# Chapter 2

# Continuous Wavelet Transform of Schwartz Tempered Distributions in $S'(\mathbb{R}^n)$

## 2.1 Introduction

In the present chapter, we define a continuous wavelet transform of a Schwartz tempered distribution $f \in S'(\mathbb{R}^n)$ with wavelet kernel $\psi \in S(\mathbb{R}^n)$ for real scale $a \neq 0$ and derive the corresponding wavelet inversion formula interpreting convergence in the weak topology of $S'(\mathbb{R}^n)$ and its associated results. In [39], it is proved that a window function $\psi(x) \in L^2(\mathbb{R}^n)$ is a wavelet if and only if the integral of $\psi$ along each of the axes is zero; therefore, any $\psi(x) \in s(\mathbb{R}^n)$ is a wavelet where $s(\mathbb{R}^n)$ is a subspace of $S(\mathbb{R}^n)$ such that every element $\phi \in s(\mathbb{R}^n)$ satisfies (1.3.2). As an example, one can easily verify that the function given by

$$\psi(x) = x_1 x_2 \dots x_n e^{-(x_1^2 + x_2^2 + \cdots + x_n^2)}$$

is a wavelet belonging to $S(\mathbb{R}^n)$.

If we take $f(x) = c \in S'(\mathbb{R}^n)$, then the wavelet transform of a constant distribution is zero. We thus realize that two elements of $S'(\mathbb{R}^n)$ having an equal wavelet transform will differ by a constant in general. For proving the inversion formula of the continuous wavelet transform, we found that the wavelet transform of a constant distribution is zero and our wavelet inversion formula is not true for constant distribution, but it is true for a non-constant distribution which is not equal to the sum of a non-constant distribution with a non-zero constant distribution.

We organize chapter 2 in the following way:

Section 2.1 is introductory, where the brief information regarding the inversion formula of the continuous wavelet transform is given. In Section 2.2, the structure formula of generalized functions of slow growth is discussed. The wavelet transform of tempered distributions is defined and its properties examined. Using these properties finally, the inversion formula of the continuous wavelet transform of distributions is derived.

## 2.2 Wavelet transform of tempered distributions in $S'(\mathbb{R}^n)$ and its inversion

In this section, the structure formula of generalized functions of slow growth, which is given by V.S. Vladimirov [57], is discussed. Definitions and various properties of the wavelet transform of tempered distributions are given; using these results, the inversion formula of the continuous wavelet transform of distributions is investigated.

**Definition 2.2.1.** A function $f(x)$ is said to be a function of slow growth in $\mathbb{R}^n$ if, for $m \geq 0$, we have

$$\int_{\mathbb{R}^n} |f(x)| (1+|x|)^{-m} dx < \infty$$

and it determines a regular functional $f$ in $S'(\mathbb{R}^n)$ by the formula

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx, \qquad \phi \in S(\mathbb{R}^n). \tag{2.2.1}$$

It is easy to verify that the functional $f$ defined by (2.2.1) exists for all $\phi \in S(\mathbb{R}^n)$ and that it is linear as well as continuous on $S(\mathbb{R}^n)$. So, the elements of $S'(\mathbb{R}^n)$ are called tempered distributions or distributions of slow growth.

**Theorem 2.2.2.** *If $f \in S'(\mathbb{R}^n)$, then there exists a continuous function $g$ of slow growth in $\mathbb{R}^n$ and an integer $m \geq 0$ such that*

$$f(x) = D_1^m D_2^m ... D_n^m g(x), \quad \frac{\partial}{\partial x_i} \equiv D_i \tag{2.2.2}$$

*or, equivalently,*

$$f(x) = D^m g(x) \qquad (D := D_1 D_2 D_3 ... D_n). \tag{2.2.3}$$

The n-dimensional wavelet inversion formula for tempered distributions will now be proved very simply by using the structure Formula (2.2.3). This structure formula enables us to reduce the wavelet analysis problem relating to tempered distributions to the classical wavelet analysis problem of $L^2(\mathbb{R}^n)$ functions. The wavelet inversion formula of $L^2(\mathbb{R}^n)$ functions will be used quite successfully in order to derive the wavelet inversion formula for the wavelet transform of tempered distributions.

