proposed generalized thermoelasticity theory by Yu et al. (2018). This theory is developed as a modification of the temperature rate-dependent theory introduced by Green and Lindsay (1972). The Green-Lindsay thermoelasticity theory (GL theory) modified the Biot's theory and successfully overcame the paradox of infinite speed of thermal wave propagation by altering the conventional theory with the interesting fact of keeping Fourier's law intact in the case of a centrally symmetric body. Here the constitutive relations were modified by including two thermal relaxation parameters and temperature rate terms as compared to the other thermoelasticity theories like, Lord-Shulman theory or the dual-phase-lag theory. Like LS theory, this theory has been investigated by several researchers to study various problems involving thermoelastic interactions arisen due to thermomechanical loads in elastic media. Detailed comparison of results predicted by classical theory, LS theory, and GL theory have also been reported in literature as mentioned in the introduction section. However, it

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has been reported by some researchers that GL theory predicts discontinuity in the displacement field for transient motion (see Chandrasekharaiah and Srikantiah (1986; 1987), Dhaliwal and Rokne (1989), Ignaczak and Mr' owka-Matejewska (1990), Yu et al. (2018), and references therein). Discontinuity in the displacement field disobeys the continuum hypothesis (Chandrasekharaiah (1998)). Recently, thermodynamic principles are employed differently by Yu et al. (2018) who have developed a modified version of GL theory by incorporating strain-rate terms along with the temperature-rate terms in constitutive relations. Hence, this theory is called as modified Green-Lindsay (MGL) theory. The strain-rate term is usually neglected in constitutive relations of linear theory by assuming it to be relatively small. However, this cannot be an appropriate assumption for extreme conditions such as in ultra-fast heating. Hence, MGL theory is developed with the help of extended thermodynamic principles and generalized dissipation inequality. This theory is a new and alternative modification of the classical thermoelasticity theory and is yet to get attention of researchers. Hence, it is worth investigating some aspects of this recent theory which is also referred to as strain and temperature rate-dependent thermoelasticity theory.

In the present chapter, the linearized theory for isotropic and homogeneous thermoelastic material under modified Green-Lindsay (MGL) model is taken into consideration. The chapter finds the Galerkin-type representation of the solution in the context of this theory. Representation of solution in terms of elementary functions such as harmonic, biharmonic, metaharmonic, etc. helps in solving various boundary value problems in the field of elasticity and thermoelasticity. Some interesting works related to this chapter can be found in the following references: Ciarletta (1991; 1995; 1999), Svanadze (1993), Svanadze and Boer (2005), Mukhopadhyay et al. (2010), etc. The presentation of work in this chapter is organized as follows. In Section 4.2, the basic governing equations and constitutive relations for the MGL model in the presence of body forces and heat sources are presented. Next, in Section 4.3, a Galerkin-type solution of equations of motion is presented followed by a Galerkin-type solution for the system of equations of steady oscillations given in Section 4.4. Lastly, in Section 4.5, the general solution for the homogeneous system of equations for steady oscillations is obtained.

4.2 Governing Equations

Let $\boldsymbol{x} = (x_1, x_2, x_3)$ represents an arbitrary point in three-dimensional Euclidean space and t be the time variable. An isotropic elastic homogeneous medium is considered to analyze a thermoelasticity theory. The medium occupies a bounded region Ξ of Euclidean three-dimensional space at $t = 0$. In presence of body forces and heat sources, the basic equations in the context of modified Green-Lindsay (MGL) linear thermoelasticity theory given by Yu et al. (2018) are presented as follows:

Heat conduction law:

$$\boldsymbol{q} = -K \,\mathrm{grad}\,\theta. \tag{4.2.1}$$

Energy equation:

$$-\mathrm{div}\,\boldsymbol{q} + R = \rho\, T_0 \dot{S}. \tag{4.2.2}$$

Entropy equation:

$$T_0\,\rho\, S = \rho\, c_E \left(\theta + \tau_0\,\dot{\theta}\right) + \beta\, T_0 \left(\mathrm{tr}\boldsymbol{E} + \tau_0\,\mathrm{tr}\dot{\boldsymbol{E}}\right). \tag{4.2.3}$$

Equation of motion:

$$\mathrm{div}\,\boldsymbol{\Gamma} + \rho\,\boldsymbol{H} = \rho\,\ddot{\boldsymbol{u}}. \tag{4.2.4}$$

Stress-strain-temperature relation:

$$\boldsymbol{\Gamma} = \lambda \left(\mathrm{tr}\boldsymbol{E} + \tau_1\,\mathrm{tr}\dot{\boldsymbol{E}}\right)\boldsymbol{I} + 2\mu\left(\boldsymbol{E} + \tau_1\dot{\boldsymbol{E}}\right) - \beta\left(\theta + \tau_1\dot{\theta}\right)\boldsymbol{I}. \tag{4.2.5}$$

Strain-displacement relation:

$$\boldsymbol{E} = \frac{1}{2}\left(\mathrm{grad}\boldsymbol{u} + (\mathrm{grad}\boldsymbol{u})^T\right). \tag{4.2.6}$$

Here, τ_0 and τ_1 represent two relaxation times with the condition, $\tau_1 \geq \tau_0 > 0$.

