CHAPTER 2

THEORETICAL ANALYSIS OF THE THERMOELASTICITY THEORY BASED ON EXACT HEAT CONDUCTION LAW WITH SINGLE DELAY

2.1 Some Theorems on Linear Theory of Thermoelasticity based on Exact Heat Conduction Model with Single Delay for Anisotropic Medium[1]

2.1.1 Introduction

In the literature, some pioneering work on thermoelasticity theory have been reported by eminent researchers like, Nickel and Sackman (1968), Ieşan (1966; 1974), Ignaczak (1963), Gurtin (1964), etc. and it has been shown that the state of dynamics of a thermoelastic system can be determined by using the variational method which describes it as the extremum of a functional or function. Ignaczak (1963) and Gurtin (1964) explained the variational principles for the initial-boundary value problems by incorporating the initial conditions into the field equations. With the help of this formulation,

[1]The content of this subchapter is published in (Ghosh D., Giri D., Mohapatra R., Savas E., Sakurai K., Singh L. (eds)) *Mathematics and Computing. ICMC 2018. Communications in Computer and Information Science,*834, Springer, Singapore,

Ieşan (1966; 1974) and then Nickell and Sackman (1968), established convolution type variational principle for the linear coupled thermoelasticity. Subsequently, the variational theorem of Gurtin type for solids with micro-structure was presented by Ieşan (1967). It is worth to be mentioned that the variational principle also plays an important role in the theoretical foundation of the numerical techniques for solving the various thermo-mechanical problems.

Moreover, it has been observed that the reciprocity theorem is used to derive various methods of integrating the elasticity equations in terms of Green's function. It has significant practical applications in finding the numerical solution of engineering problems (Nowacki (1975b)) as reciprocity theorem states the relation between two sets of thermoelastic loadings and the corresponding thermoelastic configurations. Maizel (1951) developed the Betti–Maxwell reciprocity theorem for the static problems in theory of thermoelasticity. Later, the reciprocity theorem was extended to uncoupled thermoelasticity, coupled thermoelasticity and coupled thermoelasticity for anisotropic homogeneous material by Predeleanu (1959), Ionescu-Cazimir (1964) and Nowacki (1975b), respectively. Ieşan (1967) presented the first reciprocal relation without using the Laplace transform. Convolution type reciprocity theorems were also derived by Ieşan (1966; 1974). Scalia (1990) used a method to deduce reciprocity relations without using the Laplace transform and without incorporation of the initial data in the field equations. An exhaustive treatment of the variational principles in thermoelasticity is available in the books by Lebon (1980), Carlson (1973), Hetnarski and Ignaczak (2010), and Hetnarski et al. (2009). Recently, the convolution type variational principles and reciprocal relations on different theories of thermoelasticity were given by Chiriţă and Ciarletta (2010), Mukhopadhyay et al. (2011b), Kothari and Mukhopadhyay (2013a), and Kumari and Mukhopadhyay(2017c). Further, Shivay and Mukhopadhyay (2019) presented Somigliano and Green's theorem based on reciprocity theorem in the context of the generalized thermoelasticity model with single delay. Moreover, Jangid and Mukhopadhyay (2020) focused on variational and reciprocal principles for modified temperature-rate dependent two-temperature thermoelasticity theory.

The present chapter of the thesis aims at analyzing the thermoelasticity theory based on the exact heat conduction model with a single delay as proposed by Quintanilla (2011) very recently. In the article, Quintanilla (2011) presented one modified heat conduction model to overcome the stability complexities under the three-phase-lag heat conduction theory (Roychoudhuri (2007a)) as mentioned by researchers including Dreher et al. (2009). The author proved the well posedness of the problem under this heat conduction theory and elaborated the recurrent scheme to obtain the explicit form of the solution. Later, the author extended the results of well posedness to the system of equations in thermoelasticity theory and proposed an alternative thermoelasticity theory with a single delay (τ) time parameter. Subsequently, Leseduarte and Quintanilla (2013) represented Phragm$\acute{e}$n Lindel$\ddot{o}$f type alternative for the forward-in-time (Eq. (1.3.25)) and backward-in-time (Eq. (1.3.26)) version of the model given by Quintanilla (2011).

In this subchapter, some important theorems are established in the context of this new thermoelasticity model (2011) for homogeneous and anisotropic medium. The subchapter starts with describing the basic governing equations and constitutive relations for anisotropic medium in the context of the present theory and considers a mixed initial-boundary value problem with non-homogeneous initial conditions. Then, the work is progressed in the direction to prove the uniqueness of solution of the mixed problem by using the specific internal energy function. Next, the alternative formulation of the mixed initial-boundary value problem using convolution is presented. The benefit of this formulation is that it incorporates the initial conditions into the field equations, due to which there is no need to consider the initial conditions separately. Lastly, using this formulation, the variational principle of convolution type and a reciprocity theorem is exhibited.

2.1.2 Basic Equations and Problem Formulation

Following Quintanilla (2011) and Leseduarte and Quintanilla (2013), the basic governing equations and the constitutive relations in context of of Quintanilla's thermoelasticity model for a homogeneous and anisotropic material can be written as follows:

The equation of motion:

$$\sigma_{ij,j} + \rho H_i = \rho \ddot{u}_i. \tag{2.1.1}$$

The equation of energy:

$$\rho T_0 \dot{S} = -q_{i,i} + \rho R. \tag{2.1.2}$$

The constitutive relations:

$$\sigma_{ij} = C_{ijkl} e_{kl} - \beta_{ij} \theta, \tag{2.1.3}$$

$$\rho S = \rho c_E \frac{\theta}{T_0} + \beta_{ij} e_{ij}, \tag{2.1.4}$$

$$\dot{q}_i = -\left[K_{ij} \frac{\partial}{\partial t} + K^*_{ij} (1 - \tau \frac{\partial}{\partial t} + \frac{\tau^2}{2} \frac{\partial^2}{\partial t^2}) \right] \eta_j. \tag{2.1.5}$$

The geometrical relations:

$$\eta_j = \theta_{,j}, \tag{2.1.6}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = u_{(i,j)}. \tag{2.1.7}$$

In this system of equations, a rectangular coordinate system x_k in three dimensional Euclidean space with usual indicial notations and η_i denotes the components of temperature gradient.

Mixed Initial Boundary Value Problem

Now, considering $\overline{V}$ as the closure of an open, bounded, connected domain with boundary, ∂V, enclosing an homogeneous and anisotropic thermoelastic material. Let V denote the interior of $\overline{V}$ and n_i be the components of an outward drawn unit normal to ∂V. Let B_i, $(i = 1, 2, 3, 4)$ be the subsets of ∂V such that $B_1 \cup B_2 = B_3 \cup B_4 = \partial V$ and $B_1 \cap B_2 = B_3 \cap B_4 = \phi$. The motion relative to an undistorted stress free reference state is considered for the present study.

For a mixed initial and boundary value problem, the field equations and constitutive relations are given by Eqs. (2.1.1-2.1.7) defined on $V \times [0, \infty)$ together with the following initial conditions and boundary conditions:

Initial conditions: On V

$$\left.\begin{array}{c} u_i(\boldsymbol{x}, 0) = u_{0_i}(\boldsymbol{x}),\ \dot{u}_i(\boldsymbol{x}, 0) = v_i(\boldsymbol{x}), \\ \theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}),\ \dot{\theta}(\boldsymbol{x}, 0) = \theta_1(\boldsymbol{x}),\ q_i(\boldsymbol{x}, 0) = q_{0_i}(\boldsymbol{x}). \end{array}\right\} \tag{2.1.8}$$

Boundary conditions:

$$\left.\begin{array}{c} u_i = \tilde{u}_i(\boldsymbol{x}, t) \text{ on } \ B_1 \times [0, \infty), \\ \sigma_i = \sigma_{ij} n_j = \tilde{\sigma}_i(\boldsymbol{x}, t) \text{ on } \ B_2 \times [0, \infty), \\ q = q_i n_i = \tilde{q}(\boldsymbol{x}, t) \text{ on } \ B_3 \times [0, \infty), \\ \theta = \tilde{\theta}(\boldsymbol{x}, t) \text{ on } \ B_4 \times [0, \infty). \end{array}\right\} \tag{2.1.9}$$

Here, u_{0_i}, v_i, θ_0, θ_1, q_{0_i} represent the specified initial displacement component, velocity component, temperature, rate of temperature, and heat-flux, respectively together with $\tilde{u}_i$, $\tilde{\sigma}_i$, $\tilde{\theta}$, $\tilde{q}$, which denote the known surface displacement component, component of traction vector, temperature and normal heat-flux, respectively. The smoothness requirements and other regularity assumptions on the ascribable functions are also considered as hypotheses on data. Also, assumptions are made that u_{0_i}, v_i, θ_0, θ_1, q_{0_i} are continuous on $\overline{V}$, H_i, and R are continuously differentiable on $\overline{V} \times [0, \infty)$. $\tilde{q}$ and $\tilde{\sigma}$ are piecewise continuous on $B_3 \times [0, \infty)$ and $B_2 \times [0, \infty)$, respectively. $\tilde{u}_i$ and $\tilde{\theta}$ are continuous on $B_1 \times [0, \infty)$ and $B_4 \times [0, \infty)$, respectively.

Further, C_{ijkl}, β_{ij}, K_{ij}, and K^*_{ij} are assumed to be smooth on $\overline{V}$ and satisfy

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}, \quad \beta_{ij} = \beta_{ji}, \quad K_{ij} = K_{ji}, \quad K^*_{ij} = K^*_{ji}, \tag{2.1.10}$$

$$C_{ijkl} e_{ij} e_{kl} > 0, \text{ for all } e_{ij} \text{ on } \overline{V} \times [0, \infty), \tag{2.1.11}$$

$$K_{ij} \varphi_i \varphi_j > 0 \text{ for any real } \varphi_i \text{ on } \overline{V} \times [0, \infty), \tag{2.1.12}$$

$$K^*_{ij} \psi_i \psi_j > 0 \text{ for any real } \psi_i \text{ on } \overline{V} \times [0, \infty). \tag{2.1.13}$$

The material constants and delay time parameters satisfy the following inequalities:

$$\rho > 0, \ c_E > 0, \ T_0 > 0, \ \tau > 0, \ K_{ij} - \tau K^*_{ij} > 0 \quad \text{on} \quad V. \tag{2.1.14}$$

Now, defining an admissible state as $\mathfrak{R} = \{u_i, \theta, \eta_i, e_{ij}, \sigma_{ij}, q_i, S\}$, which is an ordered array of functions u_i, θ, η_i, e_{ij}, σ_{ij}, q_i, S defined on $\overline{V} \times [0, \infty)$ with the properties that $u_i \in C^{2,2}$, $\theta \in C^{1,2}$, $\eta_i \in C^{0,2}$, $\sigma_{ij} \in C^{1,0}, q_i \in C^{1,1}$, $S \in C^{0,1}$ and $e_{ij} = e_{ji}$, $\sigma_{ij} = \sigma_{ji}$ on $\overline{V} \times [0, \infty)$. Further, defining two operations, addition of two admissible states and multiplication of an admissible state with a scalar as follows:

$\mathfrak{R} + \mathfrak{R}' = \{u_i + u_i, \theta + \theta', ..., S + S'\}$,

$\lambda^* \mathfrak{R}' = \{\lambda^* u_i, \lambda^* \theta, ..., \lambda^* S\}$, where λ^* is any scalar. Then the set of all admissible states is clearly a linear space.

