Chapter 1

Introduction

The Fourier transform is an effective and efficient tool for the study of those functions that may be represented by the sum of simpler trigonometric functions. The aforesaid theory came into light from the work of Joseph Fourier (1822), who showed that a periodic function can be expressed as the sum of trigonometric functions. The origination of wavelet transform was initially developed by exploiting the theory of Fourier transform. So, the wavelet transform enables us to provide the local and global information of signal at a time. The entire wavelet theory is encompassed by the theory of Fourier transformation. Many authors exploited Fourier transform technique and developed continuous and discrete wavelet transform on various functional spaces and studied many properties. This theory is useful in problems of image processing, signal processing and other areas of mathematics and engineering. Exploiting the theory of wavelet transform, the inversion formula, Parseval's formula and many other important results were discussed by many researchers and used this theory in the problem of Sobolev type spaces which are very beneficial to solve the higher-order partial differential equations and other problems of mathematics.

Exploiting the theory of Fourier transform, the inversion formula of the continuous wavelet transform and related results were discussed by Chui (2016), Pathak (2009) and Levedeva et al. (2014, 2016) on $L^2(\mathbb{R})$-space. Later on, the same formula was considered by Weisz (2013, 2015) on L^p-space, $1 < p < \infty$ and Weiner-amalgam spaces. Without taking the admissibility condition, the characterization of the inversion formula of the continuous wavelet transform was derived by Postinikov et al. (2014). In $n-$dimensional setting, the inversion formula of the continuous wavelet transform was discussed by Daubechies (1992), Meyer (1992), Keinert (2003) and Pathak (2009). Using the theory of window function and Fourier transform, Pandey and Upadhyay (2015) studied continuous wavelet transformation in the classical sense and obtained the inversion formula of continuous wavelet transform and other important results. The characterizations of continuous wavelet transform of distributions and its related results were investigated by Holschneider (1995) and Pathak (2004). Later on, Pandey and Upadhyay (2019), found the inversion formula of continuous wavelet transform of Schwartz tempered distributions by exploiting the Fourier transform.

The Hankel transform is considered an important tool to find the solution of cylindrical boundary value problems. Researchers exploited the aforesaid theory and explored the research works on Zemanian space and other functional spaces, and found many important and interesting observations. The theory of Hankel convolution is heavily dependent on the Hankel transform technique. Using this technique, Zemanian (1968), Betancor (1995), J. de Sousa Pinto (1985), Pathak (2003, 2011) and others discussed the theory of Hankel convolution and studied many important properties. The aforesaid theory is very important for the development of the Bessel wavelet transform.

Exploiting the theory of Hankel transform and Hankel convolution, which was introduced by Haimo (1965), Hirschman (1960) and Cholewinski (1965), Pathak and Dixit (2003) investigated the Bessel wavelet transform and discussed its various properties. Later on, by considering the Zemanian theory of Hankel transform, the continuous Bessel wavelet transform and its various properties were found by Upadhyay et al. (2012). Many results of the continuous Bessel wavelet transform are obtained by Upadhyay and Singh (2015, 2017, 2018, 2020) by using the Zemanian Hankel transform tool.

Motivated from the aforesaid results, in the present thesis, the author will consider the following observations:

(i) The inversion formula of the continuous wavelet transform of Schwartz tempered distributions in $S'(\mathbb{R}^n)$ will be investigated.

(ii) Exploiting the theory of Hankel transform, the continuity and boundedness properties of continuous Bessel wavelet transform of distributions will be discussed in Sobolev type space and other spaces.

(iii) The characterizations of the continuous Bessel wavelet transform of distributions in Besov and Triebel-Lizorkin spaces will be given.

(iv) Using the representation of distributions in $H'_\mu(\mathbb{R}^+)$-space, the inversion formula of the continuous Bessel wavelet transform will be obtained and its associated results derived.

(v) The inversion formula of the continuous Bessel wavelet transform of distributions in β'_μ-space and its associated results will be discussed.