Further, eliminating $\boldsymbol{q}$, $\boldsymbol{E}$, $\boldsymbol{\Gamma}$, and S from Eqs. (4.2.1-4.2.6) gives the following field equations in the context of modified Green-Lindsay thermoelasticity theory given by Yu et al. (2018) :

$$\begin{aligned}
&\mu\left(\nabla^2\boldsymbol{u} + \tau_1\nabla^2\dot{\boldsymbol{u}}\right) + (\lambda+\mu)\left(\mathrm{grad}\,\mathrm{div}\boldsymbol{u} + \tau_1\mathrm{grad}\,\mathrm{div}\dot{\boldsymbol{u}}\right)\\
&\qquad -\beta\left(\mathrm{grad}\,\theta + \tau_1\,\mathrm{grad}\,\dot{\theta}\right) + \rho\,\boldsymbol{H} = \rho\,\ddot{\boldsymbol{u}},
\end{aligned} \tag{4.2.7}$$

$$K\nabla^2\theta = \beta\,T_0\left(\mathrm{div}\,\dot{\boldsymbol{u}} + \tau_0\,\mathrm{div}\,\ddot{\boldsymbol{u}}\right) + \rho\,c_E\left(\dot{\theta} + \tau_0\,\ddot{\theta}\right) - R. \tag{4.2.8}$$

Now, the following notations and operators are introduced:

$$\begin{aligned}
&\ell_1\left(\nabla^2, T\right) = m_2\Box_1\nabla^2 - T^2, \quad \ell_2\left(\nabla^2, T\right) = K\,\nabla^2 - \rho\,c_E\Box_2,\\
&\Box_1(T) = 1 + \tau_1 T, \quad \Box_2(T) = T + \tau_0 T^2, \quad T = \frac{\partial}{\partial t}, \quad T^2 = \frac{\partial^2}{\partial t^2},\\
&m_1 = \left(\frac{\lambda+\mu}{\rho}\right), \quad m_2 = \frac{\mu}{\rho}, \quad m_3 = \frac{\beta}{\rho}.
\end{aligned}$$

Therefore, Eq. (4.2.7) and Eq. (4.2.8) take the forms as follows:

$$m_1\Box_1\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + \ell_1\,\boldsymbol{u} - m_3\Box_1\mathrm{grad}\,\theta = -\boldsymbol{H}, \tag{4.2.9}$$

$$\ell_2\,\theta - \beta\,T_0\Box_2\mathrm{div}\,\boldsymbol{u} = -R. \tag{4.2.10}$$

4.3 Galerkin-Type Solution of Equations of Motion

By virtue of Eq. (4.2.9) and Eq. (4.2.10), the matrix differential operator is introduced as following:

$$
\begin{aligned}
&\boldsymbol{\Omega}\left(\boldsymbol{D_x},T\right)=\begin{bmatrix}\boldsymbol{\Omega^{(1)}} & \boldsymbol{\Omega^{(2)}}\\ \boldsymbol{\Omega^{(3)}} & \boldsymbol{\Omega^{(4)}}\end{bmatrix},\\
&\boldsymbol{\Omega^{(1)}}\left(\boldsymbol{D_x},T\right)=\left[\Omega^{(1)}_{pq}\right]_{3\times3},\ \ \boldsymbol{\Omega^{(2)}}=\left[\Omega^{(2)}_{p1}\right]_{3\mathrm{x}1},\ \ \boldsymbol{\Omega^{(3)}}=\left[\Omega^{(3)}_{1q}\right]_{1\times3},\ \ \boldsymbol{\Omega^{(4)}}=\left[\Omega_{44}\right]_{1\times1},\\
&\Omega^{(1)}_{pq}\left(\boldsymbol{D_x},T\right)=\ell_1\delta_{pq}+m_1\Box_1\frac{\partial^2}{\partial x_p\partial x_q},\\
&\Omega^{(2)}_{p1}\left(\boldsymbol{D_x},T\right)=-m_3\,\Box_1\frac{\partial}{\partial x_p},\\
&\Omega^{(3)}_{1q}\left(\boldsymbol{D_x},T\right)=\left(-\beta\,T_0\Box_2\right)\frac{\partial}{\partial x_q},\quad \Omega_{44}(\boldsymbol{D_x},T)=\ell_2,
\end{aligned}
\tag{4.3.1}
$$

where, the notations are used as; $\boldsymbol{D_x}=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)$ and δ_{pq} as the Kronecker delta for $p,q=1,2,3$.

Therefore, Eq. (4.2.9) and Eq. (4.2.10), can be written as

$$\boldsymbol{\Omega}\left(\boldsymbol{D_x},T\right)\boldsymbol{U}\left(\boldsymbol{x},t\right)=\boldsymbol{\mathcal{F}}\left(\boldsymbol{x},t\right), \tag{4.3.2}$$

where, $\boldsymbol{U}=(\boldsymbol{u},\theta)$, $\boldsymbol{\mathcal{F}}=(-\boldsymbol{H},-R)$ and $(\boldsymbol{x},t)\in\mathit{\Xi}\times(0,+\infty)$.

Now, the following system of equations are introduced:

$$m_1\Box_1\mathrm{grad\,div}\boldsymbol{u}+\ell_1\,\boldsymbol{u}-\beta\,T_0\,\Box_2\,\mathrm{grad}\,\theta=\boldsymbol{\mathcal{F}'}, \tag{4.3.3}$$

$$\ell_2\,\theta-m_3\Box_1\mathrm{div}\,\boldsymbol{u}=\mathcal{F}_0, \tag{4.3.4}$$

where, $\boldsymbol{\mathcal{F}'}=(\mathcal{F}'_1,\mathcal{F}'_2,\mathcal{F}'_3)$ is the vector function with $\mathcal{F}_0$ and $\mathcal{F}'_i\,(i=1,2,3)$ as scalar functions on $\mathit{\Xi}\times(0,+\infty)$.

Hence, in term of matrix operator, system (4.3.3-4.3.4) can be expressed in the form

$$\boldsymbol{\Omega^T}\left(\boldsymbol{D_x},T\right)\boldsymbol{U}\left(\boldsymbol{x},T\right)=\boldsymbol{\mathcal{H}}\left(\boldsymbol{x},t\right), \tag{4.3.5}$$

where, $\boldsymbol{\Omega^T}$ is the transpose of matrix $\boldsymbol{\Omega}$ and $\boldsymbol{\mathcal{H}} = (\boldsymbol{\mathcal{F}'}, \mathcal{F}_0)$.