Further, an admissible state is the solution of the present mixed problem if it satisfies all the field Eqs. (2.1.1-2.1.7), the initial conditions (2.1.8) and the boundary conditions (2.1.9).

2.1.3 Uniqueness of Solution

For the uniqueness of solution, the specific internal energy for the present initial-boundary value problem is considered which is in the form

$$\mathfrak{E} = \frac{1}{2} C_{ijkl} \dot{e}_{kl} \dot{e}_{ij} + \frac{\rho c_E}{2T_0} \dot{\theta}^2. \tag{2.1.15}$$

Clearly, from Eq. (2.1.11) and Eq. (2.1.14), it can be stated that the specific internal energy (Eq. (2.1.15)) is positive definite and using Eq. (2.1.3), Eq. (2.1.4), and Eq. (2.1.10), it is attained that

$$\dot{\mathfrak{E}} = \dot{\sigma}_{ij}\ddot{e}_{ij} + \rho\ddot{S}\dot{\theta}. \tag{2.1.16}$$

Now, relations (2.1.2), (2.1.5), (2.1.6), and (2.1.7) together give

$$\begin{aligned}
\dot{\mathfrak{E}} &= \dot{\sigma}_{ij}\ddot{u}_{i,j} - \frac{1}{T_0}\dot{q}_{i,i}\dot{\theta} + \frac{\rho\dot{R}}{T_0}\dot{\theta} \\
&= (\dot{\sigma}_{ij}\ddot{u}_i)_{,j} - \dot{\sigma}_{ij,j}\ddot{u}_i - \frac{1}{T_0}(\dot{q}_i\dot{\theta})_i + \frac{1}{T_0}(\dot{q}_i\dot{\eta}_i) + \frac{\rho\dot{R}}{T_0}\dot{\theta} \\
&= (\dot{\sigma}_{ij}\ddot{u}_i)_{,j} - \frac{1}{T_0}(\dot{q}_i\dot{\theta})_{,i} + \rho\dot{H}_i\ddot{u}_i + \frac{\rho\dot{R}}{T_0}\dot{\theta} - \rho\dddot{u}_i\ddot{u}_i \\
&\quad -\frac{\dot{\eta}_i}{T_0}\left[K_{ij}\dot{\eta}_j + K^*_{ij}\eta_j - \tau K^*_{ij}\dot{\eta}_j + \frac{\tau^2}{2}K^*_{ij}\ddot{\eta}_j\right].
\end{aligned} \tag{2.1.17}$$

Integrating both sides of Eq. (2.1.17) over V, using divergence theorem and applying (2.1.1), acquire

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_V \left(\mathfrak{E} + \frac{\rho}{2}\ddot{u}_i\ddot{u}_i + \frac{K^*_{ij}}{2T_0}\eta_i\eta_j + \frac{\tau^2 K^*_{ij}}{4T_0}\dot{\eta}_i\dot{\eta}_j\right) dV + \frac{1}{T_0}\int_V \left(K_{ij} - \tau K^*_{ij}\right)\dot{\eta}_i\dot{\eta}_j dV \\
&\qquad = \int_V \left(\rho\dot{H}_i\ddot{u}_i + \frac{\rho\dot{R}\dot{\theta}}{T_0}\right) dV + \int_A \left(\dot{\tilde{\sigma}}_i\ddot{u}_i - \frac{1}{T_0}\dot{\theta}\dot{\tilde{q}}\right) dA.
\end{aligned} \tag{2.1.18}$$

Now, the uniqueness of solution of the present mixed initial-boundary value problem is established by the following uniqueness theorem.

Theorem-2.1.3.1 (Uniqueness theorem):

Statement: The mixed initial-boundary value problem given by Eqs. (2.1.1-2.1.7), which satisfies the initial conditions (2.1.8) and boundary conditions (2.1.9) has at most one solution.

Proof: Assume that there are two sets of solutions $u_i^{(\gamma)}$, $\theta^{(\gamma)}$, $e_{ij}^{(\gamma)}$, $\sigma_{ij}^{(\gamma)}$, $q_i^{(\gamma)}$, $S^{(\gamma)}$ for $\gamma = 1, 2.$ and construct the difference between these two sets of functions as

$$\overline{u_i} = u_i^{(1)} - u_i^{(2)},\ \overline{\theta} = \theta^{(1)} - \theta^{(2)}, ..., \overline{S} = S^{(1)} - S^{(2)}. \tag{2.1.19}$$

Since, the set of all admissible states is a linear space, so the difference functions defined by (2.1.19) also satisfy the Eqs. (2.1.1-2.1.7) with zero body forces and heat source, the initial conditions (2.1.8) and the boundary conditions (2.1.9) in their homogeneous form and hence, Eq. (2.1.18) too. Therefore, Eq. (2.1.18) yields

$$\frac{\partial}{\partial t}\int\limits_V \left(\overline{\mathfrak{E}} + \frac{\rho}{2}\ddot{\overline{u}}_i\ddot{\overline{u}}_i + \frac{K_{ij}^*}{2T_0}\overline{\eta}_i\overline{\eta}_j + \frac{\tau^2 K_{ij}^*}{4T_0}\dot{\overline{\eta}}_i\dot{\overline{\eta}}_j\right) dV + \frac{1}{T_0}\int\limits_V \left(K_{ij} - \tau K_{ij}^*\right)\dot{\overline{\eta}}_i\dot{\overline{\eta}}_j dV = 0. \tag{2.1.20}$$

Integrating the above equation over time interval $(0, t)$ after interchanging the variable t with ξ and using the homogeneous initial conditions for difference functions give the following equation:

$$\int\limits_V \left(\overline{\mathfrak{E}} + \frac{\rho}{2}\ddot{\overline{u}}_i\ddot{\overline{u}}_i + \frac{K_{ij}^*}{2T_0}\overline{\eta}_i\overline{\eta}_j + \frac{\tau^2 K_{ij}^*}{4T_0}\dot{\overline{\eta}}_i\dot{\overline{\eta}}_j\right) dV + \frac{1}{T_0}\int\limits_0^t\int\limits_V \left(K_{ij} - \tau K_{ij}^*\right)\dot{\overline{\eta}}_i\dot{\overline{\eta}}_j dV d\xi = 0. \tag{2.1.21}$$

From Eq. (2.1.11), Eq. (2.1.12), Eq. (2.1.13), and Eq. (2.1.14), it is observed that the component in each term present on the left hand side of Eq. (2.1.21) is non-negative. Thus it can be concluded that each term in Eq. (2.1.21) must be zero which implies that

$$\ddot{\overline{u}}_i = 0, \quad \dot{\overline{\theta}} = 0 \quad \text{on } \overline{V} \times [0, \infty). \tag{2.1.22}$$

i.e.,

$$\frac{\partial^2 \overline{u}_i}{\partial t^2} = 0,\ \ \frac{\partial \overline{\theta}}{\partial t} = 0,\ \text{on } \overline{V} \times [0, \infty). \tag{2.1.23}$$

Therefore, in view of the initial conditions $\overline{u}_i(\boldsymbol{x}, 0) = 0$, $\dot{\overline{u}}_i(\boldsymbol{x}, 0) = 0$, and $\overline{\theta}(\boldsymbol{x}, 0) = 0$, Eq. (2.1.23) yields

$$\overline{u}_i = 0, \quad \overline{\theta} = 0 \quad \text{on } \overline{V} \times [0, \infty),$$

i.e.,

$$u_i^{(1)} = u_i^{(2)}, \quad \theta^{(1)} = \theta^{(2)} \quad \text{on } \overline{V} \times [0, \infty).$$

This completes the proof of the uniqueness theorem.

2.1.4 Alternative Formulation of Mixed Problem

This subsection discusses the alternative formulation of the above mixed initial-boundary value problem in which the initial conditions are combined into the field equations (Gurtin (1964)). For this purpose, the following results are used:

Let ϕ and ψ be two functions defined on $\overline{V} \times [0, \infty)$ such that both are continuous on [0,∞) for each $x \in V$. Then the convolution $\phi * \psi$ of ϕ and ψ is defined as

$$[\phi * \psi](\boldsymbol{x}, t) = \int_0^t \phi(\boldsymbol{x}, t-\tau)\psi(\boldsymbol{x}, \tau)d\tau, \quad (\boldsymbol{x}, t) \in \overline{V} \times [0, \infty).$$

The commutativity, associativity, and distributivity properties of convolution and the property that

$$\phi * \psi = 0 \Rightarrow \phi = 0 \text{ or } \psi = 0 \tag{2.1.24}$$

are used.

Now, the functions g and l are defined on [0,∞) as

$$g(t) = t, \ l(t) = 1. \tag{2.1.25}$$

Also, let functions f_i and W be defined on $\overline{V} \times [0, \infty)$ as

$$f_i = g * \rho H_i + \rho(tv_i + u_{0_i}), \tag{2.1.26}$$

$$W = l * \frac{\rho R}{T_0} + \rho c_E \frac{\theta_0}{T_0} + \beta_{ij} u_{0_{i,j}}, \tag{2.1.27}$$

and let

$$N_i = l * (tq_{0_i} + t\theta_{0,j} K_{ij} - t\tau\theta_{0,j} K^*_{ij} + t\theta_{1,j} \frac{\tau^2}{2} K^*_{ij} + \theta_{0,j} \frac{\tau^2}{2} K^*_{ij}). \tag{2.1.28}$$

Consider $p(x, t)$ and $\dot{p}(x, t)$, two functions defined on $\overline{V} \times [0, \infty)$ such that both are continuous and differentiable on [0,∞). Then the following results hold clearly:

$$g * \ddot{p}(\boldsymbol{x}, t) = p(\boldsymbol{x}, t) - [t\dot{p}(\boldsymbol{x}, 0) + p(\boldsymbol{x}, 0)], \tag{2.1.29}$$

$$l * \dot{p}(B,t) = p(\boldsymbol{x},t) - p(\boldsymbol{x},0), \tag{2.1.30}$$

$$g * \dot{p}(\boldsymbol{x},t) = l * (l * \dot{p}(\boldsymbol{x},t)) = l * [p(\boldsymbol{x},t) - p(\boldsymbol{x},0)] = l * p(\boldsymbol{x},t) - tp(\boldsymbol{x},0). \tag{2.1.31}$$

Using this formulation, the following theorem is obtained that characterizes the considered mixed problem in an alternative way.