Henceforth, we assume that $a \neq 0$ implies each of the component $a_i \neq 0$ for all $i = 1, 2, 3, ..., n$ and $a > 0$ means each of the component $a_i$ of $a$ is greater than zero.

$|a| > \epsilon$ will mean that $|a_i| > \epsilon$ for all $i = 1, 2, 3, ..., n$.

**Definition 2.2.3.** Let $\psi(x) = \psi(x_1, x_2, ...x_n) \in S(\mathbb{R}^n)$, then $\psi(x)$ is a window function and is a wavelet if and only if

$$\int_{-\infty}^{\infty} \psi(x_1, x_2, \ldots, x_i, \ldots x_n) dx_i = 0, \qquad (\forall\ i = 1, 2, 3, \ldots, n).$$

**Definition 2.2.4.** We take $\psi\left(\frac{x-b}{a}\right) \equiv \psi\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}, \ldots, \frac{x_n-b_n}{a_n}\right)$, where $a_i, b_i$ are real numbers and none of the $a_i$ is zero. Then the wavelet transform $W_f(a, b)$ of $f \in S'(\mathbb{R}^n)$ with respect to the kernel $\frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right)$ is defined by

$$W_f(a, b) = \left\langle f(x), \frac{1}{\sqrt{|a|}}\psi\Big(\frac{x-b}{a}\Big)\right\rangle, \tag{2.2.4}$$

where

$$|a| = |a_1 a_2 a_3 \ldots a_n| \qquad (a_i \neq 0\ (i = 1, 2, 3, \ldots, n)).$$

We now prove the following lemmas which will be used to prove the main inversion formula.

**Lemma 2.2.5.** *Let $\phi \in S(\mathbb{R}^n)$ and $\psi$ be a wavelet belonging to $S(\mathbb{R}^n)$.*

$$\frac{1}{C_\psi} \int_{a\in\mathbb{R}^n} \int_{b\in\mathbb{R}^n} \int_{t\in\mathbb{R}^n} (-D_t)^m\, \phi(t) \bar{\psi}\Big(\frac{t-b}{a}\Big) \psi\Big(\frac{x_0-b}{a}\Big) \frac{dt\ db\ da}{a^2|a|}$$

$$= (-D_x)^m\, \phi(x)|_{x=x_0} \qquad (\forall\ x_0 \in \mathbb{R}^n).$$

*This is called point-wise convergence of the wavelet inversion formula.*

***Proof.*** The proof of the Lemma can be seen from [37, Theorem 1, p. 4]. □

**Lemma 2.2.6.** *Let $\phi \in S(\mathbb{R}^n)$ and $\psi$ be a wavelet belonging to $S(\mathbb{R}^n)$. Then*

$$\frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} \int\limits_{t\in\mathbb{R}^n} (-D_t)^m \phi(t)\bar{\psi}\Big(\frac{t-b}{a}\Big)\psi\Big(\frac{x-b}{a}\Big)\frac{dt\, db\, da}{a^2|a|}$$

*converges to $(-D_x)^m \phi(x)$ uniformly for all $x \in \mathbb{R}^n$.*

***Proof.*** Let $f(t) = (-D_t)^m \phi(t)$ and

$$\hat{f}(\omega) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} (-D_t)^m \phi(t)\, e^{-i\,\omega.t}\, dt,$$

be the Fourier transform of $f(t) = (-D_t)^m \phi(t)$. In view of [39, Theorem 4.2, pp. 4770-4772], putting $(-D_t)^m \phi(t) = f(t)$, we have

