Chapter 3

Continuous Bessel Wavelet Transform of Distributions

3.1 Introduction

The Hankel transform played an important role to solve many problems of physics, engineering and mathematical sciences. Zemanian [60–63], Koh [29, 30], Pathak et al. [40, 41, 43, 44], Lee [31], Dube and Pandey [17], Betancor [2–4, 6] explored the theory of Hankel transform on various function spaces and characterized many results. Betancor and Marrero [7–10, 33], Betancor and González [5], Betancor and Rodríguez - Mesa [11, 12] studied properties of functional spaces by taking the theory of Hankel transform and Hankel convolution. Aforesaid tools are also very effective in the problems of wavelet theory.

In 2003, Pathak and Dixit [42] introduced the continuous Bessel wavelet transform and studied its properties by exploiting the theory of Haimo [21], Hirschman [24] and Cholewinski [15]. Later on, considering Zemanian theory of Hankel transform, Upadhyay et al. [51–56] investigated continuous Bessel wavelet transform and their properties.

Motivated from the results of [46, 52, 56], our main concern in the present chapter is to study continuous Bessel wavelet transform in $H'_\mu(\mathbb{R}^+)$ by exploiting the Hankel transform and Hankel convolution. Later on, we shall study the boundedness properties of continuous Bessel wavelet transform in L^p-Sobolev type space.

3.2 The continuous Bessel wavelet transform of distributions

In this section, the properties of continuous Bessel wavelet transform in distributional sense are discussed by using the Hankel transform technique.
The test function space $\widetilde{H}_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$ is defined to be the space of all functions $\phi \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ such that for $l, k, \alpha, \beta \in \mathbb{N}_0$ and $\mu \geq -\frac{1}{2}$,

$$\gamma^{\mu}_{k,l,\alpha,\beta}(\phi) = \sup_{\substack{(b,a)\in\mathbb{R}^+\times\mathbb{R}^+ \\ k+\alpha\leq 2\left(l+\beta+\mu+\frac{1}{2}\right)}} \left| a^k b^\alpha \left(a^{-1}D_a\right)^l \left(b^{-1}D_b\right)^\beta (ab)^{-\mu-\frac{1}{2}} \phi(b,a) \right| < \infty. \quad (3.2.1)$$

Lemma 3.2.1. *If $\psi \in H_\mu(\mathbb{R}^+)$ and $k, l, \delta \in \mathbb{N}_0$, $\mu \geq -\frac{1}{2}$, then the relation*

$$\begin{aligned} &a^k \left(a^{-1}D_a\right)^l \left(\omega^{-1}D_\omega\right)^\delta (a\omega)^{-\mu-\frac{1}{2}} \left(h_\mu\psi\right)(a\omega) \\ &\quad = (-1)^{l+k+2\delta} \omega^{2l-2\delta-k} \\ &\quad\quad \times \int_0^\infty \left\{ (a\omega y)^{-\mu-l} J_{\mu+l+k+2\delta}(a\omega y)\, y^{2\mu+2l+2\delta+k+1} \left(y^{-1}D_y\right)^{k+\delta} y^{-\mu-\frac{1}{2}} \psi(y) \right\} dy \end{aligned}$$

holds for $a, \omega \in \mathbb{R}^+$.

Proof. By putting $t = a\omega$ and using change of variable, we get the differentiation

$$
\begin{aligned}
a^k &\left(a^{-1}D_a\right)^l \left(\omega^{-1}D_\omega\right)^\delta (a\omega)^{-\mu-\frac{1}{2}} \left(h_\mu\psi\right)(a\omega) \\
&= a^k a^{2\delta}\omega^{2l} \left(t^{-1}D_t\right)^{l+\delta} t^{-\mu-\frac{1}{2}} \left(h_\mu\psi\right)(t) \\
&= a^{k+2\delta}\omega^{2l} \int_0^\infty \left(t^{-1}D_t\right)^{l+\delta} t^{-\mu} J_\mu(yt) y^{\frac{1}{2}}\psi(y)dy \\
&= a^{k+2\delta}\omega^{2l} \int_0^\infty t^{-\mu-l-\delta} J_{\mu+l+\delta}(yt)(-1)^{l+\delta} y^{l+\delta+\frac{1}{2}}\psi(y)dy \\
&= \omega^{2l-2\delta-k}(-1)^{l+\delta} \int_0^\infty t^{-\mu-l+k+\delta} y^{\mu+l+\delta} J_{\mu+l+\delta}(yt) y^{-\mu+\frac{1}{2}}\psi(y)dy.
\end{aligned}
$$

Using (1.7.7) and integrating by parts, the above expression yields

$$
\begin{aligned}
&a^k \left(a^{-1}D_a\right)^l \left(\omega^{-1}D_\omega\right)^\delta (a\omega)^{-\mu-\frac{1}{2}} \left(h_\mu\psi\right)(a\omega) \\
&= \omega^{2l-2\delta-k}(-1)^{l+\delta} \int_0^\infty t^{-\mu-l} \left\{\left(y^{-1}D_y\right)^{k+\delta} y^{\mu+l+k+2\delta} J_{\mu+l+k+2\delta}(yt)\right\} y^{-\mu+\frac{1}{2}}\psi(y)dy \\
&= \omega^{2l-2\delta-k}(-1)^{l+k+2\delta} \int_0^\infty t^{-\mu-l} y^{\mu+l+k+2\delta+1} J_{\mu+l+k+2\delta}(yt) \left(y^{-1}D_y\right)^{k+\delta} y^{-\mu-\frac{1}{2}}\psi(y)dy \\
&= \omega^{2l-2\delta-k}(-1)^{l+k+2\delta} \\
&\quad\times \int_0^\infty (yt)^{-\mu-l} J_{\mu+l+k+2\delta}(yt)\; y^{2\mu+2l+2\delta+k+1} \left(y^{-1}D_y\right)^{k+\delta} y^{-\mu-\frac{1}{2}}\psi(y)dy.
\end{aligned}
$$

