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It is certified that the work contained in the thesis titled "**Mathematical Aspects of Some Non-Conventional Thermoelastic Models**" by **Manushi Gupta** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy and SOTA for the award of Ph.D. Degree.

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I, Manushi Gupta, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of **Prof. Santwana Mukhopadhyay** from July, 2016 to April, 2021 at the **Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi**. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, etc., reported in journals, books, magazines, reports dissertations, theses, etc., or available at websites and have not included them in this thesis and have not cited as my own work.

Date: 16.04.2021
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It is certified that the above statement made by the student is correct to the best of my/our knowledge.

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Dedicated

to

My Mother

Mrs. Surbhi Gupta

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Manushi
16/04/21

MANUSHI GUPTA

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LIST OF SYMBOLS

$\boldsymbol{\Gamma} = (\sigma_{ij})$	Stress tensor, Pa or N m^{-2} or kg $m^{-1}s^{-2}$
$\boldsymbol{u} = (u_i)$	Displacement vector, m
$\boldsymbol{E} = (e_{ij})$	Strain tensor, dimensionless
$\boldsymbol{q} = (q_i)$	Heat-flux, W m^{-2}
T_0	Uniform reference temperature, K
θ	Temperature variation from the uniform reference temperature, K
ν	Thermal displacement, K s
$\boldsymbol{H} = H_i$	Body force per unit mass, N kg^{-1}
R	Heat source per unit mass, J s^{-1} kg^{-1} or N m s^{-1} kg^{-1}
S	Entropy per unit mass, J K^{-1} kg^{-1}
ρ	Mass density, kg m^{-3}
K_{ij}/ K	Thermal conductivity tensor/ Thermal conductivity constant, W m^{-1} K^{-1}
K^*_{ij}/ K^*	Thermal conductivity rate tensor/ Thermal conductivity rate constant, W m^{-1} K^{-1} s^{-1}
C_{ijkl}	Elasticity tensor, kg $m^{-1}s^{-2}$

λ, μ	Lame's constants of material, kg m^{-1}s^{-2}
β_{ij}	Thermoelasticity tensor, kg m^{-1} s^{-2} K^{-1}
$\beta = (3\lambda + 2\mu)\beta_\theta$	Thermoelasticity constant, kg m^{-1} s^{-2} K^{-1}
β_θ	Linear thermal expansion of the material, K^{-1}
c_E	Specific heat at constant strain, J kg^{-1} K^{-1}
τ_θ	Phase-lag of temperature gradient, s
τ_q	Phase-lag of heat-flux vector, s
τ_ν	Phase-lag of thermal displacement, s
$\boldsymbol{x} = (x_1, x_2, x_3)/(x, y, z)$	Cartesian coordinates, m
t	Physical time, s
$\boldsymbol{\nabla}$	Gradient operator
∇^2	Laplacian operator
δ_{ij}	Kronecker delta, dimensionless

Note: Throughout the thesis, the subscripted comma notations are used to denote the partial derivatives with respect to the space variables. The over-headed dots denote partial derivatives with respect to time variable, t. The bold notation is used for vector or tensor quantities. Subscripts i, j, k, l take the values 1, 2, 3 and summation is implied by index repetition.

ABBREVIATIONS

BPN	**B**oussinesq-**P**apkovitch-**N**euber
BSG	**B**oussinesq-**S**omigliana-**G**alerkin
CKS	**C**auchy-**K**ovlevski-**S**omigliana
DOI	**D**omain **O**f **I**nfluence
DPL	**D**ual-**P**hase-**L**ag Thermoelasticity Theory
ETE	**E**xtended **T**hermoelasticity **T**heory
GLa	**G**reen-**La**me
GL	**G**reen-**L**indsay Thermoelasticity Theory
GN	**G**reen-**N**aghdi Thermoelasticity Theory
LS	**L**ord-**S**hulman Thermoelasticity Theory
MGL	**M**odified **G**reen-**L**indsay Thermoelasticity Theory
TPL	**T**hree-**P**hase-**L**ag Thermoelasticity Theory