Theorem -2.1.4.1:

Statement: The function u_i, θ, η_i, e_{ij}, σ_{ij}, q_i, S satisfy Eq. (2.1.1), Eq. (2.1.2) and Eq. (2.1.5) and the initial conditions (2.1.8) if and only if

$$g * \sigma_{ij,j} + f_i = \rho u_i, \tag{2.1.32}$$

$$\rho S = -l * \frac{q_{i,i}}{T_0} + W, \tag{2.1.33}$$

$$L_1 * q_i = -L_1 * K_{ij}\eta_j - L_2 * K^*_{ij}\eta_j + N_i, \tag{2.1.34}$$

where, $L_1 = l * l$ and $L_2 = l * (g + \tau l + \frac{\tau^2}{2})$, f_i, W and N_i are given by Eq. (2.1.26), Eq. (2.1.27), and Eq. (2.1.28), respectively.

Proof: Firstly, assuming that the governing Eq. (2.1.1), Eq. (2.1.2) and Eq. (2.1.5), and initial conditions (2.1.8) hold good. Then, taking the convolution of Eq. (2.1.1) with g and using the results from Eq. (2.1.29) and Eq. (2.1.8), the Eq. (2.1.32) is obtained. Similarly, taking the convolution of the Eq. (2.1.2) with l and using Eq. (2.1.30), Eq. (2.1.4), and Eq. (2.1.8), the Eq. (2.1.33) is acquired. Again, taking the convolution of Eq. (2.1.5) with $l * g$, and using the relation from (2.1.29), (2.1.31) and (2.1.8), the Eq. (2.1.34) is yielded.

Similarly, the converse of the above theorem can be proved using reverse arguments. Hence, presenting the following theorem.

Theorem-2.1.4.2:

Statement: Let $\mathfrak{R} = \{u_i, \theta, \eta_i, e_{ij}, \sigma_{ij}, q_i, S\}$ be an admissible state. Then $\mathfrak{R}$ is a solution of the mixed problem if and only if it satisfies the Eqs. (2.1.32-2.1.34), Eq. (2.1.3), Eq. (2.1.4), Eq. (2.1.6), Eq. (2.1.7) and the boundary conditions (2.1.9).

2.1.5 Variational Theorem

Using the alternative formulation and the theorem established in the previous subsection, a variational principle on linear theory of thermoelasticity for anisotropic and homogeneous medium under the present heat conduction model given by Quintanilla (2011) is formulated in the following way:

Theorem -2.1.5.1:

Statement: Let Λ be a linear space of all admissible states with addition and scalar multiplication as describe in Subsection-2.1.2. If for each $t \in [0, \infty)$ and for every $\mathfrak{R} = \{u_i,\, \theta,\, \eta_j,\, e_{ij},\, \sigma_{ij},\, q_i,\, S\} \in \Lambda$, a functional $F_t\{\mathfrak{R}\}$ on Λ is defined by

$$
\begin{aligned}
&F_t\{\mathfrak{R}\}\\
&= \int_V \Big[\frac{1}{2} L_1 * g * C_{ijkl} e_{kl} * e_{ij} - \frac{1}{2} L_1 * \rho u_i * u_i - L_1 * g * \sigma_{ij} * e_{ij} - L_1 * g * l * \frac{1}{T_0} q_i * \eta_i\\
&+ L_1 * u_i * (\rho u_i - g * \sigma_{ij,j} - f_i) - L_1 * g * \theta * \left(\rho S + l * \frac{q_{i,i}}{T_0} - W\right)\\
&+ g * l * \frac{1}{T_0}\left(-L_1 * \frac{1}{2} K_{ij}\eta_j - L_2 * \frac{1}{2} K^*_{ij}\eta_j + N_i\right) * \eta_i\\
&+ \frac{T_0}{2\rho c_E} L_1 * g * (\rho S - \beta_{rs} e_{rs}) * (\rho S - \beta_{ij} e_{ij})\Big] dV + \int_{B_1} L_1 * g * \tilde{u_i} * \sigma_i dA\\
&+ \int_{B_2} L_1 * g * (\sigma_i - \tilde{\sigma_i}) * u_i dA + \frac{1}{T_0}\int_{B_3} L_1 * g * l * q * \tilde{\theta} dA\\
&+ \frac{1}{T_0}\int_{B_4} M_1 * g * l * (q - \tilde{q}) * \theta dA, \qquad (2.1.35)
\end{aligned}
$$

then the variation of this functional,

$$\delta F_t\{\mathfrak{R}\} = 0, \quad t \in [0, \infty), \qquad (2.1.36)$$

if and only if, $\mathfrak{R}$ is a solution of the mixed initial-boundary value problem given by Eqs. (2.1.1-2.1.7) with the initial conditions (2.1.8) and the boundary conditions (2.1.9).

Proof: Let $\mathfrak{R}' = \{u_i', \theta', \eta_i', e_{ij}', \sigma_{ij}', q_i', S'\} \in \Lambda$, which implies that $\mathfrak{R} + \lambda\mathfrak{R}' \in \Lambda$, for every real λ. Then, Eq. (2.1.35) together with properties of convolution, definition of variation, and the divergence theorem, implies

$$\begin{aligned}
&\delta_{\mathfrak{R}'}\Omega_t\{\mathfrak{R}\} \\
&= \int_V \Bigg[L_1 * g * \left\{C_{ijkl}e_{kl} - \frac{T_0\beta_{ij}}{\rho c_E}(\rho S - \beta_{rs}e_{rs}) - \sigma_{ij}\right\} * e_{ij}' \\
&+ L_1 * g * \left\{\frac{T_0}{\rho c_E}((\rho S - \beta_{rs}e_{rs})) - \theta\right\} * \rho S' \\
&+ g * l * \frac{1}{T_0}\left(-L_1 * K_{ij}\eta_j - L_2 * K_{ij}^*\eta_j + N_i - L_1 * q_i\right) * \eta_i'\Bigg] dV \\
&- \int_V \left[L_1 * (g * \sigma_{ij,j} + f_i - \rho u_i) * u_i' + L_1 * g * \left(\rho S + l * \frac{q_{i,i}}{T_0} - W\right) * \theta'\right] dV \\
&- \int_V \left[L_1 * g * \left(e_{ij} - u_{(i,j)}\right) * \sigma_{ij}' - L_1 * g * l * \frac{1}{T_0}(\theta_{,i} - \eta_i) * q_i'\right] dV \\
&+ \int_{B_1} L_1 * g * (\tilde{u}_i - u_i) * \sigma_i' dA + \int_{B_2} L_1 * g * (\sigma_i - \tilde{\sigma}_i) * u_i' dA \\
&+ \frac{1}{T_0}\int_{B_3} L_1 * g * l * \left(\tilde{\theta} - \theta\right) * q' dA + \frac{1}{T_0}\int_{B_4} L_1 * g * l * (q - \tilde{q}) * \theta' dA, \qquad (2.1.37)
\end{aligned}$$

for all $t \in [0, \infty)$.

Firstly, assuming that $\mathfrak{R}$ is a solution of the mixed initial-boundary value problem, then from Theorem-2.1.4.2, the relations (2.1.32) to (2.1.34) and the boundary conditions (2.1.9), the following is obtained:

$$\delta_{\mathfrak{R}'}\Omega_t\{\mathfrak{R}\} = 0,\ t \in [0, \infty) \qquad (2.1.38)$$

for every $\mathfrak{R}' = \{u_i', \theta', \gamma_i', e_{ij}', \sigma_{ij}', q_i', S'\} \in \Lambda$, and therefore yield Eq. (2.1.36). This completes the proof of the necessary part of the Theorem-2.1.5.1.

Conversely, let Eq. (2.1.36) holds true and hence, Eq. (2.1.38) holds for every $\mathfrak{R}' =$

$\{u_i^{'}, \theta^{'}, \eta_i^{'}, e_{ij}^{'}, \sigma_{ij}^{'}, q_i^{'}, S^{'}\} \in \Lambda$. Then, it is to be shown that $\mathfrak{R}$ is a solution of mixed initial-boundary value problem.

Since, Eq. (2.1.38) holds for every $\mathfrak{R}^{'} \in \Lambda$, let $\mathfrak{R}^{'} = \{u_i^{'}, 0, 0, 0, 0, 0, 0\}$ and let $u_i^{'}$ along with all the space derivatives vanish on $\partial V \times [0, \infty)$. Therefore, Eq. (2.1.37) and Eq. (2.1.38) yield

$$\int_V L_1 * (g * \sigma_{ij,j} + f_i - \rho u_i) * u_i^{'} dV = 0, \qquad \text{for } t \in [0, \infty). \tag{2.1.39}$$

Further, by using Lemma-1 (see Gurtin (1964)) and convolution properties, it is found that Eq. (2.1.32) holds.

Again, by choosing $\mathfrak{R}^{'} = \{0, \theta^{'}, 0, 0, 0, 0, 0\}$ and letting $\theta^{'}$ along with all the space derivatives vanish on $\partial V \times [0, \infty)$, Eq. (2.1.37) and Eq. (2.1.38) imply the following:

$$\int_V L_1 * g * \left(\rho S + l * \frac{q_{i,i}}{T_0} - W\right) * \theta^{'} dV = 0, \qquad \text{for } t \in [0, \infty).$$

Therefore, by using Lemma-1 (see Gurtin (1964)) and convolution properties, Eq. (2.1.33) is obtained.