From Sneddon [49], Pathak [47], Pandey and Upadhyay [39], Zemanian [61, 63], Betancor and Marrero [10], Macaulay-Owen [32], Wing [58], Kerr [28], Pathak and

Dixit [42], Upadhyay et al. [56], Betancor et al. [9], Koh [29, 30], Hardy et al. [22] and Triebel [50], we are giving some important definitions, formulae and properties in the form of sections that will be used in the subsequent chapters.

1.1 The Fourier Transform

Motivated from the work of Sneddon [49], Pathak [47] and Pandey et al. [39], in this section, various definitions and properties of the Fourier transform are given below:

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(\omega) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i(t,\omega)} f(t) dt. \tag{1.1.1}$$

If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, then the inverse Fourier transform of $\hat{f}$ is given by

$$f(t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i(t,\omega)} \hat{f}(\omega) d\omega, \quad a.e. \tag{1.1.2}$$

where $(t,\omega) = t_1\omega_1 + t_2\omega_2 + \cdots + t_n\omega_n$.

Properties of the Fourier transform

(I) Let $f, g \in L^1(\mathbb{R}^n)$ and $c_1, c_2 \in \mathbb{C}$, then

$$(c_1 f + c_2 g)\hat{}(\omega) = c_1 \hat{f}(\omega) + c_2 \hat{g}(\omega), \; \omega \in \mathbb{R}^n \qquad \text{(linearity)}. \tag{1.1.3}$$

(II) Let $f \in L^1(\mathbb{R}^n)$. For any fix $a, b \in \mathbb{R}^n$ with $a_i \neq 0, i = 1, 2, \cdots n$, the functions $T_b f, M_b f$ and $D_a f$ are defined by

(i) $(T_b f)(x) = f(x+b), \; x \in \mathbb{R}^n$ (translation operator)

(ii) $(M_b f)(x) = e^{i(x,b)} f(x), \; x \in \mathbb{R}^n$ (modulation operator)

(iii) $(D_a f)(x) = |a|^{-\frac{1}{2}} f\left(\frac{x}{a}\right), \ x \in \mathbb{R}^n$ (dilation operator)

where $|a| = |a_1 a_2 \cdots a_n|$ and $\left(\frac{x}{a}\right) = \left(\frac{x_1}{a_1}, \frac{x_2}{a_2}, \cdots, \frac{x_n}{a_n}\right)$. By taking Fourier transform of above expressions the following results hold

(iv) $(T_b f\hat{)}(\omega) = (M_b \hat{f})(\omega), \ \omega \in \mathbb{R}^n$

(v) $(M_b f\hat{)}(\omega) = (T_{-b} \hat{f})(\omega), \ \omega \in \mathbb{R}^n$

(vi) $(D_{\frac{1}{a}} f\hat{)}(\omega) = (D_a \hat{f})(\omega), \ \omega \in \mathbb{R}^n$.

(III) **(Riemann-Lebesgue Lemma)** Let $f \in L^1(\mathbb{R}^n)$, then

(i) $\hat{f}$ is continuous on $\mathbb{R}^n$.

(ii) $\lim_{|\omega| \to \infty} \hat{f}(\omega) = 0$.

(iii) $f_j \to f$ in $L^1(\mathbb{R}^n)$ implies $\hat{f}_j \to \hat{f}$ uniformly on $\mathbb{R}^n$.

Fourier transform in $L^2(\mathbb{R}^n)$-space

From Pathak [47] and Pandey et al. [39], the Fourier transform and its inversion formula in $L^2(\mathbb{R}^n)$ are given by

$$\hat{f}(\omega) = \underset{N \to \infty}{l.i.m.} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-N}^{N} e^{-i(t,\omega)} f(t) dt, \tag{1.1.4}$$

for $x = (x_1, x_2, \cdots, x_n)$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n) \in \mathbb{R}^n$ and $(x, \omega) = x_1\omega_1 + x_2\omega_2 + \cdots + x_n\omega_n, \ N = (N_1, N_2, \cdots, N_n)$. $N \to \infty$ implies that each of the components of N tend to ∞ independently of each other. This defines convergence in $L^2(\mathbb{R}^n)$ and is called the limit in the mean ($l.i.m.$).

