CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Thermoelasticity: Definition and Applications

Thermoelasticity is a very interesting branch of science that considers the simultaneous effects of thermal and mechanical fields in an elastic body and is concerned with the prediction of thermomechanical behavior of the medium. It is an advancement of elasticity theory that takes into account the thermal effects such as thermal stress, strain, and deformation. The tendency of the material to change its mechanical properties with the change in temperature is referred to as thermal deformation. Therefore, thermoelasticity theory predicts the thermomechanical interactions in the elastic body. It comprises of the theory of stress and strain along with the heat conduction theory. In contrast to the classical theory of elasticity, the impact of action of internal forces on the temperature field is measured along with the effect of temperature change on deformation under the theory of thermoelasticity.

This growing area of continuum mechanics proves to have a broad interest in theoretical as well as practical research. Therefore, it has become an integral subject of science. With the rapid progress in machine and aircraft structures, the importance of thermal stresses is observed. Expansion and contraction of material are the factors of concern for the stability of structures. Hence, maintaining the structural integrity while designing buildings and structures is the main application of thermoelasticity. Moreover, thermoelasticity theory has applicability in other engineering fields and technologies such as nuclear, mining, chemical, and acoustic engineering. This theory also plays a vital role in studying micro- and nano-electromechanical resonators, where the inherent loss consists of thermoelastic effects. Thermoelasticity further forms the base for other branches of science such as electro-thermoelasticity, visco-thermoelasticity, magneto-thermoelasticity, poro-thermoelasticity, thermo-piezoelectric theory, aero-thermoelasticity, and many more.

1.2 Classical Thermoelasticity Theory and Its Drawbacks

The classical thermoelasticity theory discusses the coupling effect of deformation on temperature distribution with the effect of temperature on stress and strain distribution. Hence, it is also referred to as classical coupled thermoelasticity theory. On the other hand, the uncoupled theory is developed on the simplifying assumption that the influence of strain on the temperature field may be neglected. It is observed that when heat supplied to the body is the primary reason for the change in temperature distribution, then the mechanical coupling term in the energy balance equation can be avoided. However, it can not be neglected when the temperature variation is mainly due to the deformation of the body. In coupled theory, the stress and the temperature distribution are evaluated concurrently, whereas, in uncoupled theory, physical fields are evaluated successively. Hence, the coupled thermoelasticity theory aims to overcome the drawback in the classical "uncoupled theory of thermoelasticity" that the elastic changes have no effect on the temperature and vice versa.

It is worth to be mentioned that Duhamel (1837), the formulator of thermal stresses, speculated the notion of coupling between thermal and mechanical fields for the first time and derived the equations for the strain in an elastic body with temperature gradients. Later on, Neumann (1841) also obtained the similar results. However, the theory dealt with the thermal and mechanical effects as independent effects and total strain was determined by superimposing the elastic strain and the thermal expansion caused by the temperature distribution only. Hence, the theory did not include the interactions between the strain and the temperature distributions in a specified manner. Subsequently, in 1857, the thermodynamic arguments were taken into consideration by Thomson (1857) who was the first to use the laws of thermodynamics to determine the stresses and strains in an elastic body in response to varying temperatures. Later, Voigt (1928) and Jefferys (1930) ventured the thermodynamic documentation of the equations suggested by Duhamel (1837). However, the field of coupled thermoelasticity has been stimulated through the pioneering work by Biot (1956). In that work, the fundamental relations and laws of thermomechanics, namely, laws of conservation of mass, laws of conservation of energy, the balance of momentum, and kinematic relations, were employed in a fully justified manner to derive the basic governing equations and constitutive relations for coupled thermoelasticity. Biot's theory was consequently termed as the classical coupled (or conventional) thermoelasticity theory. Biot (1956) also presented the variational principle in the context of conventional thermoelasticity theory, which is useful in deriving Lagrangian equations. Further, the author articulated the methods to obtain general solutions to the thermoelasticity equations in an isotropic homogeneous medium.

The following are the basic governing equations and constitutive relations of the linear classical coupled thermoelasticity theory for a general anisotropic medium due to Biot (1956):

The equation of motion:

$$\sigma_{ij,j} + \rho H_i = \rho \ddot{u}_i. \tag{1.2.1}$$

The energy equation:

$$\rho T_0 \dot{S} = -q_{i,i} + \rho R. \tag{1.2.2}$$

The constitutive relations:

$$\sigma_{ij} = C_{ijkl}e_{kl} - \beta_{ij}\theta, \tag{1.2.3}$$

$$T_0\rho S = \rho c_E\theta + \beta_{ij}T_0 e_{ij}, \tag{1.2.4}$$

$$q_i = -K_{ij}\theta_{,j}. \tag{1.2.5}$$

Further, for isotropic homogeneous material, the above constitutive relations take the following forms:

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \beta\theta\delta_{ij}, \tag{1.2.6}$$

$$T_0\rho S = \rho c_E\theta + \beta T_0 e_{kk}, \tag{1.2.7}$$

$$q_i = -K\theta_{,i}. \tag{1.2.8}$$

Using above equations, the following equations can be acquired:

$$K\theta_{,jj} + \rho R = \rho c_E\dot{\theta} + T_0\beta\dot{u}_{j,j}, \tag{1.2.9}$$

$$\mu u_{i,jj} + (\lambda + \mu)\, u_{j,ji} - \beta\theta_{,i} + \rho H_i = \rho\ddot{u}_i. \tag{1.2.10}$$

Here, Eq. (1.2.9) and Eq. (1.2.10) represent the coupled field equations of classical thermoelasticity in u_i and θ, namely, heat conduction equation and displacement equation of motion, respectively.

Biot's thermoelasticity theory as described above is considered to be an elegant model to study various problems involving coupling effects of thermal and mechanical fields. Eminent researchers like Chadwick (1960), Boley and Weiner (1960), Nowacki (1962; 1975b), Parkus (2012), Nowinski (1978), Dhaliwal and Singh (1980), Chandrasekharaiah (1986b) have reported a broad and detailed discussion with appealing applications and theorems based on the Biot's theory. However, it is to be noted that this theory is based on Fourier's law (Eq. (1.2.8)) and hence, it consists of a parabolic partial differential equation for thermal distribution (Eq. (1.2.9)) and a hyperbolic partial differential equation for mechanical distribution (Eq. (1.2.10)). This hyperbolic-parabolic system of partial differential equations results in infinite speed of thermal wave propagation. This behavior suggests that the effects will be observed instantaneously at another end from the source, which is physically inadmissible.

Further, several research works carried out under this classical thermoelasticity theory have investigated apparent drawbacks of presenting unconvincing results in the case of short laser pulses and low temperature (see Lord and Shulman (1967), Green and Lindsay (1972), Francis (1972), Chandrasekharaiah (1986b), Ignaczak and Ostoja-Starzewski (2010) and references therein). Also, micro-scale technology advancement supports the thermal field motion as a wave, which opposes infinite thermal propagation, i.e., heat propagation as a wave rather than diffusion. All these have drawn the serious attention of researchers over the years to step out in modifying the concept of this theory. Some useful modifications in classical thermoelasticity theory have been proposed accordingly. These modified thermoelasticity theories are often referred to as generalized thermoelasticity theories.

1.3 Generalized Thermoelasticity Theory

Generalized thermoelasticity theories are mainly the modified forms of conventional thermoelasticity theory to overcome the paradox of infinite speed of thermal propagation. These theories can broadly be divided into two categories. The first one is based on modified heat conduction law, i.e., the Fourier's law employed in classical thermoelasticity is exchanged with an appropriate alteration in the constitutive relation of heat-flux and temperature gradient. These modified constitutive relations are mostly comprised of new constitutive variable or phase-lag parameters concerning time or space, or both. The second one is the category of thermoelasticity theories in which the conventional theory is improved in an alternative way to deduce reconditioned constitutive equations by employing thermodynamic principles. However, Fourier's law is kept unchanged in many of the theories in the second category. A brief introduction on some well established and well studied or recently proposed generalized theories is given below.