Similarly, by substituting appropriate choices of $\mathfrak{R}^{'}$ into Eq. (2.1.37) and with the help of three lemmas (1-3) (Gurtin (1964)) it can be proved that $\mathfrak{R}$ also satisfies the Eq. (2.1.33), Eq. (2.1.34), Eq. (2.1.3), Eq. (2.1.4), Eq. (2.1.6), Eq. (2.1.7) and the boundary conditions (2.1.9). Therefore, from Theorem-2.1.4.2, $\mathfrak{R}$ is the solution of the present mixed problem. Hence, the proof of the above theorem is complete.

2.1.6 Reciprocity Theorem

Now, considering two different systems of thermoelastic loadings

$$L^a = \left(H_i^{(a)}, R^{(a)}, \tilde{u_i}^{(a)}, \tilde{\theta}^{(a)}, \tilde{q}_i^{(a)}, \tilde{\sigma}_i^{(a)}, u_{0_i}^{(a)}, v_i^{(a)}, \theta_0^{(a)}, \theta_1^{(a)}, q_{0_i}^{(a)}\right), \; a = 1, 2. \tag{2.1.40}$$

The corresponding thermoelastic configurations are denoted as

$$I^a = (u_i^{(a)}, \theta^{(a)}), \tag{2.1.41}$$

that satisfy Eqs. (2.1.32-2.1.34), Eq. (2.1.3), Eq.(2.1.4), Eq. (2.1.6), Eq. (2.1.7) and the boundary conditions (2.1.9).

The aim is to establish a reciprocity theorem that states the relation between these two sets of thermoelastic loading and thermoelastic configurations. For this, the following notations are used:

$$f_i^{(a)} = \rho\left(g * H_i^{(a)} + tv_i^{(a)} + u_{0_i}^{(a)}\right), \tag{2.1.42}$$

$$W^{(a)} = l * \frac{\rho R^{(a)}}{T_0} + \rho c_E \frac{\theta_0^{(a)}}{T_0} + \beta_{ij} u_{0_{i,j}}^{(a)}, \tag{2.1.43}$$

$$N_i^{(a)} = l * (tq_{0_i}^{(a)} + tK_{ij}\theta_{0,j}^{(a)} - t\tau K_{ij}^*\theta_{0,j}^{(a)} + t\frac{\tau^2}{2}K_{ij}^*\theta_{1,j}^{(a)} + \frac{\tau^2}{2}K_{ij}^*\theta_{0,j}^{(a)}), \tag{2.1.44}$$

for $a = 1, 2$. Then, the reciprocity theorem is given as below.

Theorem -2.1.6.1 (Reciprocity theorem):

Statement: If a thermoelastic solid is associated with two different systems of thermoelastic loadings, L^a, $(a = 1,\ 2)$ and the corresponding thermoelastic configurations, I^a, $(a = 1,\ 2)$, then the following reciprocity relation holds:

$$\begin{aligned}\int_V L_1 * \left[f_i^{(1)} * u_i^{(2)} - g * W^{(1)} * \theta^{(2)}\right] dV + \int_A L_1 * g * \left[\sigma_i^{(1)} * u_i^{(2)} + \frac{1}{T_0} l * q^{(1)} * \theta^{(2)}\right] dA \\ - \int_V g * l * \left[\frac{1}{T_0} N_i^{(1)} * \eta_i^{(2)}\right] dV = \int_V L_1 * \left[f_i^{(2)} * u_i^{(1)} - g * W^{(2)} * \theta^{(1)}\right] dV \\ + \int_A L_1 * g * \left[\sigma_i^{(2)} * u_i^{(1)} + \frac{1}{T_0} l * q^{(2)} * \theta^{(1)}\right] dA - \int_V g * l * \left[\frac{1}{T_0} N_i^{(2)} * \eta_i^{(1)}\right] dV,\end{aligned} \tag{2.1.45}$$

where, $f_i^{(a)},\ W^{(a)},\ N_i^{(a)}\ \ (a = 1,\ 2)$ associated with two systems are given by Eq. (2.1.42), Eq. (2.1.43), and Eq. (2.1.44), respectively.

Proof: Using Eq. (2.1.3), it can be written as

$$\sigma_{ij}^{(a)} = C_{ijkl} e_{kl}^{(a)} - \beta_{ij}\theta^{(a)}. \tag{2.1.46}$$

Next on taking convolution of Eq. (2.1.46) for $a = 1$ with $e_{ij}^{(2)}$ and for $a = 2$ with $e_{ij}^{(1)}$ and then subtracting the results yield

$$\left(\sigma_{ij}^{(1)} + \beta_{ij}\theta^{(1)}\right) * e_{ij}^{(2)} = \left(\sigma_{ij}^{(2)} + \beta_{ij}\theta^{(2)}\right) * e_{ij}^{(1)} + C_{ijkl}\left(e_{kl}^{(1)} * e_{ij}^{(2)} - e_{kl}^{(2)} * e_{ij}^{(1)}\right).$$

Hence, the symmetric properties of C_{ijkl} give

$$C_{ijkl}\left(e_{kl}^{(1)} * e_{ij}^{(2)} - e_{kl}^{(2)} * e_{ij}^{(1)}\right) = C_{ijkl}e_{kl}^{(1)} * e_{ij}^{(2)} - C_{klij}e_{kl}^{(1)} * e_{ij}^{(2)} = 0. \tag{2.1.47}$$

Therefore,

$$\left(\sigma_{ij}^{(1)} + \beta_{ij}\theta^{(1)}\right) * e_{ij}^{(2)} = \left(\sigma_{ij}^{(2)} + \beta_{ij}\theta^{(2)}\right) * e_{ij}^{(1)}. \tag{2.1.48}$$

Again from Eq. (2.1.4), it can be written as

$$\rho S^{(a)} - \beta_{ij}e_{ij}^{(a)} = \rho c_E \frac{\theta^{(a)}}{T_0}, \qquad a = 1,\, 2. \tag{2.1.49}$$

Taking convolution of Eq. (2.1.49) for $a = 1$ with $\theta^{(2)}$ and for $a = 2$ with $\theta^{(1)}$ and subtracting, yield the equation as

$$\left(\rho S^{(1)} - \beta_{ij}e_{ij}^{(1)}\right) * \theta^{(2)} = \left(\rho S^{(2)} - \beta_{ij}e_{ij}^{(2)}\right) * \theta^{(1)}. \tag{2.1.50}$$

Eq. (2.1.48) and Eq. (2.1.50) yield

$$\left(\sigma_{ij}^{(1)} * e_{ij}^{(2)} - \rho S^{(1)} * \theta^{(2)}\right) = \left(\sigma_{ij}^{(2)} * e_{ij}^{(1)} - \rho S^{(2)} * \theta^{(1)}\right). \tag{2.1.51}$$

Next, introducing the notation

$$L_{ab} = \int_V L_1 * g * \left[\sigma_{ij}^{(a)} * e_{ij}^{(b)} - \rho S^{(a)} * \theta^{(b)}\right] dV, \qquad a, b = 1,\, 2. \tag{2.1.52}$$

Now, Eq. (2.1.7) and Eqs. (2.1.32-2.1.34) give

$$\begin{aligned}
&L_1 * g * \left(\sigma_{ij}^{(a)} * e_{ij}^{(b)} - \rho S^{(a)} * \theta^{(b)}\right) \\
&= L_1 * g * \sigma_{ij}^{(a)} * u_{i,j}^{(b)} - L_1 * g * \left(-l * \frac{q_{i,i}^{(a)}}{T_0} + W^{(a)}\right) * \theta^{(b)} \\
&= L_1 * g * \left(\sigma_{ij}^{(a)} * u_i^{(b)}\right)_{,j} - L_1 * g * \left(\sigma_{ij,j}^{(a)} * u_i^{(b)}\right) \\
&\quad + \frac{1}{T_0} L_1 * g * \left(l * q_i^{(a)} * \theta^{(b)}\right)_{,i} - \frac{1}{T_0} L_1 * g * l * q_i^{(a)} * \eta_i^{(b)} \\
&\quad - L_1 * g * W^{(a)} * \theta^{(b)},
\end{aligned}$$

$$
\begin{aligned}
& L_1 * g * \left(\sigma_{ij}^{(a)} * e_{ij}^{(b)} - \rho S^{(a)} * \theta^{(b)}\right) \\
& = L_1 * g * \left(\sigma_{ij}^{(a)} * u_i^{(b)}\right)_{,j} - L_1 * \rho u_i^{(a)} * u_i^{(b)} + L_1 * f_i^{(a)} * u_i^{(b)} \\
& + \frac{1}{T_0} L_1 * g * l * \left(q_i^{(a)} * \theta^{(b)}\right)_{,i} + \frac{1}{T_0} g * l * \Big(L_1 * K_{ij}\eta_j^{(a)} \\
& + L_2 * K_{ij}^* \eta_j^{(a)}\Big) * \eta_i^{(b)} - \frac{1}{T_0} g * l * N_i^{(a)} * \eta_i^{(b)} \\
& - L_1 * g * W^{(a)} * \theta^{(b)}.
\end{aligned} \tag{2.1.53}
$$

Therefore, from Eq. (2.1.52) and Eq. (2.1.53), the following is obtained:

$$
\begin{aligned}
& L_{ab} \\
& = \int_V L_1 * \left[f_i^{(a)} * u_i^{(b)} - g * W^{(a)} * \theta^{(b)}\right] dV + \int_{\partial V} L_1 * g * \left[\sigma_i^{(a)} * u_i^{(b)} + \frac{1}{T_0} l * q_i^{(a)} * \theta^{(b)}\right] dA \\
& - \int_V \left[L_1 * \rho u_i^{(a)} * u_i^{(b)} - \frac{1}{T_0} g * l * L_1 * K_{ij}\eta_j^{(a)} * \eta_i^{(b)} - \frac{1}{T_0} g * l * L_2 * K_{ij}^* \eta_j^{(a)} * \eta_i^{(b)}\right] dV \\
& - \int_V \left[\frac{1}{T_0} g * l * N_i^{(a)} * \eta_i^{(b)}\right] dV.
\end{aligned} \tag{2.1.54}
$$

Clearly, Eq. (2.1.51) and Eq. (2.1.52) imply

$$
L_{12} = L_{21}. \tag{2.1.55}
$$

Hence, Eq. (2.1.54) and Eq. (2.1.55) prove the reciprocity relation (2.1.45), which completes the proof of the Theorem-2.1.6.1.

2.1.7 Conclusion

In the present subchapter, some important theorems under generalized thermoelasticity model by Quintanilla (2011) are established. Uniqueness of the solution for mixed initial-boundary problem for homogeneous and anisotropic thermoelastic medium is obtained. Variational theorem of convolution type using an alternative formulation of the problem followed by reciprocity theorem is presented .