The corresponding inversion formula of the Fourier transform is defined by

$$f(t) = \underset{N \to \infty}{l.i.m.} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-N}^{N} e^{i(t,\omega)} \hat{f}(\omega) d\omega. \tag{1.1.5}$$

Let $f, g \in L^2$, then the Parseval's relation holds

$$\langle \hat{f}, \hat{g} \rangle_{L^2} = \langle f, g \rangle_{L^2} \tag{1.1.6}$$

and in particular if $f = g$, then

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}, \tag{1.1.7}$$

where the inner product in $L^2(\mathbb{R}^n)$ space is defined by

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{R}^n} f(t)\overline{g(t)}dt. \tag{1.1.8}$$

Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty$. Then the convolution of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy, \tag{1.1.9}$$

for almost every $x \in \mathbb{R}^n$ with

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p. \tag{1.1.10}$$

Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n), 1 \leq p, q \leq r \leq \infty$, then from [47, p. 130] we have

$$\|f * g\|_r \leq \|f\|_p \|g\|_q, \tag{1.1.11}$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$.

Let $f, g \in L^1(\mathbb{R}^n)$, then from [59, Proposition 4.1, p. 17] we get

$$\widehat{(f * g)}(\omega) = (2\pi)^{\frac{n}{2}} \hat{f}(\omega)\hat{g}(\omega). \tag{1.1.12}$$

1.2 Schwartz space and Tempered Distributions

Schwartz testing function space $S(\mathbb{R}^n)$ consists of C^∞ functions ϕ defined on $\mathbb{R}^n$ and satisfying the conditions

$$\gamma_{m,k}(\phi) = \sup_{x\in\mathbb{R}^n} \left| |x^m| \, \phi^{(k)}(x) \right| < \infty, \tag{1.2.1}$$

where $|m|, |k| = 0, 1, 2, \ldots$ and

$$\begin{aligned} |m| &= m_1 + m_2 + \cdots + m_n, \\ |k| &= k_1 + k_2 + \cdots + k_n, \\ |x^m| &= |x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}| \, , \\ \phi^{(k)}(x) &= \frac{\partial^{k_n}}{\partial x_n} \cdots \frac{\partial^{k_2}}{\partial x_2} \frac{\partial^{k_1}}{\partial x_1} \phi(x). \end{aligned}$$

The topology over $S(\mathbb{R}^n)$ is generated by the sequence of semi-norms $\{\gamma_{m,k}\}_{|m|,|k|=0}^{\infty}$. These collections of semi-norms in (1.2.1) are separating which means that an element $\phi \in S(\mathbb{R}^n)$ is non-zero if and only if there exists at least one of the semi-norms $\gamma_{m,k}$ satisfying $\gamma_{m,k}(\phi) \neq 0$. A sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ in $S(\mathbb{R}^n)$ tends to ϕ in $S(\mathbb{R}^n)$ if and only if $\gamma_{m,k}(\phi_\nu - \phi) \to 0$ as ν goes to ∞ for each of the subscripts $|m|, |k| = 0, 1, 2, \ldots$, are as defined above. Now, one can verify that the function $e^{-(t_1^2+t_2^2+\cdots+t_n^2)} \in S(\mathbb{R}^n)$ and the sequence

$$\frac{\nu - 1}{\nu} e^{-(t_1^2+t_2^2+\cdots+t_n^2)} \text{ converges to } e^{-(t_1^2+t_2^2+\cdots+t_n^2)}$$

in $S(\mathbb{R}^n)$ as $\nu \to \infty$.

The space of all continuous and linear functional on $S(\mathbb{R}^n)$ is called the space of tempered distributions and denoted by $S'(\mathbb{R}^n)$. A linear functional f is said to be continuous, if for any sequence of test functions $\{\phi_n\}_{n\in\mathbb{N}}$ that converges in $S(\mathbb{R}^n)$ to zero, the sequence of numbers $\{\langle f, \phi_n\rangle\}_{n\in\mathbb{N}}$ converges to zero.