□

Theorem 3.2.2. *Let $\psi \in H_\mu(\mathbb{R}^+)$, then the continuous Bessel wavelet transform $\left(B_\psi f\right)(b,a)$ is a continuous linear map from $H_\mu(\mathbb{R}^+)$ into $\widetilde{H}_\mu(\mathbb{R}^+\times\mathbb{R}^+)$ for $\mu \geq -\frac{1}{2}$.*

Proof. For $f \in H_\mu(\mathbb{R}^+)$ and $k, l, \alpha, \beta \in \mathbb{N}_0$ and using (1.5.4), we get

$$
\begin{aligned}
&a^k b^\alpha \left(a^{-1}D_a\right)^l \left(b^{-1}D_b\right)^\beta (ab)^{-\mu-\frac{1}{2}} \left(B_\psi f\right)(b,a) \\
&= a^k b^\alpha \left(a^{-1}D_a\right)^l \left(b^{-1}D_b\right)^\beta (ab)^{-\mu-\frac{1}{2}} \int_0^\infty (b\omega)^{\frac{1}{2}} J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega) \left(h_\mu\psi\right)(a\omega)d\omega \\
&= a^k \left(a^{-1}D_a\right)^l a^{-\mu-\frac{1}{2}} \int_0^\infty b^\alpha \left(b^{-1}D_b\right)^\beta b^{-\mu} J_\mu(b\omega)\omega^{-\mu} \left(h_\mu f\right)(\omega) \left(h_\mu\psi\right)(a\omega)d\omega
\end{aligned}
$$

$$= a^k \left(a^{-1}D_a\right)^l a^{-\mu-\frac{1}{2}} \int_0^\infty b^\alpha b^{-\mu-\beta} J_{\mu+\beta}(b\omega)(-1)^\beta \omega^{\beta-\mu} \left(h_\mu f\right)(\omega) \left(h_\mu \psi\right)(a\omega) d\omega$$
$$= a^k \left(a^{-1}D_a\right)^l a^{-\mu-\frac{1}{2}} \int_0^\infty (-1)^\beta b^{-\mu-\beta+\alpha} \omega^{\mu+\beta} J_{\mu+\beta}(b\omega)\omega^{-2\mu} \left(h_\mu f\right)(\omega) \left(h_\mu \psi\right)(a\omega) d\omega.$$

Using (1.7.7) and integrating by parts, we have

$$a^k b^\alpha \left(a^{-1}D_a\right)^l \left(b^{-1}D_b\right)^\beta (ab)^{-\mu-\frac{1}{2}} \left(B_\psi f\right)(b,a)$$
$$= a^k \left(a^{-1}D_a\right)^l a^{-\mu-\frac{1}{2}}$$
$$\times \int_0^\infty (-1)^\beta b^{-\mu-\beta} \left\{ \left(\omega^{-1}D_\omega\right)^\alpha \omega^{\mu+\beta+\alpha} J_{\mu+\beta+\alpha}(b\omega) \right\} \omega^{-2\mu} \left(h_\mu f\right)(\omega) \left(h_\mu \psi\right)(a\omega) d\omega$$
$$= a^k \left(a^{-1}D_a\right)^l a^{-\mu-\frac{1}{2}}$$
$$\times \int_0^\infty (-1)^{\alpha+\beta} b^{-\mu-\beta} \omega^{\mu+\beta+\alpha+1} J_{\mu+\beta+\alpha}(b\omega) \left(\omega^{-1}D_\omega\right)^\alpha \omega^{-2\mu-1} \left(h_\mu f\right)(\omega) \left(h_\mu \psi\right)(a\omega) d\omega$$
$$= a^k \left(a^{-1}D_a\right)^l a^{-\mu-\frac{1}{2}} \int_0^\infty (-1)^{\alpha+\beta} \omega^{2\mu+2\beta+\alpha+1} (b\omega)^{-\mu-\beta} J_{\mu+\beta+\alpha}(b\omega)$$
$$\times \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \left\{ \left(\omega^{-1}D_\omega\right)^{\alpha-\delta} \omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega) \left(\omega^{-1}D_\omega\right)^\delta \omega^{-\mu-\frac{1}{2}} \left(h_\mu \psi\right)(a\omega) \right\} d\omega$$
$$= \int_0^\infty (-1)^{\alpha+\beta} \omega^{2\mu+2\beta+\alpha+1} (b\omega)^{-\mu-\beta} J_{\mu+\beta+\alpha}(b\omega) \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \left\{ \left(\omega^{-1}D_\omega\right)^{\alpha-\delta} \omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega) \right.$$
$$\left. \times\, a^k \left(a^{-1}D_a\right)^l \left(\omega^{-1}D_\omega\right)^\delta (a\omega)^{-\mu-\frac{1}{2}} \left(h_\mu \psi\right)(a\omega) \right\} d\omega.$$