Next, taking divergence of Eq. (4.3.3), yields

$$\mathfrak{B}_1 \mathrm{div}\, \boldsymbol{u} - \beta\, T_0 \Box_2 \nabla^2 \theta = \mathrm{div}\, \boldsymbol{\mathcal{F}'}, \tag{4.3.6}$$

where, $\mathfrak{B}_1(\nabla^2, T) = \left(\frac{\lambda+2\mu}{\rho}\right) \Box_1 \nabla^2 - T^2$.

Therefore, the matrix representation of Eq. (4.3.4) and Eq. (4.3.6) is derived as follows:

$$\boldsymbol{\mathfrak{B}} \left(\nabla^2, T\right) \boldsymbol{V} = \tilde{\boldsymbol{\mathcal{F}}}, \tag{4.3.7}$$

where, $\boldsymbol{V} = (\mathrm{div}\ \boldsymbol{u},\, \theta)$, $\tilde{\boldsymbol{\mathcal{F}}} = (\mathrm{div}\, \boldsymbol{\mathcal{F}'}, \mathcal{F}_0)$, and

$$\boldsymbol{\mathfrak{B}}(\nabla^2, T) = \left[\mathfrak{B}_{pq}\left(\nabla^2, T\right)\right]_{2\times 2} = \begin{bmatrix} \mathfrak{B}_1 & -\beta\, T_0 \Box_2 \nabla^2 \\ -m_3 \Box_1 & \ell_2 \end{bmatrix}.$$

System (4.3.7) implies

$$\chi_1 \left(\nabla^2, T\right)\ \boldsymbol{V} = \boldsymbol{\Phi}, \tag{4.3.8}$$

with,

$$\boldsymbol{\Phi} = \left(\Phi_1, \Phi_2\right), \quad \Phi_q = \sum_{p=1}^{2} \mathfrak{B}^*_{pq} f_p, \quad \chi_1(\nabla^2, T) = \det \boldsymbol{\mathfrak{B}} \left(\nabla^2, T\right), \tag{4.3.9}$$

where, $q = 1,\, 2$ and $\mathfrak{B}^*_{pq}$ is the co-factor of the element $\mathfrak{B}_{pq}$ of the matrix $\boldsymbol{\mathfrak{B}}$.

Now, operating $\chi_1\left(\nabla^2, T\right)$ to Eq. (4.3.3), and using Eq. (4.3.8), gives the following relation:

$$\chi_1 \left(\nabla^2, T\right)\ \ell_1 \boldsymbol{u} = \boldsymbol{\Phi'}, \tag{4.3.10}$$

where,

$$\boldsymbol{\Phi'} = \chi_1 \boldsymbol{\mathcal{F}'} - \mathrm{grad}\left[m_1 \Box_1 \Phi_1 - \beta\, T_0 \Box_2 \Phi_2\right]. \tag{4.3.11}$$

Further, Eqs (4.3.8) and (4.3.10) give

$$\boldsymbol{\chi}\left(\nabla^2, T\right) \boldsymbol{U}(\boldsymbol{x}, t) = \tilde{\boldsymbol{\Phi}}, \tag{4.3.12}$$

where, $\tilde{\boldsymbol{\Phi}} = (\boldsymbol{\Phi'},\ \Phi_2)$ and

$$\begin{aligned}
\boldsymbol{\chi}(\nabla^2, T) &= \left[\chi_{pq}(\nabla^2, T)\right]_{4\times 4}, \\
\chi_{jj} &= \chi_1\left(\nabla^2, T\right)\ \ell_1, \ \ j = 1, 2, 3, \\
\chi_{44} &= \chi_1(\nabla^2, T), \ \ \chi_{pq} = 0, \ \ p,\ q = 1, 2, 3, 4\ p \neq q.
\end{aligned} \tag{4.3.13}$$

Further, introducing the operators

$$\begin{aligned}
n_{p1}\left(\nabla^2, T\right) &= -\left\{m_1 \Box_1 \mathfrak{B}^*_{p1} - \beta\, T_0 \Box_2 \mathfrak{B}^*_{p2}\right\}, \\
n_{p2}(\nabla^2, T) &= \mathfrak{B}^*_{p2}, \ \ p = 1,\ 2,
\end{aligned} \tag{4.3.14}$$

and using Eq. (4.3.9), Eq. (4.3.11) yields

$$\boldsymbol{\Phi'} = (\chi_1 \boldsymbol{I} + n_{11}\text{grad div})\, \boldsymbol{\mathcal{F}'} + n_{21}\text{grad}\ \mathcal{F}_0, \tag{4.3.15}$$

$$\Phi_2 = n_{12}\text{div}\, \boldsymbol{\mathcal{F}'} + n_{22}\mathcal{F}_0. \tag{4.3.16}$$

Thus, in view of Eq. (4.3.15) and Eq. (4.3.16), it is found that

$$\tilde{\boldsymbol{\Phi}}(\boldsymbol{x}, t) = \boldsymbol{\mathcal{L}}^{\boldsymbol{T}}\left(\boldsymbol{D}_{\boldsymbol{x}}, T\right) \boldsymbol{\mathcal{H}}(\boldsymbol{x}, t), \tag{4.3.17}$$

where,

$$\boldsymbol{\mathcal{L}} = \begin{bmatrix} \boldsymbol{\mathcal{L}^{(1)}} & \boldsymbol{\mathcal{L}^{(2)}} \\ \boldsymbol{\mathcal{L}^{(3)}} & \boldsymbol{\mathcal{L}^{(4)}} \end{bmatrix}_{4\times 4},$$