The Dirac delta function $\delta(t)$ is defined here by

$$\langle \delta(t_1 - a_1, t_2 - a_2, t_3 - a_3, \ldots, t_n - a_n), \phi(t_1, t_2, t_3, \ldots, t_n)\rangle = \phi(a_1, a_2, a_3, \ldots, a_n).$$

So, we have

$$\langle \delta(t_1, t_2, t_3, \ldots, t_n), \phi(t_1, t_2, t_3, \ldots, t_n)\rangle = \phi(0, 0, 0, \ldots, 0), \qquad \phi \in S(\mathbb{R}^n).$$

Clearly, $\delta(t_1, t_2, \ldots, t_n)$ is a continuous linear functional on $S(\mathbb{R}^n)$. Let f be a locally integrable function on $\mathbb{R}$, then f defines a distribution as follows:

$$\langle f, \phi\rangle = \int_{\mathbb{R}^n} f(t)\phi(t)dt, \;\; \phi \in S(\mathbb{R}^n). \tag{1.2.2}$$

Distributions generated by a locally integrable functions are called regular distributions. $\delta(t_1, t_2, \ldots, t_n)$ is not a regular distribution this implies that it is a singular distribution.

Some useful results

(I) $S(\mathbb{R}^n)$ is a dense subspace of $L^p(\mathbb{R}^n), 1 \leq p < \infty$.

(II) The Fourier transform is a continuous isomorphism from $S(\mathbb{R}^n)$ onto $S(\mathbb{R}^n)$; its inverse, given by (1.1.2), is also a continuous isomorphism from $S(\mathbb{R}^n)$ onto $S(\mathbb{R}^n)$.

(III) The Fourier transform $\hat{f}$ of $f \in S'(\mathbb{R}^n)$ is defined by

$$\langle \hat{f}, \phi\rangle := \langle f, \hat{\phi}\rangle, \;\; \phi \in S(\mathbb{R}^n). \tag{1.2.3}$$

And it is continuous isomorphism from $S'(\mathbb{R}^n)$ onto $S'(\mathbb{R}^n)$.

1.3 Wavelet Transform

From Pathak [47] and Pandey et al. [39], various definitions and related results of continuous wavelet transform are given by exploiting the Fourier transform technique.

A function $\psi \in L^2(\mathbb{R}^n)$ which satisfies the admissibility condition

$$C_\psi := \int_{\mathbb{R}^n} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty, \tag{1.3.1}$$

where $|\omega| = |\omega_1 \omega_2 \cdots \omega_n|$, is called a basic wavelet.

A function $\psi \in L^2(\mathbb{R}^n)$ is called a window function if it satisfies the following conditions:

(1) $x_1\psi(x), x_2\psi(x), \ldots, x_n\psi(x)$ all belong to $L^2(\mathbb{R}^n)$.

(2) $x_i x_j \psi(x) \in L^2(\mathbb{R}^n)$ for all $i, j = 1, 2, \ldots, n,\ i \neq j$.

(3) $x_i x_k x_j \psi(x) \in L^2(\mathbb{R}^n)$ for all $i, j, k = 1, 2, \ldots, n,\ i \neq j \neq k \neq i$. Note that $i \neq j \neq k \neq i$ implies that i, j, k are all different, $i \neq j \neq k$ may imply that i and k could be equal. In general, window function is defined by the following way:

(4) $x_1 x_2 x_3 \ldots x_n \psi(x) \in L^2(\mathbb{R}^n)$.

A window function ψ belonging to $L^2(\mathbb{R}^n)$ is called wavelet if

$$\text{(i)} \qquad \int_{-\infty}^{\infty} \psi(x_1, x_2, x_3, \ldots, x_i, \ldots, x_n) dx_i = 0, \qquad (\forall\ i = 1, 2, 3, \ldots, n) \tag{1.3.2}$$

and it satisfies the admissibility condition

$$\text{(ii)} \qquad \int_{\mathbb{R}^n} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty, \tag{1.3.3}$$

where

$$\hat{\psi}(\omega) = \hat{\psi}(\omega_1, \omega_2, \ldots, \omega_n), \quad |\omega| = |\omega_1 \omega_2 \ldots \omega_n|$$

and $\hat{\psi}(\omega)$ is the Fourier transform of $\psi(x) \equiv \psi(x_1, x_2, \ldots, x_n)$.