Taking Lemma 3.2.1, we get

$$a^k b^\alpha \left(a^{-1}D_a\right)^l \left(b^{-1}D_b\right)^\beta (ab)^{-\mu-\frac{1}{2}} \left(B_\psi f\right)(b,a)$$
$$= \int_0^\infty \left[(-1)^{\alpha+\beta} \omega^{2\mu+2\beta+\alpha+1} (b\omega)^{-\mu-\beta} J_{\mu+\beta+\alpha}(b\omega) \right.$$
$$\times \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \left\{ \left(\omega^{-1}D_\omega\right)^{\alpha-\delta} \omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega) \right\} \omega^{2l-2\delta-k} (-1)^{l+k+2\delta}$$
$$\left. \times \int_0^\infty \left\{ (yt)^{-\mu-l} J_{\mu+l+k+2\delta}(yt) \right\}_{t=a\omega} y^{2\mu+2l+2\delta+k+1} \left(y^{-1}D_y\right)^{k+\delta} y^{-\mu-\frac{1}{2}} \psi(y) dy \right] d\omega$$

$$= \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} (-1)^{\alpha+\beta+l+k+2\delta}$$
$$\times \int_0^\infty \Big[(b\omega)^{-\mu-\beta} J_{\mu+\beta+\alpha}(b\omega) \omega^{2\mu+2\beta+2l+\alpha-k-2\delta+1} \Big\{ \left(\omega^{-1}D_\omega\right)^{\alpha-\delta} \omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega) \Big\}$$
$$\times \int_0^\infty (a\omega y)^{-\mu-l} J_{\mu+l+k+2\delta}(a\omega y) y^{2\mu+2l+2\delta+k+1} \left(y^{-1}D_y\right)^{k+\delta} y^{-\mu-\frac{1}{2}} \psi(y) dy \Big] d\omega.$$

Therefore

$$\left| a^k b^\alpha \left(a^{-1}D_a\right)^l \left(b^{-1}D_b\right)^\beta (ab)^{-\mu-\frac{1}{2}} \left(B_\psi f\right)(b,a) \right|$$
$$\leq \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \int_0^\infty \Big[\left| (b\omega)^{-\mu-\beta} J_{\mu+\beta+\alpha}(b\omega) \right| \omega^{2\mu+2\beta+2l+\alpha-k-2\delta+1}$$
$$\times \left| \left(\omega^{-1}D_\omega\right)^{\alpha-\delta} \omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega) \right| \int_0^\infty \left| (a\omega y)^{-\mu-l} J_{\mu+l+k+2\delta}(a\omega y) \right|$$
$$\times \left| y^{2\mu+2l+2\delta+k+1} \left(y^{-1}D_y\right)^{k+\delta} y^{-\mu-\frac{1}{2}} \psi(y) \right| dy \Big] d\omega$$
$$\leq \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} A_{\mu,\alpha,\beta} \int_0^\infty \Big[\left| \omega^{2\mu+2\beta+2l+\alpha-k-2\delta+1} \left(\omega^{-1}D_\omega\right)^{\alpha-\delta} \omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega) \right| B_{\mu,l,k,\delta}$$
$$\times \int_0^\infty \left| y^{2\mu+2l+2\delta+k+1} \left(y^{-1}D_y\right)^{k+\delta} y^{-\mu-\frac{1}{2}} \psi(y) \right| dy \Big] d\omega,$$

where $A_{\mu,\alpha,\beta}$ and $B_{\mu,l,k,\delta}$ are positive constants such that

$$\left| (b\omega)^{-\mu-\beta} J_{\mu+\beta+\alpha}(b\omega) \right| \leq A_{\mu,\alpha,\beta} \quad \text{and} \quad \left| (a\omega y)^{-\mu-l} J_{\mu+l+k+2\delta}(a\omega y) \right| \leq B_{\mu,l,k,\delta}.$$

Now, for

$$k+\alpha \leq 2\Big(l+\beta+\mu+\frac{1}{2}\Big), \ \rho_1 = E\Big(\mu+\beta+l-\delta+\frac{\alpha-k+1}{2}\Big) \text{ and}$$
$$\rho_2 = E\Big(\mu+l+\delta+\frac{k+1}{2}\Big),$$

we have

$$
\begin{aligned}
&\left|a^k b^\alpha \left(a^{-1}D_a\right)^l \left(b^{-1}D_b\right)^\beta (ab)^{-\mu-\frac{1}{2}} \left(B_\psi f\right)(b,a)\right| \\
&\quad \leq \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \Big[A_{\mu,\alpha,\beta} B_{\mu,l,k,\delta} \int_0^\infty \left| \left(1+\omega^2\right)^{\rho_1+1} \left(\omega^{-1}D_\omega\right)^{\alpha-\delta} \omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega) \right| d\omega \\
&\qquad\qquad \times \int_0^\infty \left| \left(1+y^2\right)^{\rho_2+1} \left(y^{-1}D_y\right)^{k+\delta} y^{-\mu-\frac{1}{2}} \psi(y) \right| dy \Big] \\
&\quad \leq \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \Big[A_{\mu,\alpha,\beta} B_{\mu,l,k,\delta} \sum_{m=0}^{\rho_1+2} \binom{\rho_1+2}{m} \gamma^\mu_{2m,\alpha-\delta}(h_\mu f) \int_0^\infty \frac{1}{1+\omega^2} d\omega \\
&\qquad\qquad \times \sum_{r=0}^{\rho_2+2} \binom{\rho_2+2}{r} \gamma^\mu_{2r,k+\delta}(\psi) \int_0^\infty \frac{1}{1+y^2} dy \Big] \\
&\quad \leq \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \sum_{m=0}^{\rho_1+2} \binom{\rho_1+2}{m} \sum_{r=0}^{\rho_2+2} \binom{\rho_2+2}{r} \left(\frac{\pi}{2}\right)^2 A_{\mu,\alpha,\beta} B_{\mu,l,k,\delta} \gamma^\mu_{2m,\alpha-\delta}(h_\mu f) \gamma^\mu_{2r,k+\delta}(\psi).
\end{aligned}
$$