$$\boldsymbol{\mathcal{L}^{(1)}} = \left[\mathcal{L}^{(1)}_{pq}\right]_{3\times 3}, \ \boldsymbol{\mathcal{L}^{(2)}} = \left[\mathcal{L}^{(2)}_{p1}\right]_{3\text{x}1}, \ \boldsymbol{\mathcal{L}^{(3)}} = \left[\mathcal{L}^{(3)}_{1q}\right]_{1\times 3}, \ \boldsymbol{\mathcal{L}^{(4)}} = \left[\mathcal{L}_{44}\right]_{1\times 1},$$

$$\mathcal{L}^{(1)}_{pq}\left(\boldsymbol{D_x}, T\right) = \chi_1\left(\nabla^2, T\right)\delta_{pq} + n_{11}\left(\nabla^2, T\right)\frac{\partial^2}{\partial x_p \partial x_q}, \ \mathcal{L}^{(2)}_{p1}\left(\boldsymbol{D_x}, T\right) = n_{12}\left(\nabla^2, T\right)\frac{\partial}{\partial x_p},$$

$$\mathcal{L}^{(3)}_{1q}(\boldsymbol{D_x}, T) = n_{21}(\nabla^2, T)\frac{\partial}{\partial x_q}, \ \ \mathcal{L}_{44} = n_{22}(\nabla^2, T), \ \ p, q = 1,\ 2,\ 3. \tag{4.3.18}$$

Next, using Eq. (4.3.5), Eq. (4.3.12), and Eq. (4.3.17), the following equation is obtained

$$\boldsymbol{\chi U} = \boldsymbol{\mathcal{L}^T \Omega^T U},$$

which implies $\boldsymbol{\mathcal{L}^T \Omega^T = \chi}$ and hence,

$$\boldsymbol{\Omega}(\boldsymbol{D_x}, T)\boldsymbol{\mathcal{L}}(\boldsymbol{D_x}, T) = \boldsymbol{\chi}(\nabla^2, T). \tag{4.3.19}$$

Thus, the following lemma is proved.

Lemma-4.3.1:

Statement: If the matrix differential operators $\boldsymbol{\Omega}$, $\boldsymbol{\mathcal{L}}$, and $\boldsymbol{\chi}$ are defined by Eq. (4.3.1), Eq. (4.3.18), and Eq. (4.3.13), respectively, then $\boldsymbol{\Omega}$, $\boldsymbol{\mathcal{L}}$, and $\boldsymbol{\chi}$ satisfy Eq. (4.3.19).

Now, let $H'_j(\boldsymbol{x},\ t)$, $(j = 1,\ 2,\ 3)$ and $h(\boldsymbol{x},\ t)$ be functions on $\Xi \times (0, +\infty)$ with $\boldsymbol{H'} = (H'_1, H'_2, H'_3)$, and $\widetilde{\boldsymbol{H}} = (\boldsymbol{H'}, h)$.

Then, the subsequent theorem provides a Galerkin-type solution to the system by Eq. (4.2.9) and Eq. (4.2.10).

Theorem-4.3.1:

Statement: Let

$$\boldsymbol{u} = \boldsymbol{\mathcal{L}}^{(1)}\boldsymbol{H'} + \boldsymbol{\mathcal{L}}^{(2)}h, \tag{4.3.20}$$

$$\theta = \boldsymbol{\mathcal{L}}^{(3)}\boldsymbol{H'} + \boldsymbol{\mathcal{L}}^{(4)}h, \tag{4.3.21}$$

where, the fields H'_j of class C^8 and h of class C^5 satisfy

$$\chi_1(\nabla^2, T)\,\ell_1\boldsymbol{H'} = -\boldsymbol{H}, \tag{4.3.22}$$

$$\chi_1(\nabla^2, T)\,h = -R, \tag{4.3.23}$$

on $\Xi\times(0,+\infty)$. Then $\boldsymbol{U} = (\boldsymbol{u}, \theta)$ is the solution of Eq. (4.2.9) and Eq. (4.2.10).

Proof: From Eq. (4.3.20) and Eq. (4.3.21), the following is acquired:

$$\boldsymbol{U}(\boldsymbol{x}, t) = \boldsymbol{\mathcal{L}}(\boldsymbol{D_x}, T)\widetilde{\boldsymbol{H}}(\boldsymbol{x}, t). \tag{4.3.24}$$

On the other hand, from Eq. (4.3.22) and Eq. (4.3.23), it is derived that

$$\boldsymbol{\chi}(\nabla^2, T)\widetilde{\boldsymbol{H}}(\nabla^2, T) = \boldsymbol{\mathcal{F}}(\nabla^2, T). \tag{4.3.25}$$

Further, in view of Eq. (4.3.19), Eq. (4.3.24) and Eq. (4.3.25), $\boldsymbol{\Omega U} = \boldsymbol{\Omega\mathcal{L}}\widetilde{\boldsymbol{H}} = \boldsymbol{\chi}\widetilde{\boldsymbol{H}} = \boldsymbol{\mathcal{F}}$ is obtained which finalizes the proof of the theorem.