Let $f \in L^2(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n)$ be a basic wavelet, then the continuous wavelet transform of f is defined by

$$W_f(a, b) = \frac{1}{\sqrt{|a|}} \int_{\mathbb{R}^n} f(t) \overline{\psi\Big(\frac{t-b}{a}\Big)} dt, \tag{1.3.4}$$

where $|a| = |a_1 a_2 \cdots a_n|$, $a_i \neq 0$ for $i = 1, 2, \cdots, n$ and $\big(\frac{t-b}{a}\big) = \big(\frac{t_1 - b_1}{a_1}, \frac{t_2 - b_2}{a_2}, \cdots, \frac{t_n - b_n}{a_n}\big)$.

Let $W_f(a, b)$ be wavelet transform of $f \in L^2(\mathbb{R}^n)$, then the Fourier transform of $W_f(a, b)$ with respect to variable b is given by

$$F_b\{W_f(a, b)\}(\omega) = \prod_{i=1}^{n} |2\pi a_i|^{\frac{1}{2}} \hat{f}(\omega) \overline{\hat{\psi}(\omega)} \tag{1.3.5}$$

from Pandey and Upadhyay [39, pp. 4771-4772].

Let $f \in L^2(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n)$ be a basic wavelet, then the inversion of the continuous wavelet transform in classical sense is

$$f(x) = \frac{1}{C_\psi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W_f(a,b) |a|^{-\frac{1}{2}} \psi\Big(\frac{x-b}{a}\Big) \frac{dbda}{|a|^2}, \quad a, b \in \mathbb{R}^n \tag{1.3.6}$$

where $W_f(a,b)$ is defined as in (1.3.4) and $|a| = |a_1 a_2 \cdots a_n|$, $a_i \neq 0$ for $i = 1, 2, \cdots, n$.

1.4 Hankel Transform

From Zemanian [63], Betancor and Marrero [10], Pathak [47] and Macaulay-Owen [32], the definitions, properties and formulae of the Hankel transform are discussed.

Let $f \in L^1(\mathbb{R}^+)$, then the Hankel transform is defined by

$$(h_\mu f)(y) = \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy) f(x) dx, \quad \mu \geq -\frac{1}{2} \tag{1.4.1}$$

where J_μ denotes the Bessel function of first kind and of order μ.

If $f \in L^1(\mathbb{R}^+)$ and $h_\mu f \in L^1(\mathbb{R}^+)$, then the inverse Hankel transform is given by

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy)(h_\mu f)(y) dy, \text{ for } \mu \geq -\frac{1}{2}. \tag{1.4.2}$$

Let $x^{\mu+\frac{1}{2}} f(x) \in L^1(\mathbb{R}^+)$ and $x^{\mu+\frac{1}{2}} g(x) \in L^1(\mathbb{R}^+)$, then the Hankel convolution is defined by

$$(f \# g)(x) = \int_0^\infty f(y)(\tau_x g)(y) dy, \tag{1.4.3}$$

where

$$(\tau_x g)(y) = g(x,y) = \int_0^\infty g(z) D_\mu(x,y,z) dz \tag{1.4.4}$$

is the Hankel translation.