By the definition of (3.2.1), we get

$$
\begin{aligned}
\gamma^\mu_{k,l,\alpha,\beta}\left(B_\psi f\right) \leq \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \sum_{m=0}^{\rho_1+2} \binom{\rho_1+2}{m} \sum_{r=0}^{\rho_2+2} \binom{\rho_2+2}{r} \left(\frac{\pi}{2}\right)^2 A_{\mu,\alpha,\delta} B_{\mu,l,k,\delta} \\
\times \gamma^\mu_{2m,\alpha-\delta}(h_\mu f) \gamma^\mu_{2r,k+\delta}(\psi). \quad (3.2.2)
\end{aligned}
$$

This shows that $B_\psi f(b,a) \in \widetilde{H}_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$. The continuity of $B_\psi f$ follows from (3.2.2). □

Definition 3.2.3. Let $\widetilde{H}'_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$ be the dual of $\widetilde{H}_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$, then the generalized Bessel wavelet transform $B'_\psi T$ of $T \in \widetilde{H}'_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$ is defined by

$$
\left\langle B'_\psi T, \phi \right\rangle = \left\langle T, B_\psi \phi \right\rangle, \qquad \phi \in H_\mu(\mathbb{R}^+). \quad (3.2.3)
$$

Theorem 3.2.4. *Let $\psi \in H_\mu(\mathbb{R}^+)$, then the generalized Bessel wavelet transform B'_ψ is a continuous linear map from $\widetilde{H}'_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$ into $H'_\mu(\mathbb{R}^+)$ for $\mu \geq -\frac{1}{2}$.*

Proof. From (3.2.3), we have

$$\left\langle B'_\psi T, \phi \right\rangle = \langle T, B_\psi \phi \rangle \qquad \forall\, \phi \in H_\mu(\mathbb{R}^+), \tag{3.2.4}$$

where $T \in \widetilde{H}'_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$ and $\phi \in H_\mu(\mathbb{R}^+)$. From (3.2.4), $B'_\psi T$ is that functional on $H_\mu(\mathbb{R}^+)$ which assigns to each $\phi \in H_\mu(\mathbb{R}^+)$ the same number that $T \in \widetilde{H}'_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$ assigns to $B_\psi \phi \in \widetilde{H}_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$.

We see that $B'_\psi T \in H'_\mu(\mathbb{R}^+)$. For any $\phi, \theta \in H_\mu(\mathbb{R}^+)$ and $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned}\left\langle B'_\psi T, \alpha\phi + \beta\theta \right\rangle = \langle T, B_\psi (\alpha\phi + \beta\theta) \rangle &= \langle T, \alpha B_\psi \phi + \beta B_\psi \theta \rangle \\ &= \alpha \left\langle B'_\psi T, \phi \right\rangle + \beta \left\langle B'_\psi T, \theta \right\rangle .\end{aligned}$$

This implies that $B'_\psi T$ is a linear functional on $H_\mu(\mathbb{R}^+)$. Further, let $\{\phi_n\}_{n=1}^\infty$ converges to zero in $H_\mu(\mathbb{R}^+)$. Then by Theorem 3.2.2, as $n \to \infty$, $B_\psi \phi_n \to 0$ in $\widetilde{H}_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$. Therefore

$$\left\langle B'_\psi T, \phi_n \right\rangle = \langle T, B_\psi \phi_n \rangle \to 0, \text{ as } n \to \infty.$$

Hence, the above expression shows that $B'_\psi T$ is a linear continuous functional on $H_\mu(\mathbb{R}^+)$.

Thus B'_ψ is a mapping from $\widetilde{H}'_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$ into $H'_\mu(\mathbb{R}^+)$. We now prove that B'_ψ is linear and continuous. Let $T_1, T_2 \in \widetilde{H}'_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$, $\phi \in H_\mu(\mathbb{R}^+)$, and $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned}\left\langle B'_\psi (\alpha T_1 + \beta T_2), \phi \right\rangle &= \langle \alpha T_1 + \beta T_2, B_\psi \phi \rangle \\ &= \alpha \langle T_1, B_\psi \phi \rangle + \beta \langle T_2, B_\psi \phi \rangle \\ &= \alpha \left\langle B'_\psi T_1, \phi \right\rangle + \beta \left\langle B'_\psi T_2, \phi \right\rangle \\ &= \left\langle \alpha B'_\psi T_1, \phi \right\rangle + \left\langle \beta B'_\psi T_2, \phi \right\rangle .\end{aligned}$$