2.2 Galerkin-Type Solution for the Theory of Thermoelasticity under an Exact Heat Conduction Law with Single Delay[2]

2.2.1 Introduction

In this subchapter, the theoretical analysis of Quintanilla's thermoelastic model is further pursued by deriving the representation of solution into elementary functions for isotropic and homogeneous thermoelastic material under linear theory. This representation simplifies the original complicated system of differential equations and further helps to find the solution of original problem in terms of elementary functions such as harmonic, biharmonic, metaharmonic, etc. This provides aid in solving various boundary value problems in the field of elasticity and thermoelasticity. Proceeding with addressing all the governing equations and constitutive relations, a Galerkin-type solution of equations of motion under the thermoelasticity model is presented followed by a Galerkin-type solution for the system of equations of steady oscillations. Lastly, the general solution for the homogeneous system of equations for steady oscillations is acquired.

The related works available in the literature are stated as following. The Galerkin-type solution (Galerkin (1930)) of the equations of classical elastokinetics was given by Iacovache (1949). Nowacki (1964; 1969a) and Sandru (1966) discussed the representation of solutions namely Galerkin's and Papkovitch's in the classical theory of thermodynamics and micropolar elasticity. Representations of a solution such as the Boussinesq-Somigliana-Galerkin (BSG), Boussinesq-Papkovitch-Neuber (BPN), Green-Lame (GLa), and Cauchy-Kovlevski-Somigliana (CKS) (Gurtin (1972), Nowacki (1975b),

[2] The content of this subchapter is presented in *International Conference on Engineering, Computers and Natural Sciences, 2018*

Kupradze et al. (1976)) are well established (Scalia and Svanadze (2006)) in the context of classical elasticity. Chandrasekharaiah (1987a; 1989) thoroughly presented the BPN, GLa, and CKS forms of the solution in the theory of voids. Ciarletta (1991; 1995; 1999) provided the Galerkin-type representation of solutions in case of the theory of thermoelastic materials with voids, micropolar thermoelasticity without energy dissipation and the dynamical theory of binary mixture consisting of gas and an elastic solid, respectively. In the theory of binary mixtures of elastic solids and theory of porous media, Svanadze (1993) and Svanadze and De Boer (2005) presented the Galerkin-type representation of general solutions. Scalia and Svanadze (2006) gave the representation of general as well as steady oscillation solutions in the theory of thermoelasticity with micro-temperatures. Mukhopadhyay et al. (2010) presented the representation of solutions for the linear theory of three-phase-lag thermoelasticity theory (Roychoudhuri (2007a)). Later, Kothari and Mukhopadhyay (2012) established the representation theorem for the generalized theory of thermoelastic diffusion (Sherief et al. (2004)). Recently, Svanadze (2014; 2017) derived the Galerkin-type solution in the case of linear thermoviscoelasticity theory for Kelvin-Voigt materials with voids and linear theory of micropolar viscoelasticity, respectively. For understanding applications of this representation of solution, it is worth referring the recent article by Giorgashvili et al. (2015).

2.2.2 Governing Equations

Let $\boldsymbol{x} = (x_1, x_2, x_3)$ represents an arbitrary point in three-dimensional Euclidean space and t be the time variable. An isotropic elastic homogeneous medium is considered to analyze a thermoelasticity theory. The medium occupies a bounded region Ξ of Euclidean three-dimensional space at $t = 0$. Following Quintanilla (2011) and Leseduarte and Quintanilla (2013), the basic equations in the context of considered thermoelasticity theory are as follows:

Heat conduction law:

$$\dot{\boldsymbol{q}} = -\left\{K\frac{\partial}{\partial t} + K^{*}\left(1 - \tau\frac{\partial}{\partial t} + \frac{\tau^{2}}{2}\frac{\partial^{2}}{\partial t^{2}}\right)\right\}\operatorname{grad}\theta. \tag{2.2.1}$$

Energy equation:

$$-\operatorname{div}\boldsymbol{q} + \rho R = \rho T_0 \dot{S}. \tag{2.2.2}$$

Entropy equation:

$$T_0\,\rho\,S = \rho\,c_E\theta + \beta\,T_0\operatorname{tr}\boldsymbol{E}. \tag{2.2.3}$$

Equation of motion:

$$\operatorname{div}\boldsymbol{\Gamma} + \rho\,\boldsymbol{H} = \rho\,\ddot{\boldsymbol{u}}. \tag{2.2.4}$$

Stress-strain-temperature relation:

$$\boldsymbol{\Gamma} = \lambda\operatorname{tr}\boldsymbol{E}\ \boldsymbol{I} + 2\mu\boldsymbol{E} - \beta\theta\boldsymbol{I}. \tag{2.2.5}$$

Strain-displacement relation:

$$\boldsymbol{E} = \frac{1}{2}\left(\operatorname{grad}\boldsymbol{u} + (\operatorname{grad}\boldsymbol{u})^{T}\right). \tag{2.2.6}$$

Further, eliminating $\boldsymbol{q}$, $\boldsymbol{E}$, $\boldsymbol{\Gamma}$, and S from Eqs. (2.2.1-2.2.6) gives the following field equations in the context of thermoelasticity theory under the heat conduction model given by Quintanilla (2011):

$$\mu\,\nabla^{2}\boldsymbol{u} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\boldsymbol{u} - \beta\operatorname{grad}\theta + \rho\,\boldsymbol{H} = \rho\,\ddot{\boldsymbol{u}}, \tag{2.2.7}$$

$$\left\{K\frac{\partial}{\partial t} + K^{*}\left(1 - \tau\frac{\partial}{\partial t} + \frac{\tau^{2}}{2}\frac{\partial^{2}}{\partial t^{2}}\right)\right\}\nabla^{2}\theta = \beta\,T_0\operatorname{div}\ddot{\boldsymbol{u}} + \rho\,c_E\,\ddot{\theta} - r. \tag{2.2.8}$$

where, $r = \rho\dot{R}$ is the external rate of heat source.

Now, introducing the following notations and operators:

$$\ell_1\left(\nabla^2, T\right) = m_2\nabla^2 - T^2, \quad \ell_2\left(\nabla^2, T\right) = \left\{K\frac{\partial}{\partial t} + K^*\left(1 - \tau\frac{\partial}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2}{\partial t^2}\right)\right\}\nabla^2 - \rho\, c_E T^2,$$
$$T = \frac{\partial}{\partial t}, \quad T^2 = \frac{\partial^2}{\partial t^2}, \quad m_1 = \left(\frac{\lambda + \mu}{\rho}\right), \quad m_2 = \frac{\mu}{\rho}, \quad m_3 = \frac{\beta}{\rho}.$$

Therefore, Eq. (2.2.7) and Eq. (2.2.8) take the forms as follows:

$$m_1 \text{grad div}\, \boldsymbol{u} + \ell_1\, \boldsymbol{u} - m_3 \text{grad}\, \theta = -\boldsymbol{H}, \tag{2.2.9}$$

$$\ell_2\, \theta - \beta\, T_0 T^2 \text{div}\, \boldsymbol{u} = -r. \tag{2.2.10}$$

2.2.3 Galerkin-Type Solution of Equations of Motion

In virtue of Eq. (2.2.9) and Eq. (2.2.10), presenting the matrix differential operator as following:

$$\boldsymbol{\Omega}\left(\boldsymbol{D_x}, T\right) = \begin{bmatrix} \boldsymbol{\Omega^{(1)}} & \boldsymbol{\Omega^{(2)}} \\ \boldsymbol{\Omega^{(3)}} & \boldsymbol{\Omega^{(4)}} \end{bmatrix}$$
$$\boldsymbol{\Omega^{(1)}}\left(\boldsymbol{D_x}, T\right) = \left[\Omega^{(1)}_{pq}\right]_{3\times 3}, \quad \boldsymbol{\Omega^{(2)}} = \left[\Omega^{(2)}_{p1}\right]_{3\text{x}1}, \quad \boldsymbol{\Omega^{(3)}} = \left[\Omega^{(3)}_{1q}\right]_{1\times 3}, \quad \boldsymbol{\Omega^{(4)}} = \left[\Omega_{44}\right]_{1\times 1},$$
$$\Omega^{(1)}_{pq}\left(\boldsymbol{D_x}, T\right) = \ell_1\delta_{pq} + m_1\frac{\partial^2}{\partial x_p \partial x_q},$$
$$\Omega^{(2)}_{p1}\left(\boldsymbol{D_x}, T\right) = -m_3\frac{\partial}{\partial x_p},$$
$$\Omega^{(3)}_{1q}\left(\boldsymbol{D_x}, T\right) = \left(-\beta\, T_0 T^2\right)\frac{\partial}{\partial x_q}, \quad \Omega_{44}(\boldsymbol{D_x}, T) = \ell_2. \tag{2.2.11}$$

where, the notations are defined as; $\boldsymbol{D_x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ and δ_{pq} as the Kronecker delta for $p, q = 1, 2, 3$.

Therefore, Eq. (2.2.9) and Eq. (2.2.10), can be written as

$$\boldsymbol{\Omega}\left(\boldsymbol{D_x}, T\right)\boldsymbol{U}\left(\boldsymbol{x}, t\right) = \boldsymbol{\mathcal{F}}\left(\boldsymbol{x}, t\right), \tag{2.2.12}$$

where, $\boldsymbol{U} = (\boldsymbol{u}, \theta)$, $\boldsymbol{\mathcal{F}} = (-\boldsymbol{H}, -r)$ and $(\boldsymbol{x}, t) \in \Xi \times (0, +\infty)$.

Now, the following system of equations is introduced:

$$m_1 \text{grad div}\, \boldsymbol{u} + \ell_1\, \boldsymbol{u} - \beta\, T_0\, T^2\, \text{grad}\, \theta = \boldsymbol{\mathcal{F}'}, \tag{2.2.13}$$

$$\ell_2\, \theta - m_3 \text{div}\, \boldsymbol{u} = \mathcal{F}_0, \tag{2.2.14}$$

where, $\boldsymbol{\mathcal{F}'} = (\mathcal{F}_1', \mathcal{F}_2', \mathcal{F}_3')$ is the vector function with $\mathcal{F}_0$ and $\mathcal{F}_i'\,(i = 1, 2, 3)$ as scalar functions on $\Xi \times (0, +\infty)$.