$D_\mu(x,y,z)$ is a basic function which is defined by

$$D_\mu(x,y,z) = \int_0^\infty t^{-\mu-\frac{1}{2}}(xt)^{\frac{1}{2}}J_\mu(xt)(yt)^{\frac{1}{2}}J_\mu(yt)(zt)^{\frac{1}{2}}J_\mu(zt)dt,\ \ x,y,z \in \mathbb{R}^+ \quad (1.4.5)$$

and holds the following properties:

(i) $$D_\mu(x,y,z) \geq 0,\ \ x,y,z \in \mathbb{R}^+. \quad (1.4.6)$$

(ii) $$\int_0^\infty t^{-\mu-\frac{1}{2}}(zt)^{\frac{1}{2}}J_\mu(zt)D_\mu(x,y,z)dz = (xt)^{\frac{1}{2}}J_\mu(xt)(yt)^{\frac{1}{2}}J_\mu(yt),\ \ t \in \mathbb{R}^+. \quad (1.4.7)$$

(iii) $$\int_0^\infty D_\mu(x,y,z)z^{\mu+\frac{1}{2}}dz = \frac{(xy)^{\mu+\frac{1}{2}}}{2^\mu\Gamma(\mu+1)}. \quad (1.4.8)$$

If $x^{\mu+\frac{1}{2}}f(x) \in L^1(\mathbb{R}^+)$ and $x^{\mu+\frac{1}{2}}g(x) \in L^1(\mathbb{R}^+)$, then

$$h_\mu(f\#g)(x) = x^{-\mu-\frac{1}{2}}h_\mu f(x)h_\mu g(x). \quad (1.4.9)$$

From Wing [58] and Kerr [28], for $\mu \geq -\frac{1}{2}$, the Hankel transform of function $f \in L^2(\mathbb{R}^+)$ is defined by

$$(h_\mu f)(y) = \underset{N\to\infty}{l.i.m.} \int_0^N (xy)^{\frac{1}{2}}J_\mu(xy)f(x)dx, \quad (1.4.10)$$

and the corresponding inverse Hankel transform is

$$f(x) = \underset{N\to\infty}{l.i.m.} \int_0^N (xy)^{\frac{1}{2}}J_\mu(xy)h_\mu f(y)dy, \quad (1.4.11)$$

where $l.i.m.$ denote convergence in $L^2(\mathbb{R}^+)$.

h_μ is isometric on $L^2(\mathbb{R}^+)$, $h_\mu^{-1}h_\mu f = f$, then the Parseval's formula of the Hankel transformation for $f, g \in L^2(\mathbb{R}^+)$ is given by

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty (h_\mu f)(y)(h_\mu g)(y)dy. \tag{1.4.12}$$

1.5 Bessel Wavelet Transform

In this section, from the concept of Pathak and Dixit [42] and Upadhyay et al. [56], definition and properties of the continuous Bessel wavelet transform are given below:

A function $\psi \in L^2(\mathbb{R}^+)$ is called the basic Bessel wavelet if it satisfies the admissibility condition

$$C_{\mu,\psi} := \int_0^\infty \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}}d\omega < \infty, \quad \mu \geq -\frac{1}{2}. \tag{1.5.1}$$

The continuous Bessel wavelet transform of a function $f \in L^2(\mathbb{R}^+)$ with respect to the basic Bessel wavelet $\psi \in L^2(\mathbb{R}^+)$ is defined by

$$(B_\psi f)(b,a) = \int_0^\infty f(t)\tau_b\psi_a(t)dt = a^{\mu-\frac{1}{2}}\int_0^\infty f(t)\psi\Big(\frac{t}{a},\frac{b}{a}\Big)dt, \tag{1.5.2}$$

where

$$\psi\Big(\frac{t}{a},\frac{b}{a}\Big) = \int_0^\infty \psi(z)D_\mu\Big(\frac{t}{a},\frac{b}{a},z\Big)dz. \tag{1.5.3}$$

The relationship of $(B_\psi f)(b,a)$ with the Hankel transformation of f and ψ is

$$(B_\psi f)(b,a) = \int_0^\infty (bx)^{\frac{1}{2}}J_\mu(bx)x^{-\mu-\frac{1}{2}}(h_\mu f)(x)(h_\mu\psi)(ax)dx, \tag{1.5.4}$$

where $a > 0$, $b > 0$ and $\mu \geq -\frac{1}{2}$.

Remark 1.5.1. A function $\psi \in L^2(\mathbb{R}^+)$ is called the "Bessel" wavelet because the Bessel functions of first kind of order μ is heavily involved to construct the wavelet. It can be clarified from (1.4.5) and (1.5.3).

