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A. Appendix A

A.1 Proof of the Lemma 1.4

Proof. As the part (ii) is clearly followed from part (i), we provide the proof only for the part (i).

Let $\mathbf{A} = [\underline{a}, \bar{a}]$ and $\mathbf{B} = [\underline{b}, \bar{b}]$.

Here we recall the representation (1.2) and

$$\mathbf{A} \ominus_{gH} \mathbf{B} = \left[\min\left\{\underline{a} - \underline{b}, \bar{a} - \bar{b}\right\}, \max\left\{\underline{a} - \underline{b}, \bar{a} - \bar{b}\right\}\right].$$

Let $\mathbf{A} \preceq \mathbf{B}$. Then, by Definition 1.4.2, we note that

$$\begin{aligned}
&\mathbf{A} \preceq \mathbf{B} \\
&\Longrightarrow \underline{a} + t(\bar{a} - \underline{a}) = a(t) \leq b(t) = \underline{b} + t(\bar{b} - \underline{b}) \text{ for all } t \in [0, 1] \\
&\Longrightarrow a(0) \leq b(0) \text{ and } \ \bar{a}(1) \leq \bar{b}(1) \\
&\Longrightarrow \underline{a} \leq \underline{b} \text{ and } \ \bar{a} \leq \bar{b} \\
&\Longrightarrow \underline{a} - \underline{b} \leq 0 \text{ and } \bar{a} - \bar{b} \leq 0 \\
&\Longrightarrow \mathbf{A} \ominus_{gH} \mathbf{B} \preceq \mathbf{0}.
\end{aligned}$$

Conversely, let $\mathbf{A} \ominus_{gH} \mathbf{B} \preceq \mathbf{0}$. Then, $\underline{a} - \underline{b} \leq 0$ and $\bar{a} - \bar{b} \leq 0$, i.e., $\underline{a} \leq \underline{b}$ and $\bar{a} \leq \bar{b}$.

Depending on $\underline{b} < \bar{a}$ or $\bar{a} \leq \underline{b}$, we break the analysis into two cases.

- *Case* 1. Let $\underline{b} < \bar{a}$.

Then, $\underline{a} \leq \underline{b} < \bar{a} \leq \bar{b}$. We prove that $a(t) \leq b(t)$ for all $t \in [0, 1]$.

On contrary, let there exists $t_0 \in [0, 1]$, such that $a(t_0) > b(t_0)$.

Since $\underline{a} \leq \underline{b}$ and $\bar{a} \leq \bar{b}$, therefore $t_0 \neq 0$ and $t_0 \neq 1$. Thus, $\frac{1}{t_0} > 1$.

Note that from $a(t_0) = \underline{a} + t_0(\bar{a} - \underline{a})$, we have

$$\bar{a} = \tfrac{1}{t_0}\, a(t_0) - \left(\tfrac{1}{t_0} - 1\right) \underline{a}.$$

Similarly

$$\bar{b} = \tfrac{1}{t_0}\, b(t_0) - \left(\tfrac{1}{t_0} - 1\right) \underline{b}.$$

As $a(t_0) > b(t_0)$, $\frac{1}{t_0} > 1$ and $\underline{a} \leq \underline{b}$, we see that

$$\bar{a} \;=\; \tfrac{1}{t_0}\, a(t_0) - \left(\tfrac{1}{t_0} - 1\right) \underline{a} \;>\; \tfrac{1}{t_0}\, b(t_0) - \left(\tfrac{1}{t_0} - 1\right) \underline{b} = \bar{b}.$$

This is contradictory to $\bar{a} \leq \bar{b}$. Hence, for any $t \in [0, 1]$, $a(t) \leq b(t)$. Thus, $\mathbf{A} \preceq \mathbf{B}$.

- *Case* 2. Let $\bar{a} \leq \underline{b}$.

Since $a(t)$ and $b(t)$ are increasing functions, for any $t \in [0, 1]$ we have

$$a(t) \leq a(1) = \bar{a} \leq \underline{b} = b(0) \leq b(t).$$

Hence, $\mathbf{A} \preceq \mathbf{B}$ and the proof is complete. □

A.2 Proof of the Lemma 1.5

Proof. Let $\mathbf{A} = [\underline{a}, \overline{a}]$, $\mathbf{B} = [\underline{b}, \overline{b}]$, $\mathbf{C} = [\underline{c}, \overline{c}]$ and $\mathbf{D} = [\underline{d}, \overline{d}]$.

(i) Suppose the inequality $\mathbf{B} \nprec \mathbf{A} \ominus_{gH} (\mathbf{A} \ominus_{gH} \mathbf{B})$ is not true. Then,

$$\mathbf{B} \prec \mathbf{A} \ominus_{gH} (\mathbf{A} \ominus_{gH} \mathbf{B}). \tag{A.1}$$

Now, we have the following two cases.