Hence, in term of matrix operator, system (2.2.13-2.2.14) can be expressed in the form

$$\boldsymbol{\Omega^T}\left(\boldsymbol{D_x}, T\right) \boldsymbol{U}\left(\boldsymbol{x}, T\right) = \boldsymbol{\mathcal{H}}\left(\boldsymbol{x}, t\right). \tag{2.2.15}$$

where, $\boldsymbol{\Omega^T}$ is the transpose of matrix $\boldsymbol{\Omega}$ and $\boldsymbol{\mathcal{H}} = (\boldsymbol{\mathcal{F}'}, \mathcal{F}_0)$.

Next, taking the divergence of Eq. (2.2.13) yields

$$\mathfrak{B}_1 \text{div}\, \boldsymbol{u} - \beta\, T_0 T^2 \nabla^2 \theta = \text{div}\; \boldsymbol{\mathcal{F}'}, \tag{2.2.16}$$

where, $\mathfrak{B}_1(\nabla^2, T) = \left(\frac{\lambda + 2\mu}{\rho}\right) \nabla^2 - T^2$.

Therefore, the matrix representation of Eq. (2.2.14) and Eq. (2.2.16) is derived as follows:

$$\boldsymbol{\mathfrak{B}}\left(\nabla^2, T\right) \boldsymbol{V} = \tilde{\boldsymbol{\mathcal{F}}}. \tag{2.2.17}$$

where, $\boldsymbol{V} = (\text{div}\, \boldsymbol{u},\, \theta)$, $\tilde{\boldsymbol{\mathcal{F}}} = (\text{div}\, \boldsymbol{\mathcal{F}'}, \mathcal{F}_0)$, and

$$\boldsymbol{\mathfrak{B}}(\nabla^2, T) = \left[\mathfrak{B}_{pq}\left(\nabla^2, T\right)\right]_{2\times 2} = \begin{bmatrix} \mathfrak{B}_1 & -\beta\, T_0 T^2 \nabla^2 \\ -m_3 & \ell_2 \end{bmatrix}.$$

System (2.2.17) implies

$$\chi_1\left(\nabla^2, T\right)\, \boldsymbol{V} = \boldsymbol{\Phi}, \tag{2.2.18}$$

with,

$$\boldsymbol{\Phi} = \left(\Phi_1, \Phi_2\right), \quad \Phi_q = \sum_{p=1}^{2} \mathfrak{B}^*_{pq} f_p, \quad \chi_1(\nabla^2, T) = \det \boldsymbol{\mathfrak{B}}\left(\nabla^2, T\right), \tag{2.2.19}$$

where, $q = 1,\, 2$ and $\mathfrak{B}^*_{pq}$ is the co-factor of the element $\mathfrak{B}_{pq}$ of the matrix $\boldsymbol{\mathfrak{B}}$.

Now, operating $\chi_1\left(\nabla^2, T\right)$ to Eq. (2.2.13), and using Eq. (2.2.18), gives the following relation:

$$\chi_1\left(\nabla^2, T\right)\, \ell_1 \boldsymbol{u} = \boldsymbol{\Phi'}, \tag{2.2.20}$$

where,

$$\boldsymbol{\Phi'} = \chi_1 \boldsymbol{\mathcal{F}'} - \text{grad}\left[m_1 \Phi_1 - \beta\, T_0 T^2 \Phi_2\right]. \tag{2.2.21}$$

Further, in view of Eq. (2.2.18) and Eq. (2.2.20), it is acquired that

$$\boldsymbol{\chi}\left(\nabla^2, T\right) \boldsymbol{U}\left(\boldsymbol{x}, t\right) = \tilde{\boldsymbol{\Phi}}, \tag{2.2.22}$$

where, $\tilde{\boldsymbol{\Phi}} = (\boldsymbol{\Phi'},\, \Phi_2)$ and

$$\begin{aligned} \boldsymbol{\chi}(\nabla^2, T) &= \left[\chi_{pq}(\nabla^2, T)\right]_{4\times 4} \\ \chi_{jj} &= \chi_1\left(\nabla^2, T\right)\, \ell_1, \;\; j = 1, 2, 3 \\ \chi_{44} &= \chi_1(\nabla^2, T), \;\; \chi_{pq} = 0, \;\; p,\, q = 1, 2, 3, 4\; p \neq q. \end{aligned} \tag{2.2.23}$$

Further, introducing the operators

$$\begin{aligned} n_{p1}\left(\nabla^2, T\right) &= -\left\{m_1 \mathfrak{B}^*_{p1} - \beta\, T_0 T^2 \mathfrak{B}^*_{p2}\right\}, \\ n_{p2}(\nabla^2, T) &= \mathfrak{B}^*_{p2}, \;\; p = 1,\, 2, \end{aligned} \tag{2.2.24}$$

it can be obtained from Eq. (2.2.19), Eq. (2.2.21) that

$$\boldsymbol{\Phi'} = \left(\chi_1 \boldsymbol{I} + n_{11}\text{grad div}\right) \boldsymbol{\mathcal{F}'} + n_{21}\text{grad}\, \mathcal{F}_0, \tag{2.2.25}$$

$$\Phi_2 = n_{12}\text{div}\, \boldsymbol{\mathcal{F}'} + n_{22}\mathcal{F}_0. \tag{2.2.26}$$

Thus, in view of Eq. (2.2.25) and Eq. (2.2.26), it is found that

$$\tilde{\boldsymbol{\Phi}}(\boldsymbol{x},t) = \boldsymbol{\mathcal{L}}^{\boldsymbol{T}}\left(\boldsymbol{D}_{\boldsymbol{x}},T\right)\boldsymbol{\mathcal{H}}\left(\boldsymbol{x},t\right), \tag{2.2.27}$$

where,

$$\boldsymbol{\mathcal{L}} = \begin{bmatrix} \boldsymbol{\mathcal{L}}^{(1)} & \boldsymbol{\mathcal{L}}^{(2)} \\ \boldsymbol{\mathcal{L}}^{(3)} & \boldsymbol{\mathcal{L}}^{(4)} \end{bmatrix}_{4\times 4},$$

$$\boldsymbol{\mathcal{L}}^{(1)} = \left[\mathcal{L}^{(1)}_{pq}\right]_{3\times 3}, \;\; \boldsymbol{\mathcal{L}}^{(2)} = \left[\mathcal{L}^{(2)}_{p1}\right]_{3\text{x}1}, \;\; \boldsymbol{\mathcal{L}}^{(3)} = \left[\mathcal{L}^{(3)}_{1q}\right]_{1\times 3}, \;\; \boldsymbol{\mathcal{L}}^{(4)} = \left[\mathcal{L}_{44}\right]_{1\times 1},$$

$$\mathcal{L}^{(1)}_{pq}\left(\boldsymbol{D}_{\boldsymbol{x}},T\right) = \chi_1\left(\nabla^2,T\right)\delta_{pq} + n_{11}\left(\nabla^2,T\right)\frac{\partial^2}{\partial x_p \partial x_q}, \;\; \mathcal{L}^{(2)}_{p1}\left(\boldsymbol{D}_{\boldsymbol{x}},T\right) = n_{12}\left(\nabla^2,T\right)\frac{\partial}{\partial x_p},$$

$$\mathcal{L}^{(3)}_{1q}(\boldsymbol{D}_{\boldsymbol{x}},T) = n_{21}(\nabla^2,T)\frac{\partial}{\partial x_q}, \;\; \mathcal{L}_{44} = n_{22}(\nabla^2,T), \quad p,q = 1,\, 2,\, 3. \tag{2.2.28}$$

Next, using Eq. (2.2.15), Eq. (2.2.22), and Eq. (2.2.27), following is obtain

$$\boldsymbol{\chi}\boldsymbol{U} = \boldsymbol{\mathcal{L}}^{\boldsymbol{T}}\boldsymbol{\Omega}^{\boldsymbol{T}}\boldsymbol{U},$$

which implies $\boldsymbol{\mathcal{L}}^{\boldsymbol{T}}\boldsymbol{\Omega}^{\boldsymbol{T}} = \boldsymbol{\chi}$ and hence,

$$\boldsymbol{\Omega}(\boldsymbol{D}_{\boldsymbol{x}},T)\boldsymbol{\mathcal{L}}(\boldsymbol{D}_{\boldsymbol{x}},T) = \boldsymbol{\chi}(\nabla^2,T). \tag{2.2.29}$$

Thus, the following lemma is proved.

Lemma-2.2.3.1:

Statement: If the matrix differential operators $\boldsymbol{\Omega}$, $\boldsymbol{\mathcal{L}}$, and $\boldsymbol{\chi}$ are defined by Eq. (2.2.11), Eq. (2.2.28), and Eq. (2.2.23), respectively, then $\boldsymbol{\Omega}$, $\boldsymbol{\mathcal{L}}$, and $\boldsymbol{\chi}$ satisfy Eq. (2.2.29).

Now, let $H'_j(\boldsymbol{x},\,t)$, $(j = 1,\,2,\,3)$ and $h(\boldsymbol{x},\,t)$ be functions on $\Xi \times (0,+\infty)$ with $\boldsymbol{H'} = (H'_1, H'_2, H'_3)$, and $\widetilde{\boldsymbol{H}} = (\boldsymbol{H'}, h)$. Then, the subsequent theorem provides a Galerkin-type solution to the system by Eq. (2.2.3) and Eq. (2.2.4).

Theorem-2.2.3.1:

Statement: Let

$$\boldsymbol{u} = \boldsymbol{\mathcal{L}}^{(1)}\boldsymbol{H'} + \boldsymbol{\mathcal{L}}^{(2)}h, \tag{2.2.30}$$

$$\theta = \boldsymbol{\mathcal{L}}^{(3)}\boldsymbol{H'} + \boldsymbol{\mathcal{L}}^{(4)}h, \tag{2.2.31}$$

where, the fields H'_j of class C^6 and h of class C^4 satisfy

$$\chi_1(\nabla^2, T)\,\ell_1\boldsymbol{H'} = -\boldsymbol{H}, \tag{2.2.32}$$

$$\chi_1(\nabla^2, T)\,h = -r, \tag{2.2.33}$$

on $\mathit{\Xi}\times(0,+\infty)$. Then $\boldsymbol{U} = (\boldsymbol{u}, \theta)$ is the solution of Eq. (2.2.9) and Eq. (2.2.10).