4.4 Galerkin-Type Solution of System of Equations for Steady Oscillations

In this section, the steady state oscillations are considered. Hence, the solution and external loads can be assumed in the following forms:

$$\boldsymbol{u}(\boldsymbol{x},t) = Re[\tilde{\boldsymbol{u}}(\boldsymbol{x})\, e^{-\mathrm{i}\omega t}], \qquad \boldsymbol{H}(\boldsymbol{x},t) = Re[\tilde{\boldsymbol{H}}(\boldsymbol{x})\, e^{-\mathrm{i}\omega t}],$$
$$\theta(\boldsymbol{x},t) = Re[\tilde{\theta}(\boldsymbol{x})\, e^{-\mathrm{i}\omega t}], \qquad R(\boldsymbol{x},t) = Re[\tilde{R}(\boldsymbol{x})\, e^{-\mathrm{i}\omega t}].$$

Therefore, from Eq. (4.2.7) and Eq. (4.2.8), the system of equations of the steady oscillations for MGL thermoelasticity theory are derived as follows:

$$\begin{aligned}\mu(\nabla^2\tilde{\boldsymbol{u}} - \mathrm{i}\,\tau_1\,\omega\,\nabla^2\tilde{\boldsymbol{u}}) + (\lambda+\mu)(\mathrm{grad}\,\mathrm{div}\;\tilde{\boldsymbol{u}} - \mathrm{i}\,\tau_1\,\omega\,\mathrm{grad}\,\mathrm{div}\tilde{\boldsymbol{u}})&\\ -\beta(\mathrm{grad}\,\tilde{\theta} - \mathrm{i}\,\tau_1\,\omega\,\mathrm{grad}\,\tilde{\theta}) + \rho\,\tilde{\boldsymbol{\mathfrak{f}}} = -\omega^2\,\rho\,\tilde{\boldsymbol{u}},&\end{aligned} \tag{4.4.1}$$

$$[K\,\nabla^2 + \rho\,c_E(\mathrm{i}\,\omega + \omega^2\,\tau_0)]\tilde{\theta} + \beta\,T_0(\mathrm{i}\,\omega\,\mathrm{div}\,\tilde{\boldsymbol{u}} + \tau_0\,\omega^2\,\mathrm{div}\,\tilde{\boldsymbol{u}}) = -\tilde{R}, \tag{4.4.2}$$

where, $(\boldsymbol{x},t) \in \varXi \times (0,+\infty)$, $\mathrm{i} = \sqrt{-1}$, and $\omega(>0)$ denotes the frequency of oscillation. The above system can further be expressed as

$$\begin{aligned}\left[\rho\,\omega^2 + \mu(1-\mathrm{i}\,\tau_1\,\omega)\nabla^2\right]\tilde{\boldsymbol{u}} + (\lambda+\mu)\,[1-\mathrm{i}\,\tau_1\,\omega]\,\mathrm{grad}\,\mathrm{div}\;\tilde{\boldsymbol{u}}&\\ -\beta\,[1-\mathrm{i}\,\tau_1\,\omega]\,\mathrm{grad}\,\tilde{\theta} = -\rho\,\tilde{\boldsymbol{H}},&\end{aligned} \tag{4.4.3}$$

$$\left[K\,\nabla^2 + (\mathrm{i}\,\omega + \omega^2\,\tau_0)\rho\,c_E\right]\tilde{\theta} + \beta\,T_0(\mathrm{i}\,\omega + \tau_0\,\omega^2)\mathrm{div}\;\tilde{\boldsymbol{u}} = -\tilde{R}. \tag{4.4.4}$$

In the following, the succeeding notations are used

$$\mathfrak{C}(\nabla^2) = |\mathfrak{C}_{pq}(\nabla^2)|_{2\mathrm{x}2} = \begin{bmatrix} \rho\,\omega^2 + (\lambda+2\mu)\,[1-\mathrm{i}\,\tau_1\,\omega]\,\nabla^2 & \beta\,T_0(\mathrm{i}\,\omega + \tau_0\omega^2)\nabla^2 \\ -\beta\,[1-\mathrm{i}\,\tau_1\,\omega] & K\,\nabla^2 + \rho\,c_E(\mathrm{i}\,\omega + \omega^2\,\tau_0) \end{bmatrix}_{2\times 2}.$$

Now, let

$$\begin{aligned}\tilde{\chi}_1(\nabla^2) &= \det\,\mathfrak{C}(\nabla^2),\\ m_{p1}(\nabla^2) &= -\left[(\lambda+\mu)(1-\mathrm{i}\,\tau_1\,\omega)\mathfrak{C}^*_{p1} + \beta\,T_0(\mathrm{i}\,\omega + \omega^2\,\tau_0)\mathfrak{C}^*_{p2}\right],\\ m_{p2}(\nabla^2) &= \mathfrak{C}^*_{p2},\;\; p = 1,\,2.\end{aligned}$$

It can be verified that λ_1^2 and λ_2^2 are the roots of the equation $\tilde{\chi}_1(-\lambda^*) = 0$, such that $\tilde{\chi}_1(\nabla^2) = (\nabla^2 + \lambda_1^2)(\nabla^2 + \lambda_2^2)$.

Next, the matrix differential operators $\mathfrak{M}$ and $\tilde{\chi}$ are defined as

- $$\mathfrak{M} = \begin{bmatrix} \boldsymbol{\mathfrak{M}^{(1)}} & \boldsymbol{\mathfrak{M}^{(2)}} \\ \boldsymbol{\mathfrak{M}^{(3)}} & \boldsymbol{\mathfrak{M}^{(4)}} \end{bmatrix}_{4\times 4},$$
$$\boldsymbol{\mathfrak{M}^{(1)}} = \left[\mathfrak{M}^{(1)}_{lj}\right]_{3\times 3}, \ \boldsymbol{\mathfrak{M}^{(2)}} = \left[\mathfrak{M}^{(2)}_{l1}\right]_{3\times 1}, \ \boldsymbol{\mathfrak{M}^{(3)}} = \left[\mathfrak{M}^{(3)}_{1l}\right]_{3\times 1}, \ \boldsymbol{\mathfrak{M}^{(4)}} = \left[\mathfrak{M}_{44}\right]_{1\times 1},$$
$$\mathfrak{M}^{(1)}_{pq}(\boldsymbol{D_x}) = \tilde{\chi}_1(\nabla^2)\delta_{pq} + m_{11}(\nabla^2)\frac{\partial^2}{\partial x_p \partial x_q}, \ \mathfrak{M}^{(2)}_{p1}(\boldsymbol{D_x}) = m_{12}(\nabla^2)\frac{\partial}{\partial x_p},$$
$$\mathfrak{M}^{(3)}_{1p}(\boldsymbol{D_x}) = m_{21}(\nabla^2)\frac{\partial}{\partial x_p}, \ \mathfrak{M}_{44} = m_{22}(\nabla^2), \quad p, q = 1,\ 2,\ 3. \tag{4.4.5}$$