$$\begin{aligned}
&\frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} \int\limits_{t\in\mathbb{R}^n} (-D_t)^m \phi(t)\bar{\psi}\Big(\frac{t-b}{a}\Big)\psi\Big(\frac{x-b}{a}\Big)\frac{dt\, db\, da}{a^2|a|} \\
&\quad= \frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} \int\limits_{t\in\mathbb{R}^n} f(t)\bar{\psi}\Big(\frac{t-b}{a}\Big)\psi\Big(\frac{x-b}{a}\Big)\frac{dt\, db\, da}{a^2|a|} \\
&\quad= \frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} \Big\{\frac{1}{\sqrt{|a|}} \int\limits_{t\in\mathbb{R}^n} f(t)\bar{\psi}\Big(\frac{t-b}{a}\Big)dt\Big\}\psi\Big(\frac{x-b}{a}\Big)\frac{db\, da}{\sqrt{|a|}a^2}.
\end{aligned}$$

Using the definition of wavelet transform (1.3.4), we find

$$\begin{aligned}
&\frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} \int\limits_{t\in\mathbb{R}^n} (-D_t)^m \phi(t)\bar{\psi}\Big(\frac{t-b}{a}\Big)\psi\Big(\frac{x-b}{a}\Big)\frac{dt\, db\, da}{a^2|a|} \\
&\quad= \frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} W_f(a,b)\psi\Big(\frac{x-b}{a}\Big)\frac{db\, da}{\sqrt{|a|}a^2}.
\end{aligned}$$

Applying the Parseval relation (1.1.6), we get

$$\frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} \int\limits_{t\in\mathbb{R}^n} (-D_t)^m \phi(t)\bar{\psi}\Big(\frac{t-b}{a}\Big)\psi\Big(\frac{x-b}{a}\Big)\frac{dt\, db\, da}{a^2|a|}$$

$$= \frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} F_b\{W_f(a,b)\}(\omega)\overline{F_b\bar{\psi}\Big(\frac{x-b}{a}\Big)(\omega)}d\omega\frac{da}{\sqrt{|a|}a^2}.$$

Since $F_b\{W_f(a,b)\}(\omega) = (2\pi)^{\frac{n}{2}}\sqrt{|a|}\hat{f}(\omega)\bar{\hat{\psi}}(a\omega)$ and $\overline{F_b\bar{\psi}\Big(\frac{x-b}{a}\Big)(\omega)} = |a|e^{i\omega.x}\hat{\psi}(a\omega)$, therefore the above expression yields

$$\begin{aligned}
&\frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} \int\limits_{c\in\mathbb{R}^n} (-D_t)^m\,\phi(t)\bar{\psi}\Big(\frac{t-b}{a}\Big)\psi\Big(\frac{x-b}{a}\Big)\frac{dt\;db\;da}{a^2|a|} \\
&\quad= \frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} \{(2\pi)^{\frac{n}{2}}\sqrt{|a|}\hat{f}(\omega)\bar{\hat{\psi}}(a\omega)\}|a|e^{i\omega.x}\hat{\psi}(a\omega)d\omega\frac{da}{\sqrt{|a|}a^2} \\
&\quad= \frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} \{(2\pi)^{\frac{n}{2}}\hat{f}(\omega)\}e^{i\omega.x}\frac{|\hat{\psi}(a\omega)|^2}{|a|}d\omega da \\
&\quad= \frac{1}{C_\psi}\frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{b\in\mathbb{R}^n} \hat{f}(\omega)e^{i\omega.x}\Big\{(2\pi)^n \int\limits_{a\in\mathbb{R}^n} \frac{|\hat{\psi}(a\omega)|^2}{|a|}da\Big\}d\omega \\
&\quad= \frac{1}{C_\psi}\frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{b\in\mathbb{R}^n} \hat{f}(\omega)e^{i\omega.x}C_\psi d\omega \\
&\quad= \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \hat{f}(\omega)e^{i\omega.x}d\omega = f(x) = (-D_x)^m\,\phi(x),
\end{aligned}$$

here the convergence is in $L^2(\mathbb{R}^n)$.