1.3.1 Non-Fourier Generalized Thermoelasticity Theory

1.3.1.1 Lord-Shulman Thermoelasticity theory (LS): Thermoelasticity with One Thermal Relaxation Parameter

The generalized thermoelasticity theory proposed by Lord and Shulman (1967) is among the most studied modified theories of thermoelasticity till date. In this theory, the authors have considered the more general relation between heat-flux and temperature gradient in comparison to Fourier's law. The authors aim to include the time needed to accelerate the heat flow in the heat conduction law, which has been neglected when Fourier's law is employed (Onsager (1931)).

In the context of the heat conduction problem, Cattaneo (1958) and Vernotte (1958; 1961) have theoretically introduced the concept of "second sound." In other words, they have suggested the generalization of Fourier's law by including the time-lag, which on combining with the law of conservation of energy, gave hyperbolic type heat conduction equation in contrast to parabolic type diffusion equation. The modified Fourier's law of heat conduction presented by Cattaneo and Vernotte for the case of isotropic and homogeneous material is given as follows:

$$q_i + \tau_q \frac{\partial q_i}{\partial t} = -K\theta_{,i}, \tag{1.3.1}$$

which can be considered for the anisotropic medium in the form

$$q_i + \tau_q \frac{\partial q_i}{\partial t} = -K_{ij}\theta_{,j}. \tag{1.3.2}$$

Here, τ_q is the time-lag essential to acquire the steady-state of heat conduction when a temperature gradient is suddenly imposed. It is also addressed as a thermal relaxation time. Combining Eq. (1.3.1) with the energy equation

$$\rho c_E \dot{\theta} = -q_{i,i} + \rho R \tag{1.3.3}$$

yields the corresponding heat conduction equation as

$$K\theta_{,ii} = \left(1 + \tau_q \frac{\partial}{\partial t}\right)\left(\rho c_E \dot{\theta} - \rho R\right). \tag{1.3.4}$$

In 1963, Chester (1963) has given the definite physical interpretation of the heat conduction equation involving, τ_q (Eq. (1.3.4)). He has estimated the value of τ_q and suggested the following expression:

$$\tau_q = \frac{3K}{\rho c v_s^2}, \tag{1.3.5}$$

where, v_s is the speed of ordinary sound. The study by various researchers has speculated the range of τ_q for metals and gases from $10^{-14}s$ to $10^{-10}s$ (see the articles by Nettleton (1960), Chester (1963), Chester (1966), Maurer (1969), Mengi and Turhan (1978) and references therein). In view of the above expression, the value of τ_q being very small has created an urge among the researchers to neglect the second term on the left side in Eq. (1.3.2). Although, the more rigorous studies by several researchers including Baumeister and Hamill (1969; 1971), Chan et al. (1971), Maurer and Thompson (1973), Sadd and Cha (1982) have proved the relevance of the Eq. (1.3.2) in case of very high heat-flux and very short time intervals. It has been reported that the hyperbolic type heat conduction equation (Eq. (1.3.4)) corresponding to modified Fourier's law (Eq. (1.3.2)) presents more physically relevant results in these cases as compared to the parabolic type diffusion equation corresponding to Fourier's law of heat conduction.

Lord and Shulman (1967) have worked upon developing an extension of classical thermoelasticity theory around the modified Fourier's law given by Eq. (1.3.1) and have arrived at a generalized thermoelasticity theory. This theory supports the finite speed of heat propagation. This theory is also termed as extended thermoelasticity theory (ETE) or thermoelasticity with thermal relaxation. The heat conduction equation of this theory in the context of isotropic and homogeneous material can be presented as follows:

$$K\theta_{,jj} = \left(1 + \tau_q \frac{\partial}{\partial t}\right)\left(\rho c_E \dot{\theta} + T_0 \beta \dot{u}_{j,j} - \rho R\right), \tag{1.3.6}$$

whereas, the displacement equation is same as Eq. (1.2.10). Further, Eq. (1.3.6) evaluates the speed of thermal wave propagation as $\sqrt{\frac{K}{\rho c \tau_q}}$, which is evidently dependent on the thermal relaxation parameter. For the relaxation parameter (τ_q) equal to zero, the extended thermoelasticity theory corresponds to classical theory (Biot's theory) predicting an infinite speed of thermal signals. Furthermore, Lord and Shulman analyzed the one-dimensional problem for their theory and compared the results with that of conventional theory.

1.3.1.2 Green-Naghdi Thermoelasticity Theory (GN)

In the 1990s, Green and Naghdi (1991; 1992; 1993) have elaborated and extended the concept of coupled thermoelasticity in a completely different manner to present the unconventional set of thermoelasticity theories. One of their theories in linearized form has resulted in a similar heat conduction law, as has been suggested by the inertial theory of heat conduction. In Inertial theory, Cattaneo and Vernotte (1958; 1958; 1961) heat conduction law, i.e., $q_i + \tau_q \frac{\partial q_i}{\partial t} = -K\theta_{,i}$, is treated in limiting case, where, $\tau_q \to \infty$ and K/τ_q is assumed to be finite. The altered law of heat conduction obtained in this scenario is as follows:

$$\frac{q_i}{\tau_q} + \frac{\partial q_i}{\partial t} = -\frac{K}{\tau_q}\theta_{,i}, \tag{1.3.7}$$

which after applying the limits gives

$$\frac{\partial q_i}{\partial t} = -K_1^* \theta_{,i}, \tag{1.3.8}$$

where, $K_1^* = K/\tau_q$.

On the other hand, Green and Naghdi (1991) have formulated the generalized theory by introducing a new constitutive variable ν, where, $\frac{\partial \nu}{\partial t} = \theta$. In view of its definition, ν is termed as thermal displacement. They have presented three types of thermoelasticity theories, each varying based on variables involved in the balance of energy equation while deriving the theory. These theories have subsequently been referred to as thermoelasticity theories of type GN I, GN II, and GN III. The laws of heat conduction employed in these thermoelasticity theories for the anisotropic medium can be stated as follows:

- **GN I Theory:**

$$q_i = -K_{ij} \theta_{,j}, \tag{1.3.9}$$

  which is classical Fourier's law in which θ and $\theta_{,i}$ are considered as independent variables in derivation.

- **GN II Theory:**

$$q_i = -K_{ij}^* \nu_{,j}, \tag{1.3.10}$$

  where, θ, ν and $\nu_{,i}$ are considered as independent variables.

- **GN III Theory:**

$$q_i = -K_{ij} \theta_{,j} - K_{ij}^* \nu_{,j}, \tag{1.3.11}$$

  where, θ, $\theta_{,i}$, ν and $\nu_{,i}$ are considered as independent variables.

Form of Eq. (1.3.10), for the isotropic case, is the heat transportation law that is similar to that of inertial theory (K_1^* and K^* interpret the same quantity). The authors have also inferred that the relation of model GN II involves no dissipation of energy (Green and Naghdi (1993)). Hence, the corresponding theory is also called as thermoelasticity theory without energy dissipation. The anisotropic version of displacement equation of motion given by Eq. (1.2.10) along with heat conduction equation obtained on combining energy equation with Eqs. (1.3.9-1.3.11) gives the set of governing equations for the thermoelasticity theory of types GN I, GN II and GN III, respectively. In the context of the isotropic case, linearized forms of the equations for GN II and GN III have been presented by Green and Naghdi in 1993 and 1992, respectively. Furthermore, equations for the anisotropic case for both GN II and GN III thermoelasticity theories are articulated by Quintanilla (1999; 2001; 2002a).