- $$\tilde{\boldsymbol{\chi}}(\nabla^2, T) = \left[\chi_{pq}(\nabla^2)\right]_{4\times 4},$$
$$\tilde{\chi}_{jj} = \tilde{\chi}_1(\nabla^2)[\rho\,\omega^2 + \mu(1 - \mathrm{i}\,\tau_1\,\omega)\,\nabla^2], \ \ j = 1, 2, 3,$$
$$\tilde{\chi}_{44} = \tilde{\chi}_1(\nabla^2), \ \ \tilde{\chi}_{pq} = 0, \ \ p,\ q = 1, 2, 3, 4\ p \neq q. \tag{4.4.6}$$

If $\tilde{\mathcal{Q}}_j$, $(j = 1,\ 2,\ 3)$ and q be functions on $\varXi$ with $\tilde{\boldsymbol{\mathcal{Q}}} = (\tilde{\mathcal{Q}}_1, \tilde{\mathcal{Q}}_2, \tilde{\mathcal{Q}}_3)$, and $\boldsymbol{\mathcal{Q}} = (\tilde{\boldsymbol{\mathcal{Q}}}, q)$ then, in accordance with the Theorem-4.3.1, the following theorem provides a Galerkin-type solution to system by Eq. (4.4.1) and Eq. (4.4.2).

Theorem-4.4.1:

Statement: Let

$$\tilde{\boldsymbol{u}} = \boldsymbol{\mathfrak{M}}^{(1)} \tilde{\boldsymbol{\mathcal{Q}}} + \boldsymbol{\mathfrak{M}}^{(2)} q, \tag{4.4.7}$$

$$\tilde{\theta} = \boldsymbol{\mathfrak{M}}^{(3)} \tilde{\boldsymbol{\mathcal{Q}}} + \mathfrak{M}^{(4)} q, \tag{4.4.8}$$

where, the fields $\tilde{\mathcal{Q}}_j$ of class C^6 and q of class C^4 on Ω satisfy

$$\tilde{\chi}_1(\nabla^2) \left[\rho\, \omega^2 + \mu\, (1 - \mathrm{i}\, \tau_1\, \omega)\, \nabla^2\right] \tilde{\boldsymbol{\mathcal{Q}}} = -\tilde{\boldsymbol{H}}, \tag{4.4.9}$$

$$\tilde{\chi}_1(\nabla^2) q = -\tilde{R}, \tag{4.4.10}$$

on Ξ. Then $\left(\tilde{\boldsymbol{u}}, \tilde{\theta}\right)$ is the solution of Eq. (4.4.3) and Eq. (4.4.4).

4.5 General Solution of System of Equations for Steady Oscillations

In the absence of any body force and external heat source, the Eq. (4.4.3) and Eq. (4.4.4) can be written as

$$\begin{aligned}\left[\rho\, \omega^2 + \mu\, (1 - \mathrm{i}\, \tau_1\, \omega)\, \nabla^2\right] \tilde{\boldsymbol{u}} + (\lambda + \mu)\, (1 - \mathrm{i}\, \tau_1\, \omega)\, \mathrm{grad}\, \mathrm{div} \tilde{\boldsymbol{u}} \\ -\beta\, (1 - \mathrm{i}\, \tau_1\, \omega)\, \mathrm{grad}\, \tilde{\theta} = 0,\end{aligned} \tag{4.5.1}$$

$$[K\, \nabla^2 + \rho\, c_E\, \left(\mathrm{i}\, \omega + \omega^2\, \tau_0\right)]\tilde{\theta} + \beta\, T_0\, \left(\mathrm{i}\, \omega\, \mathrm{div}\, \tilde{\boldsymbol{u}} + \tau_0\, \omega^2\, \mathrm{div}\, \tilde{\boldsymbol{u}}\right) = 0. \tag{4.5.2}$$

Firstly, the following lemma in the context of above system of equations are used:

Lemma-4.5.1:

Statement: If $(\tilde{\boldsymbol{u}}, \tilde{\theta})$ is a solution of Eq. (4.5.1) and Eq. (4.5.2), then

$$\tilde{\chi}_1(\nabla^2)\text{div } \tilde{\boldsymbol{u}} = 0, \tag{4.5.3}$$

$$\tilde{\chi}_1(\nabla^2)\tilde{\theta} = 0, \tag{4.5.4}$$

$$\left[\rho\,\omega^2 + \mu\,(1 - \mathrm{i}\,\tau_1\,\omega)\,\nabla^2\right] \text{curl } \tilde{\boldsymbol{u}} = \boldsymbol{0}. \tag{4.5.5}$$

Proof: Firstly, by using the operator div to (4.5.1), the following is obtained

$$\left[\rho\,\omega^2 + (\lambda + 2\mu)\,(1 - \mathrm{i}\,\tau_1\,\omega)\,\nabla^2\right] \text{div } \tilde{\boldsymbol{u}} - \beta\,(1 - \mathrm{i}\,\tau_1\,\omega)\,\nabla^2\tilde{\theta} = 0. \tag{4.5.6}$$

Then, elimination of $\tilde{\theta}$ from Eq. (4.5.6) and Eq. (4.5.2) gives

$$\tilde{\chi}_1\text{div } \tilde{\boldsymbol{u}} = 0.$$

Again from Eq. (4.5.6) and Eq. (4.5.2), eliminating div $\tilde{\boldsymbol{u}}$, it is acquired that

$$\tilde{\chi}_1\tilde{\theta} = 0.$$

Furthermore, by applying the operator curl to (4.5.1), it is acquired that

$$\left[\rho\,\omega^2 + \mu\,(1 - \mathrm{i}\,\tau_1\,\omega)\,\nabla^2\right] \text{curl } \tilde{\boldsymbol{u}} = \boldsymbol{0}.$$

Therefore, the Eqs. (4.5.3-4.5.5) are obtained which completes the proof of Lemma-4.5.1.