The last expression implies that B'_ψ is linear. For continuity of B'_ψ, let $\{T_n\}_{n=1}^{\infty} \in \widetilde{H}'_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$ such that $T_n \to 0$ in $\widetilde{H}'_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$. Then for every $\phi \in H_\mu(\mathbb{R}^+)$

$$\left\langle B'_\psi T_n, \phi \right\rangle = \langle T_n, B_\psi \phi \rangle \to 0, \text{as } n \to \infty,$$

so that

$$B'_\psi T_n \to 0 \text{ in } H'_\mu(\mathbb{R}^+).$$

Hence, B'_ψ is continuous. □

Definition 3.2.5. Let $\psi \in H_\mu(\mathbb{R}^+)$ be Bessel wavelet. Assume that $h_\mu \psi \in C^\infty(\mathbb{R}^+)$ such that

$$\|h_\mu \psi\|_{\mu,\alpha}^{m,\rho} = \sup_{\omega \in I} (1+\omega)^{-m+\rho\alpha} \left| \left(\omega^{-1} D_\omega\right)^\alpha \omega^{-\mu-\frac{1}{2}} \left(h_\mu \psi\right)(\omega) \right| < \infty, \tag{3.2.5}$$

where $m \in \mathbb{R}, 0 \leq \rho \leq 1$ and $\alpha \in \mathbb{N}_0$.

Consider the test function space $H^1_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$, defined to be the space of all functions $\phi \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ such that for $l, k, \alpha, \beta \in \mathbb{N}_0$ and $\mu \geq -\frac{1}{2}$,

$$\gamma^\mu_{k,l,\alpha,\beta}(\phi) = \sup_{\substack{(b,a) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ k+\alpha \leq 2\left(l+\beta+\mu+\frac{1}{2}\right) \\ k+2\alpha \leq m}} \left| a^k b^\alpha \left(a^{-1} D_a\right)^l \left(b^{-1} D_b\right)^\beta (ab)^{-\mu-\frac{1}{2}} \phi(b,a) \right| < \infty. \tag{3.2.6}$$

Theorem 3.2.6. *Let $\psi \in H_\mu(\mathbb{R}^+)$ and (3.2.5) is satisfied, then the continuous Bessel wavelet transform $(B_\psi f)(b,a)$ is a continuous linear map from $H_\mu(\mathbb{R}^+)$ into $H^1_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$ for $\mu \geq -\frac{1}{2}$.*

Proof. Proceeding as in the proof of Theorem 3.2.2, for $f \in H_\mu(\mathbb{R}^+)$ and $k, l, \alpha, \beta \in \mathbb{N}_0$, we have

$$
\begin{aligned}
&a^k b^\alpha \left(a^{-1}D_a\right)^l \left(b^{-1}D_b\right)^\beta (ab)^{-\mu-\frac{1}{2}} \left(B_\psi f\right)(b,a) \\
&\quad = \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} (-1)^{\alpha+\beta} \\
&\qquad \times \int_0^\infty \Big[(b\omega)^{-\mu-\beta} J_{\mu+\beta+\alpha}(b\omega) \left\{ \omega^{2\mu+2\beta+\alpha+1} \left(\omega^{-1}D_\omega\right)^{\alpha-\delta} \omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega) \right\} \\
&\qquad\qquad \times a^k \left(a^{-1}D_a\right)^l \left(\omega^{-1}D_\omega\right)^\delta (a\omega)^{-\mu-\frac{1}{2}} \left(h_\mu \psi\right)(a\omega) \Big] d\omega \\
&\quad = \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} (-1)^{\alpha+\beta} \\
&\qquad \times \int_0^\infty \Big[(b\omega)^{-\mu-\beta} J_{\mu+\beta+\alpha}(b\omega) \left\{ \omega^{2\mu+2l+2\beta+\alpha+1-k-2\delta} \left(\omega^{-1}D_\omega\right)^{\alpha-\delta} \omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega) \right\} \\
&\qquad\qquad \times \left\{ t^{k+2\delta} \left(t^{-1}D_t\right)^{l+\delta} t^{-\mu-\frac{1}{2}} \left(h_\mu \psi\right)(t) \right\}_{t=a\omega} \Big] d\omega.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left| a^k b^\alpha \left(a^{-1}D_a\right)^l \left(b^{-1}D_b\right)^\beta (ab)^{-\mu-\frac{1}{2}} \left(B_\psi f\right)(b,a) \right| \\
&\quad \le \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \int_0^\infty \Big[\left| (b\omega)^{-\mu-\beta} J_{\mu+\beta+\alpha}(b\omega) \right| \omega^{2\mu+2l+2\beta+\alpha+1-k-2\delta} \\
&\qquad \times \left| \left(\omega^{-1}D_\omega\right)^{\alpha-\delta} \omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega) \right| \left| t^{k+2\delta} \left(t^{-1}D_t\right)^{l+\delta} t^{-\mu-\frac{1}{2}} \left(h_\mu \psi\right)(t) \right|_{t=a\omega} \Big] d\omega.
\end{aligned}
$$

Using (3.2.5), we get

$$
\begin{aligned}
&\left| a^k b^\alpha \left(a^{-1}D_a\right)^l \left(b^{-1}D_b\right)^\beta (ab)^{-\mu-\frac{1}{2}} \left(B_\psi f\right)(b,a) \right| \\
&\quad \le \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \|h_\mu \psi\|_{\mu,l+\delta}^{m,\rho} D_{\mu,\alpha,\beta} \\
&\qquad \times \int_0^\infty \left| \omega^{2\mu+2l+2\beta+\alpha+1-k-2\delta} \left(\omega^{-1}D_\omega\right)^{\alpha-\delta} \omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega) \right| (1+a\omega)^{k+2\delta+m-\rho(l+\delta)} d\omega,
\end{aligned}
$$

where $D_{\mu,\alpha,\beta}$ is positive constant such that $\left|(b\omega)^{-\mu-\beta} J_{\mu+\beta+\alpha}(b\omega)\right| \le D_{\mu,\alpha,\beta}$.