Proof: Eq. (2.2.30) and Eq. (2.2.31) yield

$$\boldsymbol{U}(\boldsymbol{x}, t) = \boldsymbol{\mathcal{L}}(\boldsymbol{D_x}, T)\widetilde{\boldsymbol{H}}(\boldsymbol{x}, t). \tag{2.2.34}$$

On the other hand, from Eq. (2.2.32) and Eq. (2.2.33), it is obtained that

$$\boldsymbol{\chi}(\nabla^2, T)\widetilde{\boldsymbol{H}}(\nabla^2, T) = \boldsymbol{\mathcal{F}}(\nabla^2, T). \tag{2.2.35}$$

In view of Eq. (2.2.29), Eq. (2.2.34), and Eq. (2.2.35), it is acquired that $\boldsymbol{\Omega U} = \boldsymbol{\Omega\mathcal{L}}\widetilde{\boldsymbol{H}} = \boldsymbol{\chi}\widetilde{\boldsymbol{H}} = \boldsymbol{\mathcal{F}}$, which finalizes the proof of the theorem.

2.2.4 Galerkin-Type Solution of System of Equations for Steady Oscillations

In this subsection, the steady state oscillations are considered. Hence, the solution and external loads can be assumed in the following forms:

$$\boldsymbol{u}(\boldsymbol{x}, t) = Re[\tilde{\boldsymbol{u}}(\boldsymbol{x})\,e^{-\mathrm{i}\omega t}], \qquad \boldsymbol{H}(\boldsymbol{x}, t) = Re[\tilde{\boldsymbol{H}}(\boldsymbol{x})\,e^{-\mathrm{i}\omega t}],$$

$$\theta(\boldsymbol{x}, t) = Re[\tilde{\theta}(\boldsymbol{x})\,e^{-\mathrm{i}\omega t}], \qquad r(\boldsymbol{x}, t) = Re[\tilde{r}(\boldsymbol{x})\,e^{-\mathrm{i}\omega t}].$$

Therefore, from Eq. (2.2.7) and Eq. (2.2.8), the system of equations of the steady oscillations for the assumed thermoelasticity theory are derived as follows:

$$\mu\,\nabla^2\tilde{\boldsymbol{u}} + (\lambda+\mu)\mathrm{grad}\,\mathrm{div}\,\tilde{\boldsymbol{u}} - \beta\mathrm{grad}\,\tilde{\theta} + \rho\,\tilde{\boldsymbol{H}} = -\omega^2\,\rho\,\tilde{\boldsymbol{u}}, \tag{2.2.36}$$

$$\left\{\left[K - \frac{\omega^2\tau\,K^*}{2} - \mathrm{i}\,\omega\,(K - K^*\tau)\right]\nabla^2 + \rho\,c_E\,\omega^2\right\}\tilde{\theta} + \beta\,T_0\,\omega^2\,\mathrm{div}\,\tilde{\boldsymbol{u}} = -\tilde{r}. \tag{2.2.37}$$

where, $(\boldsymbol{x},t)\ \in \varXi\times(0,+\infty)$, $\mathrm{i}=\sqrt{-1}$, and $\omega(>0)$ denotes the frequency of oscillation. The above system can further be expressed as

$$\left[\rho\,\omega^2 + \mu\,\nabla^2\right]\tilde{\boldsymbol{u}} + (\lambda+\mu)\mathrm{grad}\,\mathrm{div}\,\tilde{\boldsymbol{u}} - \beta\,\mathrm{grad}\,\tilde{\theta} = -\rho\,\tilde{\boldsymbol{H}}, \tag{2.2.38}$$

$$\left\{\left[K - \frac{\omega^2\tau\,K^*}{2} - \mathrm{i}\,\omega\,(K - K^*\tau)\right]\nabla^2 + \rho\,c_E\,\omega^2\right\}\tilde{\theta} + \beta\,T_0\,\omega^2\,\mathrm{div}\,\tilde{\boldsymbol{u}} = -\tilde{r}. \tag{2.2.39}$$

In the following, the underneath notations are used

$$\begin{aligned}\boldsymbol{\mathfrak{C}}(\nabla^2) &= |\mathfrak{C}_{pq}(\nabla^2)|_{2\mathrm{x}2}\\ &= \begin{bmatrix} \rho\,\omega^2 + (\lambda+2\mu)\nabla^2 & \beta\,T_0\omega^2\nabla^2 \\ -\beta & \left[K - \frac{\omega^2\tau\,K^*}{2} - i\,\omega\,(K - K^*\tau)\right]\nabla^2 + \rho\,c_E\,\omega^2 \end{bmatrix}_{2\times 2}.\end{aligned}$$

Now, let

$$\begin{aligned}\tilde{\chi}_1(\nabla^2) &= \det\,\mathfrak{C}(\nabla^2),\\ m_{p1}(\nabla^2) &= -\left[(\lambda+\mu)\mathfrak{C}^*_{p1} + \beta\,T_0\omega^2\,\mathfrak{C}^*_{p2}\right],\\ m_{p2}(\nabla^2) &= \mathfrak{C}^*_{p2},\ \ p=1,\,2.\end{aligned}$$

It can be easily verified that if λ_1^2 and λ_2^2 are the roots of the equation $\tilde{\chi}_1(-\lambda^*) = 0$, then $\tilde{\chi}_1(\nabla^2) = (\nabla^2+\lambda_1^2)(\nabla^2+\lambda_2^2)$.

Next, the matrix differential operators $\mathfrak{M}$ and $\tilde{\chi}$ are defined by

•

$$\mathfrak{M} = \begin{bmatrix} \mathfrak{M}^{(1)} & \mathfrak{M}^{(2)} \\ \mathfrak{M}^{(3)} & \mathfrak{M}^{(4)} \end{bmatrix}_{4\times 4},$$

$$\mathfrak{M}^{(1)} = \left[\mathfrak{M}^{(1)}_{lj}\right]_{3\times 3},\ \mathfrak{M}^{(2)} = \left[\mathfrak{M}^{(2)}_{l1}\right]_{3\times 1},\ \mathfrak{M}^{(3)} = \left[\mathfrak{M}^{(3)}_{1l}\right]_{3\times 1},\ \mathfrak{M}^{(4)} = \left[\mathfrak{M}_{44}\right]_{1\times 1},$$

$$\mathfrak{M}^{(1)}_{pq}(\boldsymbol{D_x}) = \tilde{\chi}_1(\nabla^2)\delta_{pq} + m_{11}(\nabla^2)\frac{\partial^2}{\partial x_p \partial x_q},\ \mathfrak{M}^{(2)}_{p1}(\boldsymbol{D_x}) = m_{12}(\nabla^2)\frac{\partial}{\partial x_p},$$

$$\mathfrak{M}^{(3)}_{1p}(\boldsymbol{D_x}) = m_{21}(\nabla^2)\frac{\partial}{\partial x_p},\ \mathfrak{M}_{44} = m_{22}(\nabla^2),\quad p, q = 1,\ 2,\ 3. \tag{2.2.40}$$

•

$$\tilde{\boldsymbol{\chi}}(\nabla^2, T) = \left[\chi_{pq}(\nabla^2)\right]_{4\times 4},$$

$$\tilde{\chi}_{jj} = \tilde{\chi}_1(\nabla^2)[\rho\,\omega^2 + \mu\,\nabla^2],\ \ j = 1, 2, 3,$$

$$\tilde{\chi}_{44} = \tilde{\chi}_1(\nabla^2),\ \ \tilde{\chi}_{pq} = 0,\ \ p,\ q = 1, 2, 3, 4\ p \neq q. \tag{2.2.41}$$

If $\tilde{\mathcal{Q}}_j$, ($j = 1, 2, 3$) and q be functions on Ξ with $\tilde{\boldsymbol{\mathcal{Q}}} = (\tilde{\mathcal{Q}}_1, \tilde{\mathcal{Q}}_2, \tilde{\mathcal{Q}}_3)$, and $\boldsymbol{\mathcal{Q}} = (\tilde{\boldsymbol{\mathcal{Q}}}, q)$ then, in accordance with the Theorem-2.2.3.1, the following theorem provides a Galerkin-type solution to system by Eq. (2.2.36) and Eq. (2.2.37).

Theorem-2.2.4.1:

Statement: Let

$$\tilde{\boldsymbol{u}} = \mathfrak{M}^{(1)}\tilde{\boldsymbol{\mathcal{Q}}} + \mathfrak{M}^{(2)}q, \tag{2.2.42}$$

$$\tilde{\theta} = \mathfrak{M}^{(3)}\tilde{\boldsymbol{\mathcal{Q}}} + \mathfrak{M}^{(4)}q, \tag{2.2.43}$$

where, the fields $\tilde{\mathcal{Q}}_j$ of class C^6 and q of class C^4 on Ω satisfy

$$\tilde{\chi}_1(\nabla^2)\left[\rho\,\omega^2+\mu\,\nabla^2\right]\tilde{\boldsymbol{\mathcal{Q}}}=-\tilde{\boldsymbol{H}}, \tag{2.2.44}$$

$$\tilde{\chi}_1(\nabla^2)q=-\tilde{r}, \tag{2.2.45}$$

on Ξ. Then $\left(\tilde{\boldsymbol{u}},\tilde{\theta}\right)$ is the solution of Eq. (2.2.38) and Eq. (2.2.39).