1.3.1.3 Dual-Phase-Lag Thermoelasticity Theory (DPL)

With the increasing popularity of generalized thermoelasticity theory, more detailed studies have been carried out to get alternative theories that exhibit the finite speed of thermal signals. In 1992, Tzou (1992) presented the idea of Cattaneo and Vernotte heat conduction law in a mathematical way and produced a new modified Fourier's law as follows:

$$q_i(\boldsymbol{x}, t+\tau_q) = -K_{ij}\theta_{,j}(\boldsymbol{x}, t). \tag{1.3.12}$$

This corresponds to modified Fourier's law given by Cattaneo and Vernotte (1958; 1958; 1961) when the left hand side of Eq. (1.3.12) is expanded to the first order Taylor's series expansion about t. Here, Tzou (1992) has interpreted Eq. (1.3.12) in presence of phase-lag (time lag), τ_q that the heat-flux at some point in the medium will be observed at time $t+\tau_q$ when the temperature gradient at that point is incorporated at time t. Further, Tzou (1995b; 1995c) has extended this idea of time-lag and proposed a new heat conduction relation, which can be stated as follows:

$$q_i(\boldsymbol{x}, t+\tau_q) = -K_{ij}\theta_{,j}(\boldsymbol{x}, t+\tau_\theta). \tag{1.3.13}$$

In this new law, Tzou has incorporated two phase-lags with respect to heat-flux and temperature gradient, i.e., τ_q and τ_θ, respectively and hence, termed it as dual-phase-lag heat conduction law. According to Eq. (1.3.13), when at a point in the medium temperature gradient is applied at time $t + \tau_\theta$, then the heat-flux will be felt at time $t + \tau_q$ at that point if $\tau_q > \tau_\theta$. However, the results are the other way round when $\tau_\theta > \tau_q$. The addition of new phase-lag, τ_θ emphasizes the micro-structural interactions caused due to heat transportation in the medium. It can be alternatively be phrased that this law brings out the microscopic effects in both space and time whereas the classical Fourier's law is macroscopic in view of space and time. By taking two different forms of Taylor series expansion of above equation, Tzou (1995b; 1995c) has introduced two different constitutive relations for heat-flux and temperature gradient vectors as follows:

- When the first-order Taylor's series expansion about time t is considered on both sides of Eq. (1.3.13), it yields dual-phase-lag heat conduction model-I:

$$\left(1 + \tau_q \frac{\partial}{\partial t}\right) q_i = -K_{ij} \left(1 + \tau_\theta \frac{\partial}{\partial t}\right) \theta_{,j}. \tag{1.3.14}$$

- When the second-order Taylor's series expansion about time t is considered on left hand side (in terms of τ_q) and the first-order expansion is taken on right side (in terms for τ_θ), it yields dual-phase-lag heat conduction model-II:

$$\left(1 + \tau_q \frac{\partial}{\partial t} + \tau_q^2 \frac{\partial^2}{\partial t^2}\right) q_i = -K_{ij} \left(1 + \tau_\theta \frac{\partial}{\partial t}\right) \theta_{,j}. \tag{1.3.15}$$

The interpretation of Eq. (1.3.14) and Eq. (1.3.15) in terms of heat conduction problem is conducted by Tzou (1995b; 1995c) in his work. According to him, Eq. (1.3.15) exhibits thermal propagation as wave in nature whereas results of Eq. (1.3.14) depends on the values of τ_q and τ_θ. More detailed elaboration of these two models and several important findings in this respect are available in the book given by Tzou (1997).

The dual-phase-lag heat conduction model is consequently extended by Chandrasekharaiah (1998), who introduced a new generalized thermoelasticity theory, namely dual-phase-lag thermoelasticity theory based on the above two modified Fourier's laws given by Tzou (1995b; 1995c). Chandrasekharaiah (1998) has conducted the analysis of Eq. (1.3.15) with respect to the thermoelasticity theory. The author combined Eq. (1.3.15) with energy law to derive the following governing relation of heat transportation for DPL thermoelasticity theory in the context of anisotropic medium:

$$\left(1+\tau_\theta\frac{\partial}{\partial t}\right)K_{ij}\theta_{,ij}=\left(1+\tau_q\frac{\partial}{\partial t}+\tau_q^2\frac{\partial^2}{\partial t^2}\right)\left(\rho c_E\dot{\theta}+T_0\beta_{ij}\dot{u}_{i,j}-\rho R\right). \tag{1.3.16}$$

The other governing equations however remained the same as the conventional theory. Likewise to Tzou, Chandrasekharaiah (1998) has also stated that the Eq. (1.3.16) and the other field equations form the hyperbolic system of partial differential equations in the context of dual-phase-lag thermoelasticity model. The author elaborates that the DPL thermoelasticity theory based on Eq. (1.3.14) will transmit thermal disturbances with finite speed only when $\tau_q > \tau_\theta \geq 0$ otherwise the paradox of infinite speed sustains in the theory. Further, the velocity of thermal signals for DPL model in terms of material parameters is expressed as follows:

$$v=\frac{1}{\tau_q}\sqrt{\frac{2K\tau_\theta}{\rho c_E}}. \tag{1.3.17}$$

1.3.1.4 Three-Phase-Lag Thermoelasticity Theory (TPL)

In early twenty first century, Roychoudhuri (2007a) has implemented the idea of phase-lag introduced by Tzou to Green-Naghdi thermoelasticity theory. The author has incorporated an additional phase-lag, represented as τ_v, for the gradient of thermal displacement, $\nabla\nu$, along with τ_q and τ_θ, which are phase-lags with respect to heat-flux and temperature gradient, respectively. Following Tzou, Roychoudhuri (2007a) has comprehended the role of τ_ν and stated that the heat-flux will be acquired at time $t+\tau_q$ at a point on the material when temperature gradient and thermal displacement gradient is observed at time $t+\tau_\theta$ and $t+\tau_\nu$, respectively. This type of setup also brings out the effect of phonon-scattering, phono-electron interactions in the macroscopic frame. The modified Fourier's law that has been employed in this theory in the context of an isotropic and homogeneous medium is stated as follows:

$$q_i(\boldsymbol{x}, t+\tau_q) = -K\theta_{,i}(\boldsymbol{x}, t+\tau_\theta) - K^*\nu_{,i}(\boldsymbol{x}, t+\tau_\nu). \tag{1.3.18}$$

Further, combining Eq. (1.3.18) with the energy equation gives the desired heat conduction equation of three-phase-lag theory. Similar to DPL theory, various Taylor's series expansion of three-phase-lag theory is the topic of interesting concern. Roychoudhuri discussed two versions of Eq. (1.3.18) in his article. The first one is obtained by considering first order Taylor's series expansion in Eq. (1.3.18) w.r.t. t and in terms of τ_q, τ_θ and τ_ν. On the other hand, the second one involves the first order Taylor's series expansion w.r.t. t for τ_θ and τ_ν , and the second order Taylor's series expansion in terms of τ_q. The later version can be stated as follows:

$$\left(1+\tau_q\frac{\partial}{\partial t}+\tau_q^2\frac{\partial^2}{\partial t^2}\right)q_i(\boldsymbol{x},t) = -K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\theta_{,i}(\boldsymbol{x},t) - K^*\left(1+\tau_\nu\frac{\partial}{\partial t}\right)\nu_{,i}(\boldsymbol{x},t). \tag{1.3.19}$$

Since, $\dot{\nu}=\theta$ and considering $\tau_\nu^* = K + K^*\tau_\nu$, Eq. (1.3.19) transforms as

$$\left(1+\tau_q\frac{\partial}{\partial t}+\tau_q^2\frac{\partial^2}{\partial t^2}\right)q_i = -\left[\tau_\nu^*\theta_{,i} + K\tau_\theta\frac{\partial\theta_{,i}}{\partial t} + K^*\nu_{,i}\right]. \tag{1.3.20}$$

Further, on taking divergence of Eq. (1.3.20) and using energy equation with relation, $\dot{\nu}=\theta$, the heat transportation equation of three-phase-lag thermoelasticity theory based on the Eq. (1.3.19) is obtained in the following form:

$$\left(1+\tau_q\frac{\partial}{\partial t}+\tau_q^2\frac{\partial^2}{\partial t^2}\right)\left(\rho c\ddot{\theta} + T_0\beta\ddot{u}_{j,j} - \rho\dot{R}\right) = \tau_\nu^*\dot{\theta}_{,ii} + K\tau_\theta\ddot{\theta}_{,ii} + K^*\theta_{,i}. \tag{1.3.21}$$

Neglecting, τ_q^2 in Eq. (1.3.21) gives the thermal field equation of three-phase-lag thermoelasticity theory based on the first version of Eq. (1.3.18) as has been discussed by Roychoudhuri (2007a). Using above mentioned heat transportation equation and different values of material parameter and phase-lags, equation for previously mentioned thermoelasticity theories can be obtained from Eq. (1.3.21) as special cases.