For $2\mu + 2l + 2\beta + \alpha + 1 - k - 2\delta \ge 0$ and $k + 2\delta + m - \rho(l+\delta) \le 0$, the last expression can be written as

$$\begin{aligned}
&\left|a^k b^\alpha \left(a^{-1}D_a\right)^l \left(b^{-1}D_b\right)^\beta (ab)^{-\mu-\frac{1}{2}} \left(B_\psi f\right)(b,a)\right| \\
&\quad \leq D_{\mu,\alpha,\beta} \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \|h_\mu \psi\|^{m,\rho}_{\mu,l+\delta} \\
&\qquad\qquad \times \int_0^\infty \left|\left(1+\omega^2\right)^{\mu+l+\beta+\frac{\alpha+1-k}{2}-\delta} \left(\omega^{-1}D_\omega\right)^{\alpha-\delta} \omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega)\right| d\omega \\
&\quad \leq D_{\mu,\alpha,\beta} \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \|h_\mu \psi\|^{m,\rho}_{\mu,l+\delta} \\
&\qquad\qquad \times \int_0^\infty \left|\left(1+\omega^2\right)^{\rho_1+2} \left(\omega^{-1}D_\omega\right)^{\alpha-\delta} \omega^{-\mu-\frac{1}{2}} \left(h_\mu f\right)(\omega)\right| \frac{1}{1+\omega^2} d\omega \\
&\quad \leq D_{\mu,\alpha,\beta} \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \|h_\mu \psi\|^{m,\rho}_{\mu,l+\delta} \sum_{r=0}^{\rho_1+2} \binom{\rho_1+2}{r} \gamma^\mu_{2r,\alpha-\delta}(h_\mu f) \int_0^\infty \frac{1}{1+\omega^2} d\omega \\
&\quad \leq D_{\mu,\alpha,\beta} \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \|h_\mu \psi\|^{m,\rho}_{\mu,l+\delta} \sum_{r=0}^{\rho_1+2} \binom{\rho_1+2}{r} \frac{\pi}{2} \gamma^\mu_{2r,\alpha-\delta}\left(h_\mu f\right),
\end{aligned}$$

where $\rho_1 = E\left(\mu + l + \beta + \frac{\alpha+1-k}{2} - \delta\right)$.

Thus, we have

$$\gamma^\mu_{k,l,\alpha,\beta}\left(B_\psi f\right) \leq \frac{\pi}{2} D_{\mu,\alpha,\beta} \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} \sum_{r=0}^{\rho_1+2} \binom{\rho_1+2}{r} \|h_\mu \psi\|^{m,\rho}_{\mu,l+\delta} \gamma^\mu_{2r,\alpha-\delta}\left(h_\mu f\right).$$

Hence, for $m \leq -k - 2\alpha$ and $k + \alpha \leq 2\left(l + \beta + \mu + \frac{1}{2}\right)$, $B_\psi f(b,a) \in H^1_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$ and continuity of $B_\psi f$ follows from the above inequality. □

Definition 3.2.7. Let $(H^1_\mu)'(\mathbb{R}^+ \times \mathbb{R}^+)$ be the dual of $H^1_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$, then for $T \in (H^1_\mu)'(\mathbb{R}^+ \times \mathbb{R}^+)$ the generalized Bessel wavelet transform B'_ψ is defined by

$$\left\langle B'_\psi T, \phi \right\rangle = \left\langle T, B_\psi \phi \right\rangle, \qquad \phi \in H_\mu(\mathbb{R}^+). \tag{3.2.7}$$

Theorem 3.2.8. *Let $\psi \in H_\mu(\mathbb{R}^+)$, then the generalized Bessel wavelet transform B'_ψ is a continuous linear map from $(H^1_\mu)'(\mathbb{R}^+ \times \mathbb{R}^+)$ into $H'_\mu(\mathbb{R}^+)$ for $\mu \geq -\frac{1}{2}$.*

Proof. From (3.2.7), we have

$$\left\langle B'_\psi T, \phi \right\rangle = \left\langle T, B_\psi \phi \right\rangle, \qquad \phi \in H_\mu(\mathbb{R}^+), \tag{3.2.8}$$

where $T \in (H^1_\mu)'(\mathbb{R}^+ \times \mathbb{R}^+)$, and $\phi \in H_\mu(\mathbb{R}^+)$. From (3.2.8), $B'_\psi T$ is that functional on $H_\mu(\mathbb{R}^+)$ which assigns to each $\phi \in H_\mu(\mathbb{R}^+)$ the same number that $T \in (H^1_\mu)'(\mathbb{R}^+ \times \mathbb{R}^+)$ assigns to $B_\psi \phi \in H^1_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$.

We see that $B'_\psi T \in H'_\mu(\mathbb{R}^+)$. For any $\phi, \theta \in H_\mu(\mathbb{R}^+)$, and $\alpha, \beta \in \mathbb{C}$

$$\left\langle B'_\psi T, \alpha\phi + \beta\theta \right\rangle = \left\langle T, B_\psi(\alpha\phi + \beta\theta) \right\rangle = \left\langle T, \alpha B_\psi \phi + \beta B_\psi \theta \right\rangle = \alpha \left\langle B'_\psi T, \phi \right\rangle + \beta \left\langle B'_\psi T, \theta \right\rangle.$$

This implies that $B'_\psi T$ is a linear functional on $H_\mu(\mathbb{R}^+)$.