- **Case** 1. If $\underline{a}-\underline{b} \leq \overline{a}-\overline{b}$, then $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a}-\underline{b}, \overline{a}-\overline{b}]$ and

$$\mathbf{A} \ominus_{gH} (\mathbf{A} \ominus_{gH} \mathbf{B}) = [\underline{b}, \overline{b}] = \mathbf{B},$$

which is contradictory to (A.1).

- **Case** 2. If $\overline{a}-\overline{b} < \underline{a}-\underline{b}$, then $\mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a}-\overline{b}, \underline{a}-\underline{b}]$ and we have the following two possibilities:

If $\mathbf{A} \ominus_{gH} (\mathbf{A} \ominus_{gH} \mathbf{B}) = [\underline{a}-(\overline{a}-\overline{b}), \overline{a}-(\underline{a}-\underline{b})]$, by (A.1), we have

$$\overline{b} \leq \overline{a}-(\underline{a}-\underline{b}) \implies \underline{a}-\underline{b} \leq \overline{a}-\overline{b},$$

which contradicts to $\overline{a}-\overline{b} < \underline{a}-\underline{b}$.

If $\mathbf{A} \ominus_{gH} (\mathbf{A} \ominus_{gH} \mathbf{B}) = [\overline{a}-(\underline{a}-\underline{b}), \underline{a}-(\overline{a}-\overline{b})]$, by (A.1), we get

$$\overline{b} \leq \underline{a}-(\overline{a}-\overline{b}) \implies \overline{a} \leq \underline{a} \implies \overline{a} = \underline{a},$$

and we have $\mathbf{A} \ominus_{gH} (\mathbf{A} \ominus_{gH} \mathbf{B}) = \mathbf{B}$, which contradicts (A.1).

Hence, from **Case** 1 and **Case** 2, we obtain $\mathbf{B} \nprec \mathbf{A} \ominus_{gH} (\mathbf{A} \ominus_{gH} \mathbf{B})$.

(ii) Since

$$\mathbf{0} \prec \mathbf{A} \implies 0 \leq \underline{a} \text{ and } 0 < \overline{a},$$

for any $\mathbf{C} \in I(\mathbb{R})$, we have

$$-\overline{c} \leq -\underline{c} \leq \underline{a}-\underline{c} \text{ and } -\overline{c} < \overline{a}-\overline{c}.$$

Therefore, we obtain

$$[-\overline{c}, -\underline{c}] \prec [\min\{\underline{a}-\underline{c}, \overline{a}-\overline{c}\}, \max\{\underline{a}-\underline{c}, \overline{a}-\overline{c}\}]$$
$$\implies (-1) \odot \mathbf{C} \prec \mathbf{A} \ominus_{gH} \mathbf{C}.$$

Hence, for any $\mathbf{B} \in I(\mathbb{R})$, we have

$$\mathbf{B} \not\prec \mathbf{A} \ominus_{gH} \mathbf{C} \implies \mathbf{B} \not\prec (-1) \odot \mathbf{C}.$$

(iii) We have the following four possible cases.

- **Case** 1. Let $\overline{a}-\overline{c} \geq \underline{a}-\underline{c}$ and $\overline{c}-\overline{b} \geq \underline{c}-\underline{b}$. Then, $\overline{a}-\overline{b} \geq \underline{a}-\underline{b}$ and

$$(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\underline{a}-\underline{c}, \overline{a}-\overline{c}] \oplus [\underline{c}-\underline{b}, \overline{c}-\overline{b}]$$
$$\implies \quad (\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\underline{a}-\underline{b}, \overline{a}-\overline{b}] = \mathbf{A} \ominus_{gH} \mathbf{B}.$$

- **Case** 2. Let $\overline{a}-\overline{c} \leq \underline{a}-\underline{c}$ and $\overline{c}-\overline{b} \leq \underline{c}-\underline{b}$. Therefore, $\overline{a}-\overline{b} \leq \underline{a}-\underline{b}$ and

$$(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\overline{a}-\overline{c}, \underline{a}-\underline{c}] \oplus [\overline{c}-\overline{b}, \underline{c}-\underline{b}]$$
$$\implies \quad (\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\overline{a}-\overline{b}, \underline{a}-\underline{b}] = \mathbf{A} \ominus_{gH} \mathbf{B}.$$

- **Case** 3. Let $\overline{a}-\overline{c} < \underline{a}-\underline{c}$ and $\overline{c}-\overline{b} > \underline{c}-\underline{b}$. Therefore,

$$(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\overline{a}-\overline{c}, \underline{a}-\underline{c}] \oplus [\underline{c}-\underline{b}, \overline{c}-\overline{b}]$$
$$\implies \quad (\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\overline{a}-\overline{c}+\underline{c}-\underline{b}, \underline{a}-\underline{c}+\overline{c}-\overline{b}].$$