2.2.5 General Solution of System of Equations for Steady Oscillations

In the absence of any body force and external heat source, the Eq. (2.2.38) and Eq. (2.2.39) can be written as

$$\left[\rho\,\omega^2+\mu\,\nabla^2\right]\tilde{\boldsymbol{u}}+(\lambda+\mu)\mathrm{grad}\,\mathrm{div}\,\tilde{\boldsymbol{u}}-\beta\,\mathrm{grad}\,\tilde{\theta}=0, \tag{2.2.46}$$

$$\left\{\left[K-\frac{\omega^2\tau\,K^*}{2}-\mathrm{i}\,\omega\,(K-K^*\tau)\right]\nabla^2+\rho\,c_E\,\omega^2\right\}\tilde{\theta}+\beta\,T_0\,\omega^2\,\mathrm{div}\,\tilde{\boldsymbol{u}}=0. \tag{2.2.47}$$

Firstly, the following lemma in the context of above system of equations is required to be proved:

Lemma-2.2.5.1:

Statement: If $(\tilde{\boldsymbol{u}},\tilde{\theta})$ is a solution of Eq. (2.2.46) and Eq. (2.2.47), then

$$\tilde{\chi}_1(\nabla^2)\mathrm{div}\,\tilde{\boldsymbol{u}}=0, \tag{2.2.48}$$

$$\tilde{\chi}_1(\nabla^2)\tilde{\theta}=0, \tag{2.2.49}$$

$$\left[\rho\,\omega^2+\mu\,\nabla^2\right]\mathrm{curl}\;\tilde{\boldsymbol{u}}=\boldsymbol{0}. \tag{2.2.50}$$

Proof: Firstly, using the operator div to Eq. (2.2.46) acquires

$$\left[\rho\,\omega^2+(\lambda+2\mu)\,\nabla^2\right]\mathrm{div}\,\tilde{\boldsymbol{u}}-\beta\,\nabla^2\tilde{\theta}=0. \tag{2.2.51}$$

Then, elimination of $\tilde{\theta}$ from Eq. (2.2.51) and Eq. (2.2.47) gthe ives

$$\tilde{\chi}_1 \text{div}\, \tilde{\boldsymbol{u}} = 0.$$

Again, from Eq. (2.2.51) and Eq. (2.2.47), eliminating div $\tilde{\boldsymbol{u}}$yields

$$\tilde{\chi}_1 \tilde{\theta} = 0.$$

Furthermore, by applying the operator curl to Eq. (2.2.46), the following is obtained

$$\left[\rho\,\omega^2 + \mu\,\nabla^2\right] \text{curl}\; \tilde{\boldsymbol{u}} = \mathbf{0}.$$

Therefore, the Eqs. (2.2.48-2.2.50) are acquired, which completes the proof of Lemma 2.2.5.1.

Theorem-2.2.5.1:

Statement: If $\left(\tilde{\boldsymbol{u}}, \tilde{\theta}\right)$ is a solution of Eq. (2.2.46) and Eq. (2.2.47), then

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \beta\,\text{grad} \sum_{p=1}^{2} \varphi_p(\boldsymbol{x}) + \boldsymbol{\Psi}(\boldsymbol{x}), \tag{2.2.52}$$

$$\tilde{\theta}(\boldsymbol{x}) = \sum_{p=1}^{2} a_p\, \varphi_p(\boldsymbol{x}), \tag{2.2.53}$$

where, $\varphi_p\,(p = 1,\, 2)$ and $\boldsymbol{\Psi} = (\Psi_1,\, \Psi_2,\, \Psi_3)$ satisfy the following equations:

$$\left(\nabla^2 + \lambda_p^2\right) \varphi_p(\boldsymbol{x}) = 0, \tag{2.2.54}$$

$$\left[\nabla^2 + \frac{\rho\,\omega^2}{\mu}\right] \boldsymbol{\Psi}(\boldsymbol{x}) = \mathbf{0},\; \boldsymbol{x} \in \boldsymbol{\Xi} \tag{2.2.55}$$

$$\text{div}\, \boldsymbol{\Psi}(\boldsymbol{x}) = 0, \tag{2.2.56}$$

and

$$a_p = -\left(\lambda + 2\mu\right) \lambda_p^2 + \rho\,\omega^2 \text{ where}, p = 1,\, 2. \tag{2.2.57}$$

Proof: Let Eq. (2.2.46) and Eq. (2.2.47) have $\left(\tilde{\boldsymbol{u}}, \tilde{\theta}\right)$ as solution. Then, taking into account $\nabla^2 \tilde{\boldsymbol{u}} = \text{grad div}\,\tilde{\boldsymbol{u}} - \text{curl curl}\,\tilde{\boldsymbol{u}}$, and using Eq. (2.2.46) give

$$\tilde{\boldsymbol{u}} = \frac{1}{\rho\,\omega^2}\left\{\text{grad}\left[-(\lambda+2\mu)\,\text{div}\;\tilde{\boldsymbol{u}} + \beta\,\tilde{\theta}\right] + \mu\,\text{curl curl}\,\tilde{\boldsymbol{u}}\right\}. \tag{2.2.58}$$

Introducing the notation

$$\boldsymbol{\Psi}(\boldsymbol{x}) = \frac{\mu}{\rho\,\omega^2}\text{curl curl}\;\tilde{\boldsymbol{u}}, \tag{2.2.59}$$

and using Eq. (2.2.50), Eq. (2.2.55) and Eq. (2.2.56) can be directly obtained asdiv curl $\tilde{\boldsymbol{u}} = 0$ for $\boldsymbol{x} \in \Xi$.

Now, let

$$\varphi_j = b_j\left[\prod_{\substack{p=1\\p\neq j}}^{2}\left(\nabla^2 + \lambda_p^2\right)\right]\tilde{\theta}, \tag{2.2.60}$$

where,

$$b_j = \left[a_j \prod_{\substack{p=1\\p\neq j}}^{2}\left(\lambda_p^2 - \lambda_j^2\right)\right]^{-1}, \; j = 1,\, 2,$$

Therefore, in view of Eq. (2.2.49), the Eq. (2.2.60) yields Eq. (2.2.54) and Eq. (2.2.53).

Next, using Eq. (2.2.47), Eq. (2.2.53), Eq. (2.2.54), and Eq. (2.2.57), give

$$\text{div}\,\tilde{\boldsymbol{u}} = -\beta\sum_{p=1}^{2}\lambda_p^2\,\varphi_p. \tag{2.2.61}$$

Hence, Eq. (2.2.58) yields

$$\tilde{\boldsymbol{u}} = \frac{1}{\rho\,\omega^2}\left\{\text{grad}\left[(\lambda+2\mu)\,\beta\sum_{p=1}^{2}\lambda_p^2\varphi_p + \beta\,\tilde{\theta}\right] + \mu\,\text{curl curl}\,\tilde{\boldsymbol{u}}\right\}. \tag{2.2.62}$$

Further, simplifying the above equation using Eq. (2.2.57) and Eq. (2.2.59), the following result is obtained

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \beta\text{grad}\sum_{p=1}^{2}\varphi_p(\boldsymbol{x}) + \boldsymbol{\Psi}(\boldsymbol{x}),$$

which completes the proof of Theorem 2.2.5.1.

Theorem-2.2.5.2:

Statement: If $\left(\tilde{\boldsymbol{u}}, \tilde{\theta}\right)$ is expressed as in Eq. (2.2.52) and Eq. (2.2.53), where φ_j and $\boldsymbol{\Psi}$ satisfies Eqs. (2.2.54-2.2.56), then $\left(\tilde{\boldsymbol{u}}, \tilde{\theta}\right)$ is the solution of Eq. (2.2.46) and Eq. (2.2.47) on Ξ.

Proof: From Eq. (2.2.52) and using Eq. (2.2.54) and Eq. (2.2.55), the following is attained

$$\nabla^2 \tilde{\boldsymbol{u}} = -\beta \sum_{p=1}^{2} \lambda_p^2 \varphi_p - \frac{\rho\,\omega^2}{\mu}\boldsymbol{\Psi},$$
$$\text{grad div}\,\tilde{\boldsymbol{u}} = -\beta \text{grad} \sum_{p=1}^{2} \lambda_p^2 \varphi_p. \tag{2.2.63}$$

Replacing $\tilde{\boldsymbol{u}}$ and $\tilde{\theta}$ as given in Eq. (2.2.52) and Eq. (2.2.53) on the left-hand side of Eq. (2.2.46) and using Eq. (2.2.54), Eq. (2.2.57), and Eq. (2.2.63), give

$$\left[\rho\,\omega^2 + \mu\,\nabla^2\right]\tilde{\boldsymbol{u}} + (\lambda+\mu)\text{grad}\,\text{div}\tilde{\boldsymbol{u}} - \beta\,\text{grad}\,\tilde{\theta}$$
$$= \rho\,\omega^2 \left[\beta\,\text{grad}\sum_{p=1}^{2}\varphi_p + \boldsymbol{\Psi}\right] - \beta\text{grad}\sum_{p=1}^{2}\left[(\lambda+2\mu)\,\lambda_p^2 + a_p\right]\varphi_p - \rho\,\omega^2\boldsymbol{\Psi}.$$

After simplification, the above equation yields

$$\left[\rho\,\omega^2 + \mu\,\nabla^2\right]\tilde{\boldsymbol{u}} + (\lambda+\mu)\text{grad}\,\text{div}\tilde{\boldsymbol{u}} - \beta\,\text{grad}\,\tilde{\theta} = 0,$$

which is the field Eq. (2.2.46).

Similarly, replacing $\tilde{\boldsymbol{u}}$ and $\tilde{\theta}$ again on the left-hand side of Eq. (2.2.47) by the expression given in (2.2.52) and (2.2.53) and using Eq. (2.2.54), Eq. (2.2.57) and Eq. (2.2.61), the following is acquired

$$\left\{\left[K-\frac{\omega^2\tau\,K^*}{2}-\mathrm{i}\,\omega\,(K-K^*\tau)\right]\nabla^2+\rho\,c_E\,\omega^2\right\}\tilde{\theta}+\beta\,T_0\,\omega^2\mathrm{div}\;\tilde{\boldsymbol{u}}$$

$$=\left\{\left[K-\frac{\omega^2\tau\,K^*}{2}-\mathrm{i}\,\omega\,(K-K^*\tau)\right]\nabla^2+\rho\,c_E\,\omega^2\right\}\left(\sum_{p=1}^{2}a_p\,\varphi_p\right)+\beta^2\,T_0\,\omega^2\left(-\sum_{p=1}^{2}\lambda_p^2\varphi_p\right)$$

$$=\sum_{p=1}^{2}\left\{a_p\left[\left(K-\frac{\omega^2\tau\,K^*}{2}-\mathrm{i}\,\omega\,(K-K^*\tau)\right)(-\lambda_p^2)+\rho\,c_E\,\omega^2\right]-\beta^2T_0\,\omega^2\lambda_p^2\right\}\varphi_p$$

$$=0.\ \left(\text{by using}\ \tilde{\chi}_1\left(-\lambda_p^2\right)=0, p=1,2\right)$$

Thus acquiring Eq. (2.2.47).

Hence, it can be confirmed that the general solution of the system of homogeneous Eq. (2.2.46) and Eq. (2.2.47) is attained in terms of the metaharmonic functions φ_p and $\boldsymbol{\Psi}$.

2.2.6 Conclusion

The present subchapter investigates a non-classical thermoelasticity model under exact heat conduction law with single delay. This includes Galerkin-type representation of solution for the system of equations of motion in terms of elementary functions. A theorem that represents Galerkin type solution of equations for steady state oscillations in the context of considered linear thermoelasticity theory is established. Finally, the representation of general solution of the system of equations in case of steady state oscillations is also acquired in terms of metaharmonic functions.