Let $\{\phi_n\}_{n=1}^\infty$ converges to zero in $H_\mu(\mathbb{R}^+)$. Then by Theorem 3.2.6, as $n \to \infty$, $B_\psi \phi_n \to 0$ in $H^1_\mu(\mathbb{R}^+ \times \mathbb{R}^+)$. Therefore,

$$\left\langle B'_\psi T, \phi_n \right\rangle = \left\langle T, B_\psi \phi_n \right\rangle \to 0.$$

Hence, $B'_\psi T$ is a linear continuous functional on $H_\mu(\mathbb{R}^+)$.

Thus B'_ψ is a mapping from $(H^1_\mu)'(\mathbb{R}^+ \times \mathbb{R}^+)$ into $H'_\mu(\mathbb{R}^+)$. We now prove that B'_ψ is linear and continuous. Now, we take $\phi \in H_\mu(\mathbb{R}^+)$, $T_1, T_2 \in (H^1_\mu)'(\mathbb{R}^+ \times \mathbb{R}^+)$, and $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned}\left\langle B'_\psi(\alpha T_1 + \beta T_2), \phi \right\rangle = \left\langle \alpha T_1 + \beta T_2, B_\psi \phi \right\rangle &= \alpha \left\langle T_1, B_\psi \phi \right\rangle + \beta \left\langle T_2, B_\psi \phi \right\rangle \\ &= \alpha \left\langle B'_\psi T_1, \phi \right\rangle + \beta \left\langle B'_\psi T_2, \phi \right\rangle \\ &= \left\langle \alpha B'_\psi T_1, \phi \right\rangle + \left\langle \beta B'_\psi T_2, \phi \right\rangle.\end{aligned}$$

This shows that B'_ψ is linear.

For continuity of B'_ψ, let $\{T_n\}_{n=1}^\infty \in (H^1_\mu)'(\mathbb{R}^+ \times \mathbb{R}^+)$ such that $T_n \to 0$ in $(H^1_\mu)'(\mathbb{R}^+ \times$

$\mathbb{R}^+$). Then for every $\phi \in H_\mu(\mathbb{R}^+)$

$$\left\langle B'_\psi T_n, \phi \right\rangle = \langle T_n, B_\psi \phi \rangle \to 0, \text{ as } n \to \infty,$$

so that

$$B'_\psi T_n \to 0 \text{ in } H'_\mu(\mathbb{R}^+).$$

Hence, B'_ψ is continuous. □

3.3 The Bessel wavelet transform on L^p- Sobolev space

In this section, we study continuous Bessel wavelet transform on the L^p- Sobolev type space.

Definition 3.3.1. For $-\infty < s < \infty$, $\mu \geq -\frac{1}{2}$ and $1 \leq p < \infty$, then L^p-Sobolev space $H^{s,p}_\mu$ is defined to be set of all $f \in H'_\mu(\mathbb{R}^+)$ such that

$$\|f\|_{H^{s,p}_\mu} = \left\| \omega^{\frac{s}{p} - \mu - \frac{1}{2}} h_\mu f \right\|_{L^p}, \qquad \omega \in \mathbb{R}^+. \tag{3.3.1}$$

For $1 \leq p, p' < \infty$ and $-\infty < s < \infty$, the space $W^{s,p'}_p$ of all measurable functions g on $\mathbb{R}^+ \times \mathbb{R}^+$ such that

$$\|g(b,a)\|_{W^{s,p'}_p} = \left(\int_0^\infty \left(\int_0^\infty |g(b,a)|^p \, db \right)^{\frac{p'}{p}} a^{-s-1} da \right)^{\frac{1}{p'}} < \infty. \tag{3.3.2}$$

Lemma 3.3.2. *Let $\psi \in H_\mu(\mathbb{R}^+)$ be the Bessel wavelet and $f \in H'_\mu(\mathbb{R}^+)$ then for $1 \leq p < \infty$, we have*

$$\int_0^\infty \int_0^\infty \left|\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(a\omega)(h_\mu f)(\omega)\right|^p d\omega\, a^{-s-1}da = C^{s,p}_{\mu,\psi} \int_0^\infty \left|\omega^{\frac{s}{p}-\mu-\frac{1}{2}}(h_\mu f)(\omega)\right|^p d\omega, \tag{3.3.3}$$

where

$$C^{s,p}_{\mu,\psi} = \int_0^\infty |(h_\mu\psi)(a\omega)|^p (a\omega)^{-s}\frac{da}{a}, \qquad a > 0. \tag{3.3.4}$$

Proof. By Fubini's theorem, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \left|\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(a\omega)(h_\mu f)(\omega)\right|^p d\omega\, a^{-s-1}da \\
&= \int_0^\infty \int_0^\infty \left|\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(a\omega)(h_\mu f)(\omega)\right|^p a^{-s-1}dad\omega \\
&= \int_0^\infty \int_0^\infty \left|\omega^{\frac{s}{p}}\omega^{-\frac{s}{p}}\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(a\omega)(h_\mu f)(\omega)\right|^p a^{-s-1}dad\omega \\
&= \int_0^\infty \left(\int_0^\infty |(h_\mu\psi)(a\omega)|^p (a\omega)^{-s}a^{-1}da\right) \left|\omega^{\frac{s}{p}-\mu-\frac{1}{2}}(h_\mu f)(\omega)\right|^p d\omega \\
&= C^{s,p}_{\mu,\psi} \int_0^\infty \left|\omega^{\frac{s}{p}-\mu-\frac{1}{2}}(h_\mu f)(\omega)\right|^p d\omega,
\end{aligned}$$

where

$$C^{s,p}_{\mu,\psi} = \int_0^\infty |(h_\mu\psi)(a\omega)|^p (a\omega)^{-s}\frac{da}{a} > 0.$$