PREFACE

Thermoelasticity is a branch of science that deals with the interaction of the mechanical field with the thermal field and vice versa. In comparison to the theory of elasticity, which has been developed in the early seventeenth century, the thermoelasticity theory is relatively younger as it was introduced by Duhamel in 1837. With the development in technology, especially in the field of aeronautics, nuclear science, steam turbines, etc., where the rising temperature caused due to high speed has significant effects on the mechanical nature of the material used, demands the thorough knowledge of thermal stresses. Moreover, the advancement in thermal processes involving ultrafast laser heating, where the study of transient behavior occurred at an extremely short time is necessary, requires the understanding of the modified theory of generalized thermoelasticity. The equation of motion, the compatibility equations, and the constitutive laws are the fundamental laws on which the thermoelasticity theory has been developed. With the progress in time, profound changes have been gradually implemented on the conventional notion of theory. The first comprehensive documentation on thermoelasticity theory has been given by Biot (1956), which comprises of Fourier's law as one of the constitutive laws. This theory is generally referred to as the classical theory of thermoelasticity. Due to the presence of Fourier's law, the classical thermoelasticity theory contains a parabolic-hyperbolic combination of partial differential equations. This combination results in infinite speed of thermal wave propagation which is a physically unrealistic behavior. Although, Biot's theory has been widely and successfully applied to study the conventional thermoelastic problems that involve large spatial dimension and when the focus is on long time behavior, but it yields unacceptable results in situations involving temperature near absolute zero, extreme thermal gradients, high heat flux conduction and short time behavior, such as laser-material interactions. Furthermore, due to the advancement of modern technology of material processing by pulsed sources, this conventional theory has been shown to be inadequate in modeling laser processing of materials and high frequency response of materials. Intense efforts are therefore put forth since last few decades to better understand the limitations of Fourier's law and conventional thermoelasticity theory. Various generalized thermoelasticity theories, which are apparently the alternative and better version of the classical theory, have continuously been proposed by researchers over time.

The remodeling of the classical thermoelasticity theory is broadly carried out on two bases. Firstly, by modifying Fourier's law and the second one is by altering the other constitutive laws, generally keeping the Fourier's law intact. The generalized theories given by Lord and Shulman (LS) (1967) and Green and Lindsay (GL) (1972) are among the most investigated approaches. In the first theory, Fourier's law is replaced by the heat conduction law consisting of one time-lag, τ_q concerning heat flux. However, the second model by Green and Lindsay (1972) alters the conventional theory by incorporating the temperature-rate term in constitutive laws. Later, in the early 1990s, Green and Naghdi (GN) (1991, 1992, 1993) have extended the idea of classical theory by incorporating the new constitutive variable, ν, called as thermal displacement. The authors have divided the model into a set of three and have discussed the various aspects of the models. These theories developed by Green and Naghdi are subsequently being called as theories of GN I, GN II, and GN III. Further, dual-phase-lag thermoelasticity theory (DPL) has been developed by Chandrasekharaiah (1998), which is based on heat conduction law with two phase-lags, τ_q and τ_θ, proposed by Tzou (1992).

Eventually, Roychoudhuri (2007) applied the idea of Tzou (1992) to the GN III theory to develop the three-phase-lag thermoelastic model (TPL) having additional time-lag, τ_ν with respect to the gradient of thermal displacement. Furthermore, to remove the ill-posedness found in the phase-lag theories, Quintanilla (2011) has treated the TPL heat conduction model differently and has presented a heat conduction model with a single delay term. Based on this model, the author has further proposed thermoelasticity theory and has tested the system's stability in that respect. Recently, to remove the discontinuity in the displacement field under the GL model, Yu et al. (2018) have proposed a modified Green-Lindsay (MGL) thermoelastic model by fusing strain-rate term along with temperature-rate term. Researchers have developed many other generalized thermoelasticity theories also by combining the concept of fractional calculus, non-local phenomenon, micro-structures etc., which can be found out in the literature.

The subject of the present thesis is to investigate various aspects of some recently developed thermoelastic models mathematically. The thesis mainly talks about three generalized thermoelasticity theories, namely, Quintanilla's theory, modified Green-Lindsay theory (MGL), and dual-phase-lag thermoelasticity theory (DPL). Hence, it is divided into three parts on this basis. The first part discusses Quintanilla's model, which includes Chapter 2 and Chapter 3, whereas the second part comprises of Chapter 4 and Chapter 5, which elaborates MGL thermoelasticity theory. Further, Chapter 6, the last part of the thesis, analyses the dual-phase-lag thermoelastic model in different scenarios. Therefore, this work arrangement aims to understand the nature of physical fields when the conventional theory is altered using either modified heat conduction law or other reformed constitutive relations.

