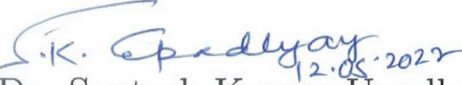


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It is certified that the work contained in the thesis titled "*Certain Wavelet Transforms of Distributions*" by *Jay Singh Maurya* has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy and SOTA for the award of Ph.D. degree.

  
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## DECLARATION BY THE CANDIDATE

I, *Jay Singh Maurya*, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of *Prof. Santosh Kumar Upadhyay* from *July, 2017* to *May, 2022* at the *Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi*. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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# List of Symbols and Abbreviations

|  |   |
|--|---|
| $\mathbb{N}$                             | Set of natural numbers                          |
| $\mathbb{N}_0$                           | Set of non-negative integers                    |
| $\mathbb{R}^+$                           | Open interval $(0, \infty)$                     |
| $\mathbb{R}$                             | Set of real numbers                             |
| $\mathbb{R}^n$                           | Usual Euclidean space of dimension $n$          |
| $\mathbb{C}$                             | Set of complex numbers                          |
| $E(x)$ or $[x]$                          | Integer part of $x$                             |
| $\ x\ $                                  | Norm of $x$                                     |
| $D_x \equiv \frac{\partial}{\partial x}$ | Partial derivative with respect to variable $x$ |
| <i>a.e.</i>                              | Almost everywhere                               |
| R.H.S.                                   | Right hand side                                 |
| L.H.S.                                   | Left hand side                                  |