1.3.1.5 Thermoelasticity Theory Based on Exact Heat Conduction Law with a Single Delay

The results of generalized thermoelasticity theories based on modified Fourier's law depend on the nature of that law. The study of these constitutive laws in terms of the heat conduction problem can help in understanding the tentative heat flow in corresponding thermoelasticity theory. For this purpose, Dreher et al. (2009) have analyzed the heat flow laws given by Eq (1.3.13) and Eq. (1.3.18) after combining them with the following energy equation:

$$-\,q_{i,i}(\boldsymbol{x},t) = \rho c_E \dot{\theta}(\boldsymbol{x},t). \tag{1.3.22}$$

The authors have marked the observation that in the point spectrum, the sequence of eigenvalues with its real part tending to infinity always exist. This inspection points out the instability and ill-posedness of the problems on dual-phase-lag and three-phase-lag heat conduction. These types of observations further shift the interest of researchers towards different theories based on Taylor's expansion of heat conduction law with phase-lag(s).

In 2011, Quintanilla (2011) differently treated the three-phase-lag heat flow law given by (1.3.18) in order to obtain the stability of the model. The author has proposed to consider, $\tau_1 = \tau_q - \tau_\theta$ and $\tau_2 = \tau_q - \tau_\nu$ in Eq. (1.3.18), which modified the equation as follows:

$$q_i(\boldsymbol{x},t) = -K\theta_{,i}(\boldsymbol{x},t-\tau_1) - K^*\nu_{,i}(\boldsymbol{x},t-\tau_2). \tag{1.3.23}$$

Here, τ_1 and τ_2 are termed as delay time parameters. The author has further studied the problem considering, $\tau_1 = 0$ and $\tau_2 = \tau > 0$, i.e.

$$q_i(\boldsymbol{x}, t) = -K\theta_{,i}(\boldsymbol{x}, t) - K^*\nu_{,i}(\boldsymbol{x}, t - \tau). \tag{1.3.24}$$

In continuation to this, the author has presented the thermoelasticity theory with a delay term based on the anisotropic version of Eq. (1.3.24) and has stated the stability of a problem for the system of thermoelasticity theory. Furthermore, he has concluded the article by showing the stability of the heat conduction problem by considering Taylor's series expansion of Eq. (1.3.24) when expanded till third order. Subsequently, Leseduarte and Quintanilla (2013) have investigated the spatial behavior of the heat conduction problem solution based on Eq. (1.3.24). The authors have later presented Phragmén Lindelöf type alternative for heat conduction problem and have extended the results to thermoelasticity theory. Further, they have also studied problems based on the forward- and backward-in-time version of (1.3.24). These two versions of the exact heat conduction law can be stated as follows:

- Forward-in-time version, i.e., when $\tau = \tau_\nu - \tau_q > 0$

$$q_i(\boldsymbol{x}, t) = -K\theta_{,i}(\boldsymbol{x}, t) - K^* \left(\nu_{,i}(\boldsymbol{x}, t) + \tau \frac{\partial}{\partial t}\nu_{,i}(\boldsymbol{x}, t) + \frac{\tau^2}{2}\frac{\partial^2}{\partial t^2}\nu_{,i}(\boldsymbol{x}, t) \right). \tag{1.3.25}$$

- Backward-in-time version, i.e., when $\tau = \tau_q - \tau_\nu > 0$

$$q_i(\boldsymbol{x}, t) = -K\theta_{,i}(\boldsymbol{x}, t) - K^* \left(\nu_{,i}(\boldsymbol{x}, t) - \tau \frac{\partial}{\partial t}\nu_{,i}(\boldsymbol{x}, t) + \frac{\tau^2}{2}\frac{\partial^2}{\partial t^2}\nu_{,i}(\boldsymbol{x}, t) \right). \tag{1.3.26}$$

It is worth noting that this thermoelasticity theory is yet to get attention. In the view of stability of the heat conduction problem based on these constitutive relations, this recent generalized thermoelasticity theory under the exact heat conduction equation with a single delay is, therefore, worth studying.

1.3.1.6 Non-Local Thermoelasticity Theory

With the emerging era of nanotechnology, the shift of studies from macro to micro level is must needed. In this respect, the non-local continuum theory has proved to be in the right direction. This type of theory involves the effect of a neighborhood of the considered point of material during the thermomechanical process (Eringen (2002)). The size factor brings out the influence of microscopic elements at a macroscopic level in the ongoing process. Moreover, the role of non-local response in space is analogous to the lagging response in time. Like the time phase-lag helps to understand the process in femtosecond domain, non-local response aids in comprehending the mechanism at the nanoscale (Tzou (1997)).

The concept of non-local response can be applied to the thermoelasticity theory either on the basis of non-local elasticity (Eringen (1974), Balta and Suhubi (1977)) or using non-local heat conduction law. In 2010, Tzou and Guo (2010) have presented a new heat conduction model which involves both phase-lag response and non-local response. In this work, the authors have firstly combined the non-local response with the single phase-lag heat conduction model (Tzou (1992)) which can be stated as:

$$q_i(\boldsymbol{x} + \boldsymbol{\lambda}_q, t + \tau_q) = -K\theta_{,i}(\boldsymbol{x}, t), \tag{1.3.27}$$

where, $\boldsymbol{\lambda}_q$ refers the correlating length vector. Further, this non-local heat conduction model has been compared with the thermomass heat conduction model (Cao and Guo (2007), Guo and Hou (2010)) in order to obtain the relation between the corresponding parameters. The phase-lag of heat-flux and correlating length have been shown equivalent to the thermal lagging and twice the length parameter in the thermomass model, respectively. Heat conduction laws based on the concept of thermomass theory have been developed by Cao and Guo (2007), Guo and Cao (2008), and Guo and Huo (2010). Thermomass is defined according to Einstein's mass-energy relation as the equivalent mass of phonon gas in dielectrics.

In order to remove singularities from model with single phase-lag, Tzou and Guo (2010) has combined non-local response with dual-phase-lag model given by Tzou (1995b; 1995c). The general form of their final modified Fourier's law is proposed as follows:

$$q_i(\boldsymbol{x} + \boldsymbol{\lambda}_q, t + \tau_q) = -K\theta_{,i}(\boldsymbol{x}, t + \tau_\theta). \tag{1.3.28}$$

Here, the authors further have analyzed the impact of various parameters on a one-dimensional heat conduction problem by considering the following Taylor's series expansion of Eq. (1.3.28):

$$q_i + (\boldsymbol{\lambda}_q.\boldsymbol{\nabla})q_i + \tau_q\frac{\partial q_i}{\partial t} = -K\theta_{,i} - K\tau_\theta\frac{\partial \theta_{,i}}{\partial t} \tag{1.3.29}$$