1.6 Zemanian Spaces and Their Duals

In this section, from Zemanian [61, 63] and Betancor et al. [9], the definitions of various test function spaces are given below:

H_μ is the space of all complex-valued function $\phi \in C^\infty(\mathbb{R}^+)$ such that

$$\gamma^{\mu}_{\alpha,\beta}(\phi) = \sup_{x\in\mathbb{R}^+} \left| x^\alpha \left(x^{-1}D_x\right)^\beta x^{-\mu-\frac{1}{2}}\phi(x)\right| < \infty, \qquad \mu \geq -\frac{1}{2},\ \alpha,\beta \in \mathbb{N}_0 \tag{1.6.1}$$

and H'_μ denotes its dual.

From Zemanian [60, 63], generalized Hankel transform is given by

$$\left\langle h'_\mu f, \phi \right\rangle = \left\langle f, h_\mu \phi \right\rangle, \quad \phi \in H_\mu(\mathbb{R}^+),\ f \in H'_\mu(\mathbb{R}^+), \tag{1.6.2}$$

where h_μ and h'_μ are automorphism on H_μ and H'_μ.

For $\mu \in \mathbb{R}$ and $c > 0$, the space $\beta_{\mu,c}$ consists of all complex-valued function $\phi \in C^\infty(\mathbb{R}^+)$ such that $\phi(x) = 0$, for $x \geq c$ and

$$\gamma^{\mu}_{k}(\phi) = \sup_{x\in\mathbb{R}^+} \left| \left(x^{-1}D_x\right)^k x^{-\mu-\frac{1}{2}}\phi(x)\right| < \infty, \quad k \in \mathbb{N}_0, \tag{1.6.3}$$

when endowed with the topology generated by the family of seminorms $\{\gamma^\mu_k\}_{k\in\mathbb{N}_0}$, $\beta_{\mu,c}$ becomes a Fréchet space. It is clear that $\beta_{\mu,c} \subset \beta_{\mu,d}$ provided that $c \leq d$. This fact allows to define $\beta_\mu = \bigcup_{c>0} \beta_{\mu,c}$ as the inductive limit of $\{\beta_{\mu,c}\}_{c>0}$. $\beta'_{\mu,c}$ and β'_μ denote duals of $\beta_{\mu,c}$ and β_μ, respectively.

Let $\eta = y + i\omega,\ y, \omega \in \mathbb{R}$ and $\mu, c \in \mathbb{R}$ with $c > 0$. Then Φ is an element of $\mathscr{Y}_{\mu,c}$ if and only if $\eta^{-\mu-\frac{1}{2}}\Phi$ is an even entire function of η such that

$$\alpha^{\mu}_{c,k}(\Phi) = \sup_{\eta} |e^{-c|\omega|}\eta^{2k-\mu-\frac{1}{2}}\Phi(\eta)| < \infty, \quad k \in \mathbb{N}_0. \tag{1.6.4}$$

The topology of $\mathscr{Y}_{\mu,c}$ is the one generated by using the seminorms $\alpha^{\mu}_{c,k},\ k = 0, 1, 2, 3, \cdots$. From Zemanian [61, p. 682, p. 685], it is clear that $\mathscr{Y}_{\mu,c}$ is a Fréchet space and $\mathscr{Y}_{\mu,c} \subset \mathscr{Y}_{\mu,d}$ for $c < d$. In view of the above fact, $\mathscr{Y}_{\mu} = \bigcup\limits_{c>0} \mathscr{Y}_{\mu,c}$ is the inductive limit of $\mathscr{Y}_{\mu,c},\ c > 0$. The dual space of $\mathscr{Y}_{\mu}$ is $\mathscr{Y}'_{\mu}$ which is the space of all continuous linear functional on $\mathscr{Y}_{\mu}$-space.

1.7 Some Useful Results

From Zemanian [61], Koh [29, 30], Hardy et al. [22] and Triebel [50], useful results and inequalities are discussed in this section.