This convergence is also uniform by a Weierstrass $M$-test because

$$\Big|\frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \hat{f}(\omega)\;e^{i\omega.x}\;d\omega\Big| \le \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} |\hat{f}(\omega)|d\omega < \infty$$

and $\hat{f}(\omega) \in S(\mathbb{R}^n)$. □

**Theorem 2.2.7.** *Let $f \in S'(\mathbb{R}^n)$ and $W_f(a,b)$ be its wavelet transform defined by*

$$W_f(a,b) = \Big\langle f(x),\;\; \frac{1}{\sqrt{|a|}}\psi\Big(\frac{x-b}{a}\Big)\Big\rangle. \tag{2.2.5}$$

*Then the inversion formula of the wavelet transform $W_f(a,b)$ is given by*

$$\Big\langle \frac{1}{C_\psi}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} W_f(a,b)\psi\Big(\frac{t-b}{a}\Big)\frac{db\,da}{\sqrt{|a|}a^2}, \phi(t)\Big\rangle = \langle f,\phi\rangle, \qquad (\forall\ \phi\in S(\mathbb{R}^n)) \tag{2.2.6}$$

*where the equality holds true almost everywhere.*

***Proof.*** Using the structure formula (2.2.3) for $f$, we find by distributional differentiation that

$$\begin{aligned} W_f(a,b) &= \Big\langle D_x^m g(x), \frac{1}{\sqrt{|a|}}\psi\Big(\frac{x-b}{a}\Big)\Big\rangle \\ &= \Big\langle g(x), (-D_x)^m\,\frac{1}{\sqrt{|a|}}\psi\Big(\frac{x-b}{a}\Big)\Big\rangle. \end{aligned}$$

Here, we have

$$(-D_x) = (-D_{x_1})\,(-D_{x_2})\,(-D_{x_3})\ldots(-D_{x_n}), \quad D_{x_i}\equiv\frac{\partial}{\partial x_i}, \qquad (i=1,2,3,\ldots,n).$$

We thus obtain

$$\begin{aligned} W_f(a,b) &= \Big\langle g(x), (D_b)^m\,\frac{1}{\sqrt{|a|}}\psi\Big(\frac{x-b}{a}\Big)\Big\rangle \\ D_b &= \frac{\partial}{\partial b_1}\frac{\partial}{\partial b_2}\cdots\frac{\partial}{\partial b_n}. \end{aligned} \tag{2.2.7}$$

Using (2.2.7), the left-hand side in (2.2.6) can be written as follows:

$$\begin{aligned} &\Big\langle \frac{1}{C_\psi}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} W_f(a,b)\psi\Big(\frac{t-b}{a}\Big)\frac{db\,da}{\sqrt{|a|}a^2}, \phi(t)\Big\rangle \\ &= \frac{1}{C_\psi}\int\limits_{t\in\mathbb{R}^n}\int\limits_{a\in\mathbb{R}^n}\int\limits_{b\in\mathbb{R}^n}\Big\langle g(x), (D_b)^m\,\frac{1}{\sqrt{|a|}}\psi\Big(\frac{x-b}{a}\Big)\Big\rangle\psi\Big(\frac{t-b}{a}\Big)\bar{\phi}(t)\frac{db\,da\,dt}{\sqrt{|a|}a^2} \\ &= \frac{1}{C_\psi}\int\limits_{t\in\mathbb{R}^n}\int\limits_{a\in\mathbb{R}^n}\int\limits_{b\in\mathbb{R}^n}\int\limits_{x\in\mathbb{R}^n} g(x)D_b^m\frac{1}{\sqrt{|a|}}\bar{\psi}\Big(\frac{x-b}{a}\Big)\psi\Big(\frac{t-b}{a}\Big)\bar{\phi}(t)\frac{dx\,db\,da\,dt}{\sqrt{|a|}a^2} \end{aligned}$$

$$= \frac{1}{C_\psi} \int\limits_{t\in\mathbb{R}^n} \int\limits_{a\in\mathbb{R}^n} \int\limits_{x\in\mathbb{R}^n} g(x)\Big[ \int\limits_{b\in\mathbb{R}^n} \Big\{D_b^m \bar{\psi}\Big(\frac{x-b}{a}\Big)\Big\}\psi\Big(\frac{t-b}{a}\Big)db\Big]\bar{\phi}(t)\frac{dx\,da\,dt}{a^2|a|}. \tag{2.2.8}$$