The outline of the thesis is as follows:

Chapter 1 introduces the topic of the thesis. It discusses the brief history of the development in the thermoelasticity theory followed by a detailed literature review. Finally, it ends with the objective of the thesis.

Chapter 2 comprises of two subchapters that deal with the Quintanilla's thermoelastic model (2011). **Subchapter 2.1** discusses the considered theory in terms of mixed type boundary and initial value problem for a homogeneous and anisotropic medium. Firstly, the uniqueness theorem is proved using a specific internal energy function. Then, an alternative formulation of the problem using convolution is established to prove the convolution type variational theorem. A reciprocal relation is also established. In continuation, **Subchapter 2.2** elaborates on the Galerkin-type representation of the solution to the system of equations of motion for an isotropic elastic homogeneous medium in presence of body force and heat source. Analogous working is followed next to establish the Galerkin-type representation of the solution in case of steady oscillation. Further, a general solution is established in terms of metaharmonic functions for steady oscillation case in the absence of body force and external heat source.

Chapter 3 aims to describe the reflection of thermoelastic waves under Quintanilla's theory for medium with variable material parameters. The mathematical model is analyzed along with other three thermoelastic theories, i.e., classical theory, LS theory, and GN III theory, to highlight the nature of thermoelastic interactions under Quintanilla's model. The propagation of elastic longitudinal and transverse wave and the thermal wave, is analyzed in case of incident longitudinal and incident transverse wave. The amplitude ratio and phase velocity for various waves are obtained and are presented graphically for various angles of incidence for the theories. The significant effect of variable material parameters on wave characteristics and other important observations are highlighted.

Chapter 4 starts the second part of the thesis, which addresses the modified Green-Lindsay thermoelastic model in the context of the Galerkin-type representation of the solution. Similar to Subchapter 2.1, Galerkin-type representation of the solution to the equation of motion followed by representation for the case of steady oscillation is given.

Following these theorems, a theorem is established which expresses the general solution of the system of homogeneous equations of steady oscillation in terms of metaharmonic functions.

Chapter 5 completes the discussion on MGL thermoelastic model by analyzing the propagation of plane harmonic waves in an unbounded isotropic homogeneous elastic medium. In order to investigate the effects of additional strain-rate term, the unified governing equations related to three thermoelastic models, namely, the classical thermoelastic model, the GL model, and the MGL model, have been considered. The behavior of different wave components for longitudinal waves is examined through graphical representation to investigate the effects of temperature-rate and strain-rate terms on the plane wave propagation. Significant differences in predictions by the new model as compared to the classical and GL models are observed.

Chapter 6 is devoted to the last part of the thesis, which elaborates behavior of physical fields under different scenarios in the context of dual-phase-lag (DPL) thermoelasticity theory. It comprises of three subchapters to discuss three different problems in the context of DPL model. **Subchapter 6.1** discusses the thermoelastic interactions in an isotropic homogeneous half-space caused due to stochastic conditions applied at the boundary. One-dimensional problem with traction free boundary subjected to two types of time-dependent thermal distributions is considered. The stochastic case solution has been obtained by using the concept of the Wiener process and stochastic simulation. Numerical analysis based on stochastic simulation is carried out for copper material along different sample paths. Comparative analysis between the deterministic and the stochastic distributions of field variables is presented. Special attention is paid to highlight the effects of considering randomness added to the boundary conditions. On the other hand, **Subchapter 6.2** deals with the natural stress-heat-flux problem in the context of dual-phase-lag thermoelasticity theory. The main aim of this subchapter is to establish the domain of influence theorem for the considered problem. In this theorem, a bounded domain, D_t is obtained such that outside this domain, no thermoelastic disturbance can be observed due to the thermomechanical loading caused by heat-flux and stress field applied at the boundary of the medium. The finite speed of propagation is shown to be dependent on material parameters and phase-lags. Lastly, **Subchapter 6.3** is concerned with the generalized thermoelasticity theory based on the recently introduced non-local heat conduction model with dual-phase-lag effects by Tzou and Guo (2010). The generalized governing equations for this thermoelasticity theory are formulated, and a one-dimensional thermal shock half-space problem is considered. Special attention is paid to investigate the effects of the non-local length parameter, λ_q, which is the characteristic of this model. The results are further compared with the corresponding results predicted by other existing models. The impacts of phase-lag parameters, τ_q and τ_θ in presence of non-local parameter are also investigated in a detailed manner.

Chapter 7 incorporates the summary of the present work as well as scopes for future work in the relevant areas.