□

Theorem 3.3.3. *Let the Bessel wavelet ψ satisfies the admissibility condition (3.3.4). Then the continuous Bessel wavelet transform is a bounded linear operator from $H^{s,p}_\mu(\mathbb{R}^+)$ into $W^{s,p'}_p(\mathbb{R}^+ \times \mathbb{R}^+)$ for $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$ and for all $s \in \mathbb{R}$. Also, for all $f \in H^{s,p}_\mu(\mathbb{R}^+)$ we have*

$$\|f\|_{H^{s,p'}_\mu} \cong \|B_\psi f(b,a)\|_{W^{s,p'}_p},$$

for all $s \in \mathbb{R}$ *and* $1 \le p \le 2, \frac{1}{p} + \frac{1}{p'} = 1.$

Proof. From (1.5.4), we have

$$h_\mu\left[B_\psi f(b,a)\right](\omega) = \omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(a\omega)(h_\mu f)(\omega). \tag{3.3.5}$$

For $1 \le p \le 2, \frac{1}{p} + \frac{1}{p'} = 1$, using [16] and (3.3.5) we get

$$\begin{aligned}\left(\int_0^\infty |B_\psi f(b,a)|^p \, db\right)^{\frac{1}{p}} &= \left(\int_0^\infty \left|h_\mu^{-1}[\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(a\omega)(h_\mu f)(\omega)](b)\right|^p db\right)^{\frac{1}{p}} \\ &\le D_{p'}\left(\int_0^\infty \left|\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(a\omega)(h_\mu f)(\omega)\right|^{p'} d\omega\right)^{\frac{1}{p'}},\end{aligned}$$

where $D_{p'} > 0$ is a constant. Multiplying both sides by a^{-s-1} and integrating from 0 to ∞, we find

$$\begin{aligned}&\int_0^\infty \left(\int_0^\infty |B_\psi f(b,a)|^p \, db\right)^{\frac{p'}{p}} a^{-s-1} da \\ &\qquad \le (D_{p'})^{p'} \int_0^\infty \left(\int_0^\infty \left|\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(a\omega)(h_\mu f)(\omega)\right|^{p'} d\omega\right) a^{-s-1} da.\end{aligned}$$

From Lemma 3.3.2, we get

$$\int_0^\infty \left(\int_0^\infty |B_\psi f(b,a)|^p \, db\right)^{\frac{p'}{p}} a^{-s-1} da \le (D_{p'})^{p'} C_{\mu,\psi}^{s,p'} \int_0^\infty \left|\omega^{\frac{s}{p'}-\mu-\frac{1}{2}}(h_\mu f)(\omega)\right|^{p'} d\omega.$$

This implies that

$$\|B_\psi f(b,a)\|_{W_p^{s,p'}} \le D_{p'}(C_{\mu,\psi}^{s,p'})^{\frac{1}{p'}} \|f\|_{H_\mu^{s,p'}}. \tag{3.3.6}$$

Further, from [16] and (3.3.5) for $1 \le p \le 2, \frac{1}{p} + \frac{1}{p'} = 1$, we get

$$\left(\int_0^\infty \left|\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(a\omega)(h_\mu f)(\omega)\right|^{p'} d\omega\right)^{\frac{1}{p'}} = \left(\int_0^\infty |h_\mu[B_\psi f(b,a)](\omega)|^{p'} d\omega\right)^{\frac{1}{p'}}$$
$$\le D_p \left(\int_0^\infty |B_\psi f(b,a)|^p \, db\right)^{\frac{1}{p}},$$

where $D_p > 0$ is a constant. Multiplying both sides by a^{-s-1} and integrating from 0 to ∞, we find

$$\int_0^\infty \left(\int_0^\infty \left|\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(a\omega)(h_\mu f)(\omega)\right|^{p'} d\omega\right) a^{-s-1} da$$
$$\le (D_p)^{p'} \int_0^\infty \left(\int_0^\infty |B_\psi f(b,a)|^p \, db\right)^{\frac{p'}{p}} a^{-s-1} da.$$

From Lemma 3.3.2, we get

$$C_{\mu,\psi}^{s,p'} \int_0^\infty \left|\omega^{\frac{s}{p'}-\mu-\frac{1}{2}}(h_\mu f)(\omega)\right|^{p'} d\omega \le (D_p)^{p'} \int_0^\infty \left(\int_0^\infty |B_\psi f(b,a)|^p \, db\right)^{\frac{p'}{p}} a^{-s-1} da.$$

This yields

$$\|f\|_{H_\mu^{s,p'}} \le \frac{D_p}{\left(C_{\mu,\psi}^{s,p'}\right)^{\frac{1}{p'}}} \|B_\psi f(b,a)\|_{W_p^{s,p'}}. \tag{3.3.7}$$

From (3.3.6) and (3.3.7), we get the results. □

3.4 Conclusions

In the present chapter, author introduced the Bessel wavelet transform in distributional sense and studied many properties related to the Bessel wavelet transform in distributions. In this chapter, author also applied the theory of Bessel wavelet transform in Sobolev space and found boundedness property of the aforesaid transform.

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