We now evaluate the integral in the big bracket by parts to find from (2.2.8) that

$$\begin{aligned}&\Big\langle \frac{1}{C_\psi}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} W_f(a,b)\psi\Big(\frac{t-b}{a}\Big)\frac{db\,da}{\sqrt{|a|}a^2}, \phi(t)\Big\rangle \\ &= \frac{1}{C_\psi} \int\limits_{t\in\mathbb{R}^n} \int\limits_{a\in\mathbb{R}^n} \int\limits_{x\in\mathbb{R}^n} g(x)\Big[ \int\limits_{b\in\mathbb{R}^n} \bar{\psi}\Big(\frac{x-b}{a}\Big)\,(-D_b)^m\,\psi\Big(\frac{t-b}{a}\Big)db\Big]\bar{\phi}(t)\frac{dx\,da\,dt}{a^2|a|}.\end{aligned}$$

Using $(-D_b)\psi\left(\frac{t-b}{a}\right) = D_t\psi\left(\frac{t-b}{a}\right)$, the above expression can be expressed as

$$\begin{aligned}&\Big\langle \frac{1}{C_\psi}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} W_f(a,b)\psi\Big(\frac{t-b}{a}\Big)\frac{db\,da}{\sqrt{|a|}a^2}, \phi(t)\Big\rangle \\ &= \frac{1}{C_\psi} \int\limits_{t\in\mathbb{R}^n} \int\limits_{a\in\mathbb{R}^n} \int\limits_{x\in\mathbb{R}^n} g(x)\Big[ \int\limits_{b\in\mathbb{R}^n} \bar{\psi}\Big(\frac{x-b}{a}\Big)\,(D_t)^m\,\psi\Big(\frac{t-b}{a}\Big)db\Big]\bar{\phi}(t)\frac{dx\,da\,dt}{a^2|a|}.\end{aligned}$$

Inverting the order of integration with respect to $a$ and $t$, the above expression yields

$$\begin{aligned}&\Big\langle \frac{1}{C_\psi}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} W_f(a,b)\psi\Big(\frac{t-b}{a}\Big)\frac{db\,da}{\sqrt{|a|}a^2}, \phi(t)\Big\rangle \\ &= \frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{t\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} \int\limits_{x\in\mathbb{R}^n} g(x)\bar{\psi}\Big(\frac{x-b}{a}\Big)\,dx\;D_t^m\;\psi\Big(\frac{t-b}{a}\Big)db\bar{\phi}(t)\frac{dt\,da}{|a|^2|a|} \\ &= \frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} \int\limits_{x\in\mathbb{R}^n} g(x)\bar{\psi}\Big(\frac{x-b}{a}\Big)dx \int\limits_{t\in\mathbb{R}^n} \psi\Big(\frac{t-b}{a}\Big)db\,(-D_t)^m\,\bar{\phi}(t)\frac{dt\,da}{|a|^2|a|}.\end{aligned} \tag{2.2.9}$$

In order to justify the inversion of the order of integration with respect to $a$ and $t$, we first perform the integration in the region $\{(a,t) : |a| > \epsilon, a, t \in \mathbb{R}^n\}$, invert the order of integration and then let $\epsilon \to 0$. This existence of the triple integral in terms of $b$, $a$ and $t$ in (2.2.9) is proved by using the Plancherel theorem with respect to the variable $b$. Thus, by using

$$C_\psi = (2\pi)^n \int\limits_{\mathbb{R}^n} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega,$$

we notice that the variable $a$ disappears from the denominator and every calculation goes on smoothly. Since the functions $\phi$ and $\psi$ are elements of $S(\mathbb{R}^n)$, the Fubini's theorem can be applied in order to justify the above interchanges of the order of integration.