1.3.2 Generalized Thermoelasticity Theory Based on Classical Fourier's Law

1.3.2.1 Green-Lindsay Thermoelasticity Theory (GL)

After LS thermoelasticity theory, it is the Green-Lindsay theory (GL theory) that gained popularity amongst researchers. In 1972, Green and Lindsay (1972) formulated an unconventional thermoelasticity theory based on entropy production inequality given by Green and Laws (1972). This inequality is a generalization of the classical entropy inequality, which involves a scalar function depending on both absolute temperature and temperature-rate. One of the key features in this theory is that it does not alter Fourier's law when the center of symmetry of the body is considered. The derivation of this theory results in modified constitutive relations involving temperature-rate terms and two relaxation times. For this reason, it is sometimes referred to as thermoelasticity theory with two relaxation parameters. Further, since it is derived including temperature-rate, it is also termed as temperature rate-dependent thermoelasticity theory. This new development in thermoelasticity theory has consequently resulted in the finite speed of propagation of both thermal and elastic waves. The constitutive relations for linear generalized thermoelasticity theory given by Green and Lindsay (1972) for the anisotropic homogeneous medium with the centre of symmetry are given as follows:

The equation of motion:

$$\sigma_{ij,j} + \rho H_i = \rho \ddot{u}_i. \tag{1.3.30}$$

The energy equation:

$$\rho T_0 \dot{S} = -q_{i,i} + \rho R. \tag{1.3.31}$$

The constitutive relations:

$$T_0 \rho S = \rho c_E(\theta + \tau_0 \dot{\theta}) + \beta_{ij} T_0 e_{ij}, \tag{1.3.32}$$

$$\sigma_{ij} = C_{ijkl} e_{kl} - \beta_{ij}(\theta + \tau_1 \dot{\theta}), \tag{1.3.33}$$

$$q_i = -K_{ij} \theta_{,j}. \tag{1.3.34}$$

Here, τ_0 and τ_1 are the two thermal relaxation time parameters included in the constitutive relations to incorporate the temperature-rate terms. Using the aforementioned Eqs. (1.3.30-1.3.34), the two field equations in the context of linear GL thermoelasticity theory can be obtained as earlier.

1.3.2.2 Modified Green-Lindsay Thermoelasticity Theory (MGL)

As explained in the previous subsection, Green-Lindsay thermoelasticity theory successfully overcomes the paradox of infinite speed of thermal wave propagation by altering the conventional thermoelasticity with the interesting fact of keeping Fourier's law intact in the case of a centrally symmetric body. Like LS theory, this theory has drawn considerable attention from researchers to investigate several problems concerning with thermoelastic interactions in thermoelastic media. Detailed comparison of results predicted by classical theory, LS theory, and GL theory have also been reported in literature. However, it has been reported in several studies that GL theory suffers discontinuity in the displacement field for transient motion (see Chandrasekharaiah and Srikantiah (1986; 1987), Dhaliwal and Rokne (1989), Chatterjee and Roychoudhuri (1990), Ignaczak and Mr' owka-Matejewska (1990)). Discontinuity in the displacement field suggests that one part of the matter penetrates the other, which disobeys the continuum hypothesis (Chandrasekharaiah (1998)). Considering this fact, recently, Yu et al. (2018) have developed a modified version of Green-Lindsay thermoelasticity theory based on strain-rate along with the temperature-rate terms. The strain-rate term is usually neglected in constitutive relations of linear theory by assuming it to be relatively small. This is not an appropriate assumption for extreme conditions such as in ultra-fast heating. Hence, the authors have proposed the new thermoelasticity model with the help of extended thermodynamics theory and generalized dissipation inequality. While developing this theory, the modification in constitutive relations are resulted as follows:

$$T_0 \rho S = \rho c_E(\theta + \tau_0 \dot{\theta}) + \beta_{ij} T_0 (e_{ij} + \tau_0 \dot{e}_{ij}), \tag{1.3.35}$$

$$\sigma_{ij} = C_{ijkl}(e_{kl} + \tau_1 \dot{e}_{kl}) - \beta_{ij}(\theta + \tau_1 \dot{\theta}). \tag{1.3.36}$$

Eq. (1.3.35) and Eq. (1.3.36) represent the addition of strain-rate term and temperature-rate term in entropy and stress-strain constitutive relations, respectively. Further, with the help of a one-dimensional problem, the authors have demonstrated the impact of this new theory and have concluded that this theory may remove the drawback of occurrence of the discontinuous displacement field. A detailed comparison of the results in the contexts of MGL thermoelasticity theory with the corresponding results under GL theory and GN (II, III) thermoelasticity theory has also been presented in the same article by Yu et al. (2018).

1.3.3 Other Generalized Thermoelasticity Theories

Apart from the aforementioned theories, there are thermoelasticity theories which have been developed on the concepts of fractional calculus, micropolar, porous media, etc. Thermoelasticity theories which are proposed on the basis of the heat conduction law with fractional derivatives are called as fractional order thermoelasticity theories. A book by Povstenko (2015) has summarized the work done in the fractional thermoelasticity from the beginning. With the help of fractional calculus, the theory can be studied in time and space non-local domain. Few of the other works on fractional thermoelasticity can be seen in the following references: Sherief et al. (2010), Youssef (2010; 2016), El-Karamany and Ezzat (2011a; 2011b), Sur and Kanoria (2012), Abbas(2014; 2015), and Povstenko (2019). Further, micropolar thermoelasticity theory is another topic that has been studied widely. Eringen (1970) has covered the fundamental of micropolar thermoelasticity in which the author expressed the constitutive thermomechanical equations for materials inheriting granular and molecular nature. The articles by Boschi and Ieşan (1973), Chandrasekharaiah (1986a), Ciarletta (1999), Sherief et al. (2005), Othman and Singh (2007), Ciarletta et al. (2007) and many more have inspected the micropolar behavior of materials under different thermoelastic models.

1.4 Literature Review

With the advancement of thermoelasticity theory as the integral subject of science, a significant range of research work has been carried out both in mathematical and physical terms. In view of overcoming the drawbacks of conventional thermoelasticity theory and attaining appropriate results in extreme thermal and mechanical conditions, gradual and continuous developments have been observed to derive various generalized thermoelasticity theories. Several review articles and books have reported these advancements upto certain extents. Few among them to be worth mentioning are as follows: Nowacki (1969b; 1975b), Chandrasekharaiah (1986b; 1998), Joseph and Prezios (1989; 1990), Straughan (2011), Parkus (2012), Hetnarski and Ignaczak (1999), and Ignaczak and Ostoja-Starzewski (2010). Moreover, in 2005, Picard (2005) has addressed the structural formulation for linear thermoelasticity in nonsmooth media. Subsequently, structural formulation for linear material laws in classical mathematical physics has been reported by Picard (2009), where class of evolutionary boundary value problems have been considered to cover a number of initial boundary value problems of classical mathematical physcis and the corresponding solution theory hase been derived. For more review and studies on specific problems under various thermoelasticity theories, the Ph.D. theses of Roushan Kumar (2010), Rajesh Prasad (2012), Shweta Kothari (2013), Rakhi Tiwari (2017), Bharti Kumari (2017), Shashi Kant (2018), and Anil Kumar (2018) can also be viewed. This section aims at reporting some literature survey to acquire the state of art in the context of generalized thermoelasticity theories as mentioned in sections.