Theorem-4.5.1:

Statement: If $\left(\tilde{\boldsymbol{u}}, \tilde{\theta}\right)$ is a solution of Eq. (4.5.1) and Eq. (4.5.2), then

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \beta\,(1 - \mathrm{i}\,\tau_1\,\omega)\,\text{grad}\sum_{p=1}^{2}\varphi_p(\boldsymbol{x}) + \boldsymbol{\Psi}(\boldsymbol{x}), \tag{4.5.7}$$

$$\tilde{\theta}(\boldsymbol{x}) = \sum_{p=1}^{2} a_p\,\varphi_p(\boldsymbol{x}), \tag{4.5.8}$$

where, $\varphi_p\,(p = 1,\, 2)$ and $\boldsymbol{\Psi} = (\Psi_1,\, \Psi_2,\, \Psi_3)$ satisfy the following equations:

$$\left(\nabla^2 + \lambda_p^2\right) \varphi_p(\boldsymbol{x}) = 0, \tag{4.5.9}$$

$$\left[\nabla^2 + \frac{\rho\, \omega^2}{\mu\, (1 - \mathrm{i}\, \tau_1\, \omega)}\right] \boldsymbol{\Psi}(\boldsymbol{x}) = \mathbf{0},\ \boldsymbol{x} \in \boldsymbol{\Xi}, \tag{4.5.10}$$

$$\operatorname{div} \boldsymbol{\Psi}(\boldsymbol{x}) = 0, \tag{4.5.11}$$

and

$$a_p = -\,(\lambda + 2\mu)\,(1 - \mathrm{i}\, \tau_1\, \omega)\, \lambda_p^2 + \rho\, \omega^2 \text{ where}, p = 1,\, 2. \tag{4.5.12}$$

Proof: Let Eq. (4.5.1) and Eq. (4.5.2) have $\left(\tilde{\boldsymbol{u}}, \tilde{\theta}\right)$ as solution. Then, taking into account $\nabla^2 \tilde{\boldsymbol{u}} = \text{grad div}\tilde{\boldsymbol{u}} - \text{curl curl}\, \tilde{\boldsymbol{u}}$ and using Eq. (4.5.1),the following is obtained

$$\tilde{\boldsymbol{u}} = \frac{1 - \mathrm{i}\, \tau_1\, \omega}{\rho\, \omega^2} \left\{ \text{grad}\left[-\,(\lambda + 2\mu)\, \text{div}\ \tilde{\boldsymbol{u}} + \beta\, \tilde{\theta}\right] + \mu\, \text{curl curl}\, \tilde{\boldsymbol{u}} \right\}. \tag{4.5.13}$$

Introducing the notation

$$\boldsymbol{\Psi}(\boldsymbol{x}) = \frac{(1 - \mathrm{i}\, \tau_1\, \omega)\, \mu}{\rho\, \omega^2} \text{curl curl}\ \tilde{\boldsymbol{u}}, \tag{4.5.14}$$

and using Eq. (4.5.5), and div curl $\tilde{\boldsymbol{u}} = 0$ for $\boldsymbol{x} \in \Xi$, Eq. (4.5.10) and Eq. (4.5.11) can be directly obtained.

Now, let

$$\varphi_j = b_j \left[\prod_{\substack{p=1 \\ p \neq j}}^{2} \left(\nabla^2 + \lambda_p^2\right) \right] \tilde{\theta}, \tag{4.5.15}$$

where,

$$b_j = \left[a_j \prod_{\substack{p=1 \\ p \neq j}}^{2} \left(\lambda_p^2 - \lambda_j^2\right) \right]^{-1},\ j = 1,\, 2.$$

Therefore, in view of Eq. (4.5.4), the Eq. (4.5.15) yields Eq. (4.5.9) and Eq. (4.5.8).

Next, using Eq. (4.5.2), Eq. (4.5.8), Eq. (4.5.9), and Eq. (4.5.12), it is acquired that

$$\operatorname{div} \tilde{\boldsymbol{u}} = -\beta\, (1 - \mathrm{i}\, \tau_1\, \omega) \sum_{p=1}^{2} \lambda_p^2\, \varphi_p. \tag{4.5.16}$$

Hence, Eq. (4.5.13) yields

$$\tilde{\boldsymbol{u}} = \frac{1 - \mathrm{i}\,\tau_1\,\omega}{\rho\,\omega^2}\left\{\mathrm{grad}\left[(\lambda + 2\mu)\,\beta\,(1 - \mathrm{i}\,\tau_1\,\omega)\sum_{p=1}^{2}\lambda_p^2\varphi_p + \beta\,\tilde{\theta}\right] + \mu\,\mathrm{curl}\,\mathrm{curl}\,\tilde{\boldsymbol{u}}\right\}. \quad (4.5.17)$$

Further, simplifying the above equation using Eq. (4.5.12) and Eq. (4.5.14), finally it is obtained that

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \beta\,(1 - \mathrm{i}\,\tau_1\,\omega)\,\mathrm{grad}\sum_{p=1}^{2}\varphi_p(\boldsymbol{x}) + \boldsymbol{\Psi}(\boldsymbol{x}),$$

which completes the proof of Theorem 4.5.1.