If possible, let

$$(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) \prec \mathbf{A} \ominus_{gH} \mathbf{B}. \tag{A.2}$$

If $\overline{a} - \overline{b} \geq \underline{a} - \underline{b}$, then from (A.2) we get

$$\begin{aligned} & [\overline{a} - \overline{c} + \underline{c} - \underline{b}, \underline{a} - \underline{c} + \overline{c} - \overline{b}] \prec [\underline{a} - \underline{b}, \overline{a} - \overline{b}] \\ \implies & \underline{a} - \underline{c} + \overline{c} - \overline{b} \leq \overline{a} - \overline{b} \\ \implies & \underline{a} - \underline{c} \leq \overline{a} - \overline{c}, \quad \text{which is an impossibility.} \end{aligned}$$

Further, if $\overline{a} - \overline{b} \leq \underline{a} - \underline{b}$, then from (A.2), we have

$$\begin{aligned} & [\overline{a} - \overline{c} + \underline{c} - \underline{b}, \underline{a} - \underline{c} + \overline{c} - \overline{b}] \prec [\overline{a} - \overline{b}, \underline{a} - \underline{b}] \\ \implies & \underline{a} - \underline{c} + \overline{c} - \overline{b} \leq \underline{a} - \underline{b} \\ \implies & \overline{c} - \overline{b} \leq \underline{c} - \underline{b}, \quad \text{which is an impossibility.} \end{aligned}$$

Thus, (A.2) is not true.

- **Case** 4. Let $\overline{a} - \overline{c} > \underline{a} - \underline{c}$ and $\overline{c} - \overline{b} < \underline{c} - \underline{b}$. Proceeding as in **Case** 3 of (iii) we can prove that (A.2) is not true. Hence,

$$(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) \not\prec \mathbf{A} \ominus_{gH} \mathbf{B}.$$

(iv) If $\mathbf{A} \not\prec \mathbf{0}$, then $\overline{a} \geq 0$. Since $\mathbf{A} \preceq \mathbf{B}$, $\overline{b} \geq 0$. Thus, $\mathbf{B} \not\prec \mathbf{0}$.

(v) According to the dominance of intervals, we have

$$\begin{aligned} & \mathbf{A} \ominus_{gH} \mathbf{B} \not\prec \mathbf{0} \\ \implies & \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\} \geq 0 \\ \implies & \underline{a} - \underline{b} \geq 0 \text{ or, } \overline{a} - \overline{b} \geq 0. \end{aligned} \tag{A.3}$$

Since $\mathbf{C} \preceq \mathbf{B}$,

$$\underline{c} \leq \underline{b} \text{ and } \overline{c} \leq \overline{b} \quad \implies \quad \underline{b} - \underline{c} \geq 0 \text{ and } \overline{b} - \overline{c} \geq 0. \tag{A.4}$$

From (A.3) and (A.4), we have

$$\text{either } \underline{a} - \underline{c} \geq 0 \text{ or } \overline{a} - \overline{c} \geq 0.$$

Therefore,

$$\mathbf{A} \ominus_{gH} \mathbf{C} \not\prec \mathbf{0}.$$

(vi) Since $\mathbf{C} \preceq \mathbf{B}$, we have

$$\begin{aligned}
& \underline{c} \leq \underline{b} \text{ and } \overline{c} \leq \overline{b} \\
\implies \quad & \underline{a} - \underline{c} \geq \underline{a} - \underline{b} \text{ and } \overline{a} - \overline{c} \geq \overline{a} - \overline{b} \\
\implies \quad & [\min\{\underline{a} - \underline{c}, \overline{a} - \overline{c}\}, \max\{\underline{a} - \underline{c}, \overline{a} - \overline{c}\}] \\
& \succeq [\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}] \\
\implies \quad & \mathbf{A} \ominus_{gH} \mathbf{B} \preceq \mathbf{A} \ominus_{gH} \mathbf{C}.
\end{aligned}$$

□

A.3 Proof of the Lemma 1.6

Proof. Let $\mathbf{A} = [\underline{a}, \overline{a}]$, $\mathbf{B} = [\underline{b}, \overline{b}]$, $\mathbf{C} = [\underline{c}, \overline{c}]$ and $\mathbf{D} = [\underline{d}, \overline{d}]$.