The classical thermoelasticity theory proposed by Biot (1956) succeeded in bringing the coupling effects of thermal and mechanical fields. This theory, therefore gained serious attention during subsequent decades. Sternberg and McDowell (1957) studied this theory for a semi-infinite elastic medium bounded by a plane in order to find steady-state thermal stresses and displacements. The authors using the method of Green showed that the stress field is plane and parallel to the boundary. Further, Sneddon and Lockett (1960) extended the study of Sternberg and McDowell (1957) to thick plate. Deresiewicz (1957) studied the propagation of plane waves in the context of classical theory proving the independence of shear waves from the thermal effect and the presence of two dilatation waves, namely, elastic wave and thermal wave. Subsequently, Lessen (1957; 1959) studied the propagation of thermoelastic waves and the thermal shock problem in the thermoelastic medium to analyze the thermal and mechanical effects in the case of conventional coupled theory given by Biot (1956). Furthermore, the thermoelastic effects in thick plates and rods were analyzed by Chadwick (1962). A survey on various methods on the basis of categories such as linear and nonlinear, isotropic and anisotropic, stationary and non-stationary, deterministic and random, was made by Parkus (1963) in the context of thermoelastic problems. Ismail and Nowinski (1965) used a perturbation scheme to solve an axially symmetric steady state thermoelastic problem with temperature-dependent material properties. Ignaczak and Nowacki (1966) represented the solution of the equations of conventional coupled thermoelasticity in terms of surface integral. The solution provided the surface potentials, which helps to reduce the basic boundary value problem to the solution of a system of singular integral equations. Books by Chadwick (1960), Boley and Wiener (1960), Carlson (1973), Nowacki (1975a), Parkus (2012), Nowinski (1978), and Dhaliwal and Singh (1980) thoroughly recorded the results and applications of this classical thermoelasticity theory. However, with the progression of the decade, the deviation of research was observed from studying classical thermoelasticity to modifying classical thermoelasticity in order to overcome the paradox of infinite speed of thermal propagation.

The generalized thermoelasticity theory given by Lord and Shulman (1967) is considered as one of the most appropriate modifications to the classical theory till date. The authors studied the one-dimensional problem for an isotropic homogeneous thermoelastic half-space with free-surface subjected to step-strain and evaluated the exact solution for a particular case. Further, they made the comparison with the previous theories to articulate the elimination of the paradox of an infinite propagation speed of thermal waves. Similarly, Achenbach (1968), Norwood and Warren (1969), and Lord and Lopez (1970) investigated the effects of step in strain and temperature; step strain, stress and temperature and step in strain at the free-surface, respectively. Nayfeh and Nemat-Nasser (1972) focused on two-dimensional Lamb's problem, highlighting the importance of relaxation times on wave speed and wave amplitudes. Puri (1973) analyzed phase velocity, specific loss, and amplitude ratio of the plane waves and approximated the expressions for very low and high frequency values. The problem of thermal shock for a circular cylinder with stress-free boundary was discussed by Wadhawan (1973). Moreover, the thermal shock problem in the case of the infinite plate and long bar were respectively studied by Kolyano and Semerak (1973) and Shashkov and Yanovskii (1977) and Szekeres (1980). Dhaliwal and Sherief (1980) further thoroughly proved the theory of LS model for general anisotropic medium. Ignaczak (1979) showed the uniqueness of the problem involving stress- heat-flux initial boundary conditions. Chandrasekharaiah (1986b) illustrated the uniqueness of the solution, domain of influence results, variational principle, and reciprocity theorem in the context of Lord-Shulman thermoelasticity theory. The solution of thermal shock problem in this context was also discussed in the same article. Sherief (1987) proved the uniqueness of the LS thermoelasticity theory for general anisotropic medium and studied the stability when initial data is perturbed. In 1988, Chandrasekharaiah (1988) presented the thermo-piezoelectricity theory related to LS theory along with the uniqueness of solution in the same context. Anwar and Sherief (1988) treated a one-dimensional problem under LS model by using state space approach and presented numerical results for two scenarios, namely, half-space domain and layered domain. On the other hand, Furukawa et al. (1990) found the short-time solution for a one-dimensional problem in an infinite body with a circular cylindrical cavity and illustrated the impact of relaxation time on physical fields. An elaborated study of plane harmonic thermoelastic waves in the homogeneous anisotropic medium was conducted by Sharma and Singh (1989) to show the existence of four waves, namely, a quasi-longitudinal, two quasi-transverse, and a thermal wave. Chand et al. (1990) made the combined study of the thermal, mechanical, and magnetic field for uniformly rotating elastic half-space. Further, Mukhopadhyay et al. (1991) investigated the thermoelastic waves in infinite solid with spherical cavity when the inner boundary is subjected to step rise in temperature and step rise in dynamic pressure on its surface. They articulated the presence of discontinuities at the corresponding wavefront of physical fields. Wang and Dhaliwal (1993) developed the fundamental solution when impulsive body force and heat source are acted at a point in the infinite domain. Moreover, Sherief and Anwar (1994) studied the two-dimensional problem under the LS model using state-space approach.

The second most studied generalized thermoelasticity theory considered by researchers was the theory by Green and Lindsay (1972). In this article, the authors showed the finite speed of thermal waves along with the uniqueness of the linearized theory. Boschi (1972) analyzed the plane waves in the context of this model. Further, in 1973, Boschi and Ieşan (1973) extended this theory to a homogeneous micropolar continuum supported by derivation based on invariance conditions under superposed rigid body motions. Agarwal (1978; 1979) studied the surface waves under LS and GL thermoelasticity and plane waves in GL theory, respectively. Further to support the existence of the finite speed of thermal waves, Ignaczak (1978) proved the domain of influence theorem for linear Green-Lindsay thermoelasticity theory. Chandrasekharaiah and Srikantaiah (1983) studied the uniqueness theorem, variational principle, and Betti-Rayleigh-type reciprocity theorem for anisotropic medium, whereas Gladysz (1985) presented Gurtin-type convolutional variational principle in the same context. Further, Chandrasekharaiah and Srikantiah (1984) did the analysis of decay coefficient, energy loss, and phase velocity for isotropic homogeneous rotating solid. Erbay and Suhubi (1986) studied the longitudinal waves when the lateral surface of an infinite cylinder is stress-free and kept at a constant ambient temperature. Chen and Wang (1988) applied a combined method of finite element and Laplace transform to work on the dynamic problem in the contexts of LS and GL thermoelasticity theories. Noda et al. (1989) analyzed variation in temperature, displacement, and stresses for an infinite solid with the cylindrical hole under both the theories given by Lord and Shulman and Green and Lindsay. Ieşan (1989) introduced a new form of reciprocity theorem in the LS model case, which led to classical reciprocity, uniqueness, and minimum principle. In 1991, Ignaczak (1991) presented a survey on the domain of influence theorems in the context of LS and GL thermoelasticity theories. Further, Chandrasekharaiah and Murthy (1991) considered a linear, homogeneous, and isotropic unbounded thermoelastic body acted upon by continuous line heat source under LS model. They employed Laplace and Hankel transformation to solve the problem and study the thermoelastic interactions. On the other hand, Hetnarski and Ignaczak (1993) discussed the one-dimensional Green's function in the case of plane heat source for infinite and semi-infinite space and presented the closed-form solutions in the physical domain after Laplace inversion. Sherief (1992) discussed the fundamental solution in the presence of a spherically symmetric point heat source in GL theory and numerically compared with the classical and LS thermoelastic model for copper material. Chandrasekharaiah and Murthy (1994) articulated mixed initial and boundary value problem using unified governing equations of Lord-Shulman and Green-Lindsay models. Sinha and Elsibai (1996) showed the effects of two relaxation times on the reflection of two types of incident waves and numerically showed the variation of reflection coefficient and partition of energy with respect to the angle of incidence. Misra et al. (1996) studied the thermoelastic interaction under LS theory using state-space approach for ramp-type heating. Sharma (1997) showed the impact of two boundary conditions, namely, thermal shock and normal load for the LS and GL model using the state-space approach. Singh and Kumar (1998) traced the variation of the reflection coefficient of thermoelastic waves at the free surface of micropolar solid half-space. Singh (2000) presented the analysis of plane wave reflection and refraction at the thermally conducting liquid interface and a micropolar generalized thermoelastic solid. They numerically highlighted the dependence of angle of incidence on amplitude ratios. Ezzat and El-Karamany (2002) extended the theory of generalized thermoelasticity by Lord-Shulman and Green-Lindsay theory to present a generalized theory of thermoviscoelasticity. The authors discussed the theory for anisotropic medium to prove the uniqueness of solution and reciprocity theorem. Othman (2003; 2004) treated the GL thermoelasticity model in two ways. In the first one, he showed the effect of temperature dependent elastic modulus using the state-space approach, whereas later, he presented the exact expression of physical fields for two different problems using normal mode analysis for rotating elastic medium under linearized theory. El-Maghraby (2005) studied the two-dimensional thick plate with traction free upper surface subjected to known temperature distribution and the thermally insulated rigid lower surface under Lord-Shulman theory and Green-Lindsay theory. Further, Youssef (2006b; 2006c) numerically discussed a two-dimensional half space problem and also a problem of infinite medium with a cylindrical cavity both subjected to ramp-type heating. Bagri and Eslami (2007a) expanded the analysis of GL thermoelasticity theory to a functionally graded sphere with inner surface symmetrically loaded with thermal shock, whereas Abbas and Abd-alla (2008) investigated thermoelastic variations for an infinite orthotropic elastic medium with a cylindrical cavity using the finite element method.