Theorem-4.5.2:

Statement: If $\left(\tilde{\boldsymbol{u}}, \tilde{\theta}\right)$ is expressed as in Eq. (4.5.7) and Eq. (4.5.8), where φ_j and $\boldsymbol{\Psi}$ satisfies Eqs. (4.5.9-4.5.11), then $\left(\tilde{\boldsymbol{u}}, \tilde{\theta}\right)$ is the solution of Eq. (4.5.1) and Eq. (4.5.2) on Ξ.

Proof: From Eq. (4.5.7) and using Eq. (4.5.9) and Eq. (4.5.10), the following is acquired

$$\begin{aligned}
\nabla^2\tilde{\boldsymbol{u}} &= -\beta\,(1 - \mathrm{i}\,\tau_1\,\omega)\,\mathrm{grad}\sum_{p=1}^{2}\lambda_p^2\varphi_p - \frac{\rho\,\omega^2}{\mu\,(1 - \mathrm{i}\,\tau_1\,\omega)}\boldsymbol{\Psi},\\
\mathrm{grad}\,\mathrm{div}\,\tilde{\boldsymbol{u}} &= -\beta\,(1 - \mathrm{i}\,\tau_1\,\omega)\,\mathrm{grad}\sum_{p=1}^{2}\lambda_p^2\varphi_p. \qquad (4.5.18)
\end{aligned}$$

Replacing $\tilde{\boldsymbol{u}}$ and $\tilde{\theta}$ as given in Eq. (4.5.7) and Eq. (4.5.8) on the left-hand side of Eq. (4.5.1) and using Eq. (4.5.9), Eq. (4.5.12), and Eq. (4.5.18), it is yielded that

$$\begin{aligned}
&\left[\rho\,\omega^2 + \mu\,(1 - \mathrm{i}\,\tau_1\,\omega)\,\nabla^2\right]\tilde{\boldsymbol{u}} + (\lambda + \mu)\,(1 - \mathrm{i}\,\tau_1\,\omega)\,\mathrm{grad}\,\mathrm{div}\tilde{\boldsymbol{u}}\\
&\quad -\beta\,(1 - \mathrm{i}\,\tau_1\,\omega)\,\mathrm{grad}\,\tilde{\theta} = \rho\,\omega^2\left[(1 - \mathrm{i}\,\tau_1\,\omega)\,\beta\,\mathrm{grad}\sum_{p=1}^{2}\varphi_p + \boldsymbol{\Psi}\right],\\
&-\beta\,(1 - \mathrm{i}\,\tau_1\,\omega)\,\mathrm{grad}\sum_{p=1}^{2}\left[(\lambda + 2\mu)\,(1 - \mathrm{i}\,\tau_1\,\omega)\,\lambda_p^2 + a_p\right]\varphi_p - \rho\,\omega^2\boldsymbol{\Psi}.
\end{aligned}$$

After simplification, the above equation yields

$$\left[\rho\,\omega^2+\mu\,(1-\mathrm{i}\,\tau_1\,\omega)\,\nabla^2\right]\tilde{\boldsymbol{u}}+(\lambda+\mu)\,(1-\mathrm{i}\,\tau_1\,\omega)\,\mathrm{grad\,div}\tilde{\boldsymbol{u}}-\beta\,(1-\mathrm{i}\,\tau_1\,\omega)\,\mathrm{grad}\,\tilde{\theta}=0,$$

which is our field Eq. (4.5.1).

Similarly, replacing $\tilde{\boldsymbol{u}}$ and $\tilde{\theta}$ again on the left-hand side of Eq. (4.5.2) by the expression given in (4.5.7) and (4.5.8) and using Eq. (4.5.9), Eq. (4.5.12), and Eq. (4.5.16) give

$$\begin{aligned}
&\left[K\,\nabla^2+\rho\,c_E\left(\mathrm{i}\,\omega+\omega^2\,\tau_0\right)\right]\tilde{\theta}+\beta\,T_0\left(\mathrm{i}\,\omega+\tau_0\,\omega^2\right)\mathrm{div}\,\tilde{\boldsymbol{u}}\\
&=\left[K\,\nabla^2+\rho\,c_E\left(\mathrm{i}\,\omega+\omega^2\,\tau_0\right)\right]\left(\sum_{p=1}^{2}a_p\,\varphi_p\right)+\beta^2\,T_0\left(\mathrm{i}\,\omega+\tau_0\,\omega^2\right)(1-\mathrm{i}\,\tau_1\,\omega)\left(-\sum_{p=1}^{2}\lambda_p^2\varphi_p\right)\\
&=\sum_{p=1}^{2}\left\{a_p\left[K\,\nabla^2+\rho\,c_E\left(\mathrm{i}\,\omega+\omega^2\,\tau_0\right)\right]-\beta^2T_0\left(\mathrm{i}\,\omega+\tau_0\,\omega^2\right)(1-\mathrm{i}\,\tau_1\,\omega)\,\lambda_p^2\right\}\varphi_p\\
&=0\ \left(\text{by using}\ \tilde{\chi}_1\left(-\lambda_p^2\right)=0,p=1,2\right).
\end{aligned}$$

Thus acquiring Eq. (4.5.2).

Hence, it can be confirmed that the general solution of the system of homogeneous Eq. (4.5.1) and Eq. (4.5.2) is attained in terms of the metaharmonic functions φ_p and $\boldsymbol{\Psi}$.

4.6 Conclusion

The present chapter is concerned with a newly proposed non-classical model referred to as modified Green-Lindsay (MGL) model that incorporates temperature and strain rate terms in the constitutive relations. The work includes representation of Galerkin-type solution for the system of equations of motion in terms of elementary functions. A theorem that represents Galerkin-type solution of equations for steady state oscillations in the context of MGL linear thermoelasticity theory is established. Finally, the representation of a general solution for the system of equations in the case of steady state oscillations is also acquired in terms of metaharmonic functions.