Now, (2.2.9) can be written as follows:

$$\begin{aligned}
&\Big\langle \frac{1}{C_\psi} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} W_f(a,b) \psi\Big(\frac{t-b}{a}\Big) \frac{db\, da}{\sqrt{|a|} a^2}, \phi(t) \Big\rangle \\
&= \Big\langle g(x), \frac{1}{C_\psi} \int\limits_{a\in\mathbb{R}^n} \int\limits_{b\in\mathbb{R}^n} \int\limits_{t\in\mathbb{R}^n} (-D_t)^m \phi(t) \bar{\psi}\Big(\frac{t-b}{a}\Big) dt\, \psi\Big(\frac{x-b}{a}\Big) \frac{db\, da}{|a|^2 |a|} \Big\rangle \\
&= \langle g(x), (-D_x)^m \phi(x) \rangle, \qquad (2.2.10)
\end{aligned}$$

by means of the wavelet inversion formula in $\mathbb{R}^n$ [39, Theorem 4.2, pp. 4770-4772] and Lemma 2.2.6. We note that the triple integral in the above expression converges uniformly to $(-D_x)^m \phi(x)$, for all $x \in \mathbb{R}^n$.

Using the distributional differentiation, (2.2.3) and (2.2.10), we get

$$\begin{aligned}
&\Big\langle \frac{1}{C_\psi} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} W_f(a,b) \psi\Big(\frac{t-b}{a}\Big) \frac{db\, da}{\sqrt{|a|} a^2}, \phi(t) \Big\rangle \\
&= \langle g(x), (-D_x)^m \phi(x) \rangle = \langle (D_x)^m g(x),\ \phi(x) \rangle = \langle f(x),\ \phi(x) \rangle.
\end{aligned}$$

□

## 2.3 Conclusions

In the present chapter, author discussed the continuous wavelet transform of Schwartz tempered distribution $f \in S'(\mathbb{R}^n)$ with the wavelet kernel $\psi \in S(\mathbb{R}^n)$ and derived the corresponding wavelet inversion formula by interpreting convergence in the weak topology of $S'(\mathbb{R}^n)$.

The author found that the wavelet transform of a constant distribution is zero and also that the wavelet inversion formula is not true for constant distribution, but it is true for a non-constant distribution which is not equal to the sum of a non-constant distribution with a non-zero constant distribution. The results and findings are proved in the form of Lemmas and Theorems.

But our wavelet kernel chosen suffers from a drawback that all its moments of even order will be zero and so using $\frac{x-b}{a} = t$ we have

$$\int_{-\infty}^{\infty} \Big(\frac{x-b}{a}\Big)^{2m} \psi\Big(\frac{x-b}{a}\Big) dx = |a| \int_{-\infty}^{\infty} t^{2m+1} e^{-t^2} dt = 0$$

where $\psi(t) = te^{-t^2}$ in dimension $n = 1$.

So two functions having the same wavelet transform with respect to the kernel $\psi(x)$ will differ by a polynomial

$$P\Big(\frac{x-b}{a}\Big) = a_0 + a_2\Big(\frac{x-b}{a}\Big)^2 + a_4\Big(\frac{x-b}{a}\Big)^4 + \dots a_{2m}\Big(\frac{x-b}{a}\Big)^{2m} \qquad m \geq 1,$$

where at least one of $a_0, a_2, \dots a_{2m}$ is non-zero. Therefore, in order that the uniqueness theorem for the inversion formula of the wavelet transform may be valid our wavelet $\psi$ should be so chosen that all its moments of order $m \geq 1$ should be non-zero. One such wavelet is $(1 + cx - 2x^2)e^{-x^2}$ where $c$ is an arbitrary nonzero