With the introduction to the three theories by Green and Naghdi (1991; 1992; 1993), several research work were focused on studying these theories in comparison with previously established theories. Chandrasekharaiah (1996d) presented the set of linearized governing equations for GN II thermoelasticity theory in terms of stress and entropy-flux to discuss the uniqueness of the solution to the initial value problem. Chandrasekharaiah (1996a) also differently discussed the uniqueness of the mixed initial-boundary value problem using the energy equation depending on temperature and velocity fields. Further, Chandrasekharaiah (1996b) analyzed the one-dimensional thermoelastic disturbances under the GN II theory when temperature and strain or stress is suddenly applied at the boundary. The closed form solution for the physical fields was derived here. Moreover, Chandrasekharaiah (1996c) articulated the behavior of plane waves in the context of the GN II thermoelastic model. Chandrasekharaiah and Srinath (1997b) investigated the thermoelastic plane waves in an unbounded body rotating with uniform angular velocity. Later, Dhaliwal et al. (1997) analyzed the thermoelastic interactions caused by continuous line heat source in the case of the GN III thermoelastic theory and pointed out that this theory demonstrates the diffusion type thermal wave propagation. Subsequently, Chandrasekharaiah and Srinath (1997a), considered the GN II thermoelastic model and studied thermoelastic problem in an isotropic and homogeneous infinite body with cylindrical cavity when acted upon by step radial stress or temperature at the boundary. In 1998, Ieşan (1998) established the fundamental solution based on Galerkin-type solution for an isotropic and homogeneous body in the context of GN II thermoelasticity theory and also showed the continuous dependence of solution on initial data and body loads. Chandrasekharaiah and Srinath (1998a; 1998b) analyzed homogeneous and isotropic unbounded bodies when acted upon by a point heat source and continuous line heat source under the GN II model. Ciarletta (1999) extended the GN II thermoelasticity theory to micro thermoelasticity and then found the Galerkin-type solution for an isotropic and homogeneous medium. Later, Svanadze et al. (2006) derived the fundamental solution based on the theory given by Ciarletta (1999). Quintanilla (1999) discussed the spatial behavior of linear thermoelasticity theory without energy dissipation for isotropic and homogeneous medium, where the author discussed the theorems on spatial energy and decay estimates. Chandrasekharaiah and Srinath (2000) presented the thermoelastic variation in the isotropic and homogeneous medium with a spherical cavity using Laplace Transform technique under GN II theory. Misra et al. (2000) studied the thermoelastic interactions under GN II thermoelastic model when ramp-type heating is applied for two cases of boundary, namely traction free and rigid. Wang and Slattery (2002) established the linear GN II thermoelasticity theory for the prestressed body, i.e., the body that has received large deformation and is at nonuniform temperature. Quintanilla (2002a), with the help of the linear theory of operators, discussed the well-posedness of a problem of anisotropic medium under GN II theory. Sharma et al. (2003) thoroughly studied the reflection of thermoelastic waves and reported the comparison of numerical results of GN theory with previously established theories in terms of the ratio of reflection coefficients and partition of energy. The authors discussed the reflection for two types of surfaces, namely, stress-free surface and rigid surface. Quintanilla and Straughan (2004) studied the complete non-linear version of GN II and GN III thermoelasticity theories to analyze thermal and mechanical waves. Roychoudhuri and Bandyopadhyay (2005) focused on analysis of time-harmonic plane thermoelastic waves in the rotating medium in the context of the GN II model. They proceeded with the perturbation method to find a solution to the dispersion equation and emphasized the effect of rotation on phase velocity. On the other hand, Kumar and Sarthi (2006) studied the reflection and refraction of thermoelastic waves at five different combinations of the interface under thermoelasticity without energy dissipation and calculated the amplitude ratio for an imperfect interface. Bagri and Eslami (2007b) considered the unified set of partial differential equations of Lord-Shulman, Green-Lindsay, and Green-Naghdi (II) thermoelasticity theories to analyze thermoelastic variations in a thick functionally graded cylinder. The authors made a detailed comparison among different theories and pointed out that in comparison to GL and GN theories, LS theory predicted larger values for temperature waves. Mallik and Kanoria (2008) viewed GN II and GN III thermoelastic models to study a two-dimensional problem for transversely isotropic medium and made the comparison of physical fields between GN II and GN III and later contrasted the numerical results for isotropic and transversely isotropic material. Abbas and Othman (2009) and Abbas (2011) undertook a problem of tracing thermoelastic waves using finite element method under the Green and Naghdi theory of type II and III for rotating homogeneous isotropic hollow cylinder and a fiber-reinforced anisotropic half-space undergoing thermal shock, respectively. Chiriţă and Ciarletta (2010) presented variational theorem and reciprocity theorem in the context of the GN II model for an inhomogeneous anisotropic medium with a center of symmetry at each point. Mukhopadhyay and Kumar (2010b) employed the state-space approach to detect the thermoelastic variation under the boundary value problem for GN III thermoelastic model. The authors undertook two kinds of boundary conditions, namely zero stress with a sudden change in temperature and zero temperature change with sudden variation in load. Sarkar and Lahiri (2012) investigated the GN II thermoelasticity model in terms of three-dimensional problem using normal mode analysis applied to homogeneous isotropic thermoelastic medium with stress-free boundary acted upon by time-dependent thermal conditions.

In the late 20th century, Chandrasekharaiah (1998) developed a thermoelasticity theory based on dual-phase-lag heat conduction model of Tzou (1995b; 1995c). Hetnarski and Ignaczak (1999) reviewed the DPL thermoelastic model along with previously established generalized thermoelasticity theories to study the wave-like propagation of the thermal signal. Quintanilla (2003) worked upon the stability conditions in terms of phase-lags , τ_q and $\tau_\theta$for one-dimensional thermoelasticity problem. The author thoroughly proved that the problem will be exponentially stable if $\tau_\theta > \frac{\tau_q}{2}$ and there exist an unstable solution when $\tau_\theta < \frac{\tau_q}{2}$. Later, Quintanilla (2004) considered Lord-Shulman and dual-phase-lag thermoelasticity theories to evaluate structural stability when the relaxation parameters tend to zero. Further, Quintanilla and Racke (2006) analyzed the dual-phase-lag thermoelasticity model by considering Eq. (1.3.15). They firstly aimed to present conditions for well-posedness and stability for the problem with more complicated boundary conditions, and then they discussed the spatial behavior of the thermoelastic solution for a semi-infinite cylinder. Roychoudhuri (2007b) investigated a one-dimensional problem in which isotropic homogeneous half-space is subjected to two types of boundary conditions, i.e., zero stress with thermal shock and zero temperature with constant step input of stress. The author highlighted the presence of two waves and discontinuities at the wavefront in the short-time approximated analytical expressions of physical fields. Prasad et al. (2010) examined the harmonic plane waves in the context of dual-phase-lag thermoelasticity theory. The authors presented the asymptotic expressions of wave characteristics, and numerically expressed the results for very high and low frequency values. Mukhopadhyay et al. (2011a) proved the propagation of thermal waves with finite speed via the domain of influence. They showed the dependence of the domain of influence on two phase-lag parameters and thermoelastic coupling constant. Abouelregal (2011) explored thermoelastic variations in the isotropic solid sphere with constrained boundary subjected to constant heat-flux. They featured the effects of phase-lags, τ_q and τ_θ, and the comparison of results predicted by GL and DPL theories through graphical representation. Singh (2013) analyzed thermoelastic plane wave propagation in a transversely isotropic medium under DPL thermoelastic model. Zenkour et al. (2013) considered the solid half-space with temperature-dependent material properties to study the reflection of thermoelastic waves under dual-phase-lag thermoelasticity theory. Further, Kothari and Mukhopadhyay (2013a) and Mukhopadhyay et al. (2014) studied uniqueness, reciprocity theorem, and variational principle for the linear theory of thermoelasticity in anisotropic medium and showed the behavior of thermoelastic waves when the medium is subjected to thermal shock. On the other hand, El-Karamany and Ezzat (2014) discussed the same theorems with a different approach for inhomogeneous anisotropic solid and also proved a continuous dependence of the solution on initial data and supply terms by using dissipative inequality. Abouelregal and Abo-Dahab (2015) picked up a two-dimensional problem for the rigidly fixed surface when subjected to thermal shock and compared the results with that of LS and classical thermoelasticity theories. In the context of DPL thermoelasticity, Sarkar (2017) used normal mode analysis and an eigenvalue approach to investigate a three-dimensional half space problem with temperature-dependent material properties and a stress-free boundary subjected to a time-dependent heat source.