From Hardy et al. [22, Theorem 201], for $p > 1$, the Minkowski integral inequality is given by

$$\left[\sum_{k=0}^{\infty}\left\{\int_0^{\infty} f_k(x)dx\right\}^p\right]^{\frac{1}{p}} \leq \int_0^{\infty}\left\{\sum_{k=0}^{\infty} f_k^p(x)\right\}^{\frac{1}{p}} dx. \tag{1.7.1}$$

From [50], for $1 \leq p < \infty$, $1 \leq q \leq \infty$ and if $\{f_k(x)\}_{k=0}^{\infty}$ is a sequence of complex-valued Borel measurable function on $\mathbb{R}^+$, then

$$\|f_k\|_{L^{\mu}_p(l_q)} = \left(\int_0^{\infty} x^{2\mu+1}\left(\sum_{k=0}^{\infty} |f_k(x)|^q\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}, \tag{1.7.2}$$

$$\|f_k\|_{l_q(L^{\mu}_p)} = \left(\sum_{k=0}^{\infty}\left(\int_0^{\infty} x^{2\mu+1}\,|f_k(x)|^p\,dx\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}, \tag{1.7.3}$$

with

$$\|f_k(x)\|_{l_\infty} = \sup_k |f_k(x)|\,, \tag{1.7.4}$$

and $f(x) = \Big(f_0(x), f_1(x), f_2(x), \ldots, f_k(x), \ldots\Big) \equiv \{f_k(x)\}_{k=0}^{\infty} \in l_q$, then

$$\|f(x)\|_{l_q} = \Big(\sum_{k=0}^{\infty} |f_k(x)|^q\Big)^{\frac{1}{q}}, \quad \|f_k\|_{L_p^\mu} = \Big(\int_0^\infty x^{2\mu+1}\,|f_k(x)|^p\,dx\Big)^{\frac{1}{p}}. \tag{1.7.5}$$

From Zemanian [63] and Koh [29], we have

$$D_x x^{-\mu} J_\mu(xy) = -yx^{-\mu} J_{\mu+1}(xy). \tag{1.7.6}$$

$$D_x x^{\mu+1} J_{\mu+1}(xy) = yx^{\mu+1} J_\mu(xy). \tag{1.7.7}$$

For $S_{\mu,x} \equiv D_x^2 - \frac{4\mu^2-1}{4x^2}$, the Bessel operator

$$h_\mu\,(S_{\mu,x}\phi) = -y^2 h_\mu \phi, \tag{1.7.8}$$

and the Bessel differential operator of order n be

$$S_{\mu,x}^n f(x) = \sum_{j=0}^{n} A_j x^{2j+\mu+\frac{1}{2}} \left(x^{-1}D_x\right)^{n+j} x^{-\mu-\frac{1}{2}} f(x), \tag{1.7.9}$$

where A_j are constants depending on j and μ.

$$\left(x^{-1}D_x\right)^n (f(x)\,g(x)) = \sum_{m=0}^{n} \binom{n}{m} \left(x^{-1}D_x\right)^m f(x) \left(x^{-1}D_x\right)^{n-m} g(x). \tag{1.7.10}$$

Theorem 1.7.1. *For every $x \in \mathbb{R}^+$, the mapping $\phi \to \tau_x\phi$ is continuous from H_μ into itself.*

Theorem 1.7.2. *For every $x \in \mathbb{R}^+$, the mapping $\phi \to S_{\mu,x}\phi$ is continuous from H_μ into itself.*

Theorem 1.7.3. *For every $x \in \mathbb{R}^+$, the mapping $\phi \to \tau_x \phi$ is continuous from $\beta_{\mu,a}$ into $\beta_{\mu,a+x}$.*

Theorem 1.7.4. *For $\mu \geq -\frac{1}{2}$, h_μ is an isomorphism from $\beta_{\mu,b}$ onto $\mathscr{Y}_{\mu,b}$.*

Theorem 1.7.5. *For $\mu \geq -\frac{1}{2}$, h_μ is an isomorphism from β_μ onto $\mathscr{Y}_\mu$.*

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