Three-phase-lag thermoelasticity theory (TPL) given by Roychoudhuri (2007a) was investigated by Kumar and Mukhopadhyay (2009) for a problem of an infinite medium with a cylindrical cavity. The authors undertook the problem of stress-free boundary acted upon by step input in temperature and obtained the solution using the Laplace transformation technique. Using the solution, the authors analytically and numerically compared the two models, namely GN-III and three-phase-lag models. Further, Kar and Kanoria (2009) discussed the TPL thermoelasticity theory for an orthotropic functionally graded hollow sphere to compare the results with Green-Naghdi theories and showed the differences in nature of physical fields due to the presence of functionally graded material. Mukhopadhyay and Kumar (2010a) did a combined study for three thermoelastic models namely DPL, TPL, and GN-III models for a thick plate subjected to the axisymmetric temperature distribution. The authors employed Laplace and Hankel transform techniques to find the solution of the problem and analyzed the discontinuities in field variables with the help of Boley's theorem. Mukhopadhyay et al. (2010) demonstrated the representation of the Galerkin-type solution of the coupled system of equations for TPL theory. They further obtained the Galerkin type solution of quations for steady oscillations, using which a general solution for steady oscillations was also derived. Further, Kothari et al. (2010) considered TPL model and found the fundamental solutions in presence of concentrated body force and heat sources with the aid of Galerkin-type representation of solution of the problem. Moreover, the behavior of harmonic plane thermoelastic waves in the presence of three phase-lags in an isotropic homogeneous medium was studied by Kumar and Mukhopadhyay (2010). Furthermore, Kumar and Chawla (2011) traced the occurrences of elastic and thermal waves in an anisotropic medium under dual-phase-lag and three-phase-lag thermoelasticity theories by numerically computing the wave characteristics. Prasad et al. (2011) explored an infinite isotropic homogeneous solid for thermoelastic vibrations experienced due to a continuous line heat source at the boundary of the medium. The authors highlighted the effects of heat source through analytical expressions of different field variables and pointed out the similarities and dissimilarities of results with respect to the corresponding results predicted by previous theories for copper material. El-Karamany and Ezzat (2013) extended the three-phase-lag thermoelasticity theory to micropolar thermoelasticity theory for an anisotropic and inhomogeneous medium. The authors proved the uniqueness and variational principle for the considered problem and also discussed the results of continuous dependence on initial data. Later, Kothari and Mukhopadhyay (2013b) considered a functionally graded hollow disk to pursue a combined study of GN II, DPL, and TPL thermoelasticity theories using Laplace transform and finite element method. Moreover, Akbarzdeh et al. (2014) reported a unified study of different theories for a problem of functionally graded infinite hollow cylinder with material properties varying along the radial direction as per power-law distribution. Recently, the domain of influence results for three-phase-lag thermoelasticity theory has been investigated by Kumar and Kumar (2015).

Continuous and gradual mathematical analyses of different thermoelasticity theories aid in judging the application of the theories in real-time scenarios. Recently, Mukhopadhyay et al. (2016; 2017) studied various models of thermoelasticity theory and showed that these models can be treated within the common structural framework of evolutionary equations. By considering the flexibility of the structural perspective, they obtained well-posedness results for a large class of generalized models allowing for more general material properties such as anisotropies, inhomogeneities, etc. The recently proposed generalized thermoelasticity theories namely, Quintanilla's theory (2011) and modified Green-Lindsay theory (2018) are yet to receive attention of researchers. Few recent studies using these theories can be mentioned as follows. Kant and Mukhopadhyay (2016) discussed the behavior of the physical fields in the thermoelastic setup of a thick plate under Quintanilla's heat conduction model using potential function approach along with Laplace and Hankel transform. On the other hand, Kumar and Mukhopadhyay (2016) used the state space approach to tackle a boundary value problem in this context due to sudden temperature change and zero stress at the boundary surface of an elastic half space. Later on, Kumar and Mukhopadhyay (2017) analyzed the effects of temperature-dependent material properties on thermoelastic interactions in a spherical shell with three different boundary conditions. Further, Kumari and Mukhopadhyay (2017b) presented the fundamental solutions for the cases of a concentrated heat source and a concentrated body force in the isotropic and homogeneous unbounded medium under forward in time version (1.3.25) of Quintanilla's model. In order to analyze the continuous dependence of the solution on initial data and study the exponential decay of solution, Quintanilla (2018) made a qualitative analysis on MGL thermoelasticity theory. Singh and Mukhopadhyay (2020) considered isotropic homogeneous unbounded medium with a cylindrical cavity to explore MGL theory along with LS and GL thermoelasticity theories. The authors presented here the analytical results using short-time approximation and highlighted some important findings.

1.5 Objective of the Thesis

The main objective of the present thesis is to investigate some recently developed generalized thermoelasticity theories using mathematical tools and studying some unsolved problems involving thermomechanical interactions. In order to understand various aspects of the thermoelasticity theories from the view point of their real-time applicability, the mathematical examination of the theories can be worth pursuing. The thesis is broadly divided into three parts on the basis of the generalized thermoelasticity theories. The first part deals with the thermoelasticity theory developed on the basis of Quintanilla's heat conduction law (2011). The second part explores the novel thermoelastic model, namely the modified Green-Lindsay (MGL) model presented by Yu et al. (2018). Lastly, the thesis deals with the dual-phase-lag (DPL) thermoelasticity theory to explore its behavior in random and natural conditions. Moreover, the last part extends the DPL theory on the basis of non-local heat conduction model (Tzou and Guo (2010)). As illustrated in previous sections, out of these three theories, the second one is based on Fourier's heat conduction law while the others involve non-Fourier law.

Focusing on various aspects of these theories, it is aimed at understanding the behavior of physical fields involved in various problems of coupled thermoelasticity when the conventional theory is modified using either altered heat conduction law or other altered constitutive relations. Galerkin-type representation of the models is articulated for the models, which is further applicable in finding the solution of the boundary value problems. Furthermore, the propagation of plane waves is investigated in details to highlight the characteristics of elastic-mode and thermal-mode waves generated in the medium. Some specific problems involving coupled thermoelastic interactions have been investigated in detailed way and thorough comparisons among the predictions by the present theories and by previously established theories have been highlighted. The present analysis of different thermomechanical problems illustrates all the essential aspects of the recently introduced generalized thermoelasticity theories, which serve the purpose of the thesis.