(i) If possible, let the inequality (i) be not true. Therefore, there exists a pair of intervals $\mathbf{A}$ and $\mathbf{B}$ for which

$$\|\mathbf{A}\|_{I(\mathbb{R})} - \|\mathbf{B}\|_{I(\mathbb{R})} > \|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})}.$$

Then,

$$\|\mathbf{A}\|_{I(\mathbb{R})} > \|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} + \|\mathbf{B}\|_{I(\mathbb{R})} \geq \|(\mathbf{A} \ominus_{gH} \mathbf{B}) \oplus \mathbf{B}\|_{I(\mathbb{R})}$$

$$\text{i.e.,} \quad \|\mathbf{A}\|_{I(\mathbb{R})} > \|(\mathbf{A} \ominus_{gH} \mathbf{B}) \oplus \mathbf{B}\|_{I(\mathbb{R})}. \tag{A.5}$$

According to the definition of gH-difference, we have

$$\text{either } \mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}] \tag{A.6}$$

$$\text{or } \quad \mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]. \tag{A.7}$$

If (A.6) is true, then

$$(\mathbf{A} \ominus_{gH} \mathbf{B}) \oplus \mathbf{B} = [\underline{a} - \underline{b} + \underline{b}, \overline{a} - \overline{b} + \overline{b}] = [\underline{a}, \overline{a}] = \mathbf{A}$$

$$\text{i.e.,} \quad \|\mathbf{A}\|_{I(\mathbb{R})} = \|(\mathbf{A} \ominus_{gH} \mathbf{B}) \oplus \mathbf{B}\|_{I(\mathbb{R})},$$

which contradicts (A.5).

If (A.7) is true, then

$$(\mathbf{A} \ominus_{gH} \mathbf{B}) \oplus \mathbf{B} = [\overline{a} - \overline{b} + \underline{b}, \underline{a} - \underline{b} + \overline{b}]. \tag{A.8}$$

We now consider the following two cases.

- **Case** 1. Let $\|A\|_{I(\mathbb{R})} = |\underline{a}|$.

Since $\underline{a} \leq \overline{a}$ and $|\underline{a}| \geq |\overline{a}|$, $\underline{a}$ must be nonpositive, i.e., $\underline{a} \leq 0$.

In view of the relations (A.5) and (A.8), we have

$$\begin{aligned} &\mid \underline{a} \mid > \max\{\mid \overline{a} - \overline{b} + \underline{b} \mid, \mid \underline{a} - \underline{b} + \overline{b} \mid\} \\ \text{i.e., } &\mid \underline{a} \mid > \mid \overline{a} - \overline{b} + \underline{b} \mid. \end{aligned} \tag{A.9}$$

By (A.7), we have $\overline{a} - \overline{b} \leq \underline{a} - \underline{b}$, or, $\overline{a} - \overline{b} + \underline{b} \leq \underline{a} \leq 0$.

Therefore,

$$\mid \underline{a} \mid \leq \mid \overline{a} - \overline{b} + \underline{b} \mid,$$

which contradicts the relation (A.9).

- **Case** 2. Let $\|A\|_{I(\mathbb{R})} = |\overline{a}|$.

Then, $\overline{a} \geq 0$ and from (A.5) and (A.8) we obtain

$$\mid \overline{a} \mid > \max\{\mid \overline{a} - \overline{b} + \underline{b} \mid, \mid \underline{a} - \underline{b} + \overline{b} \mid\}.$$

Thus,

$$\mid \overline{a} \mid > \mid \underline{a} - \underline{b} + \overline{b} \mid. \tag{A.10}$$

According to (A.7) we have $\overline{a} - \overline{b} \leq \underline{a} - \underline{b}$, which implies $0 \leq \overline{a} \leq \underline{a} - \underline{b} + \overline{b}$.

Therefore,

$$\mid \overline{a} \mid \leq \mid \underline{a} - \underline{b} + \overline{b} \mid,$$

which contradicts the relation (A.10).

Hence, (i) must be true for all $\mathbf{A},\ \mathbf{B} \in I(\mathbb{R})$.

(ii) If possible, let the inequality (ii) be not true. Therefore, there exist three intervals $\mathbf{A}, \mathbf{B}$ and $\mathbf{C} = [\underline{c}, \overline{c}]$ such that

$$\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} > \|(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B})\|_{I(\mathbb{R})}. \tag{A.11}$$

According to the definition of gH-difference of two intervals,

$$\text{either} \quad \mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}] \quad \text{or} \quad \mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]. \tag{A.12}$$

Similarly,

$$\text{either} \quad \mathbf{A} \ominus_{gH} \mathbf{C} = [\underline{a} - \underline{c}, \overline{a} - \overline{c}] \quad \text{or} \quad \mathbf{A} \ominus_{gH} \mathbf{C} = [\overline{a} - \overline{c}, \underline{a} - \underline{c}]$$

and

$$\mathbf{C} \ominus_{gH} \mathbf{B} = [\underline{c} - \underline{b}, \overline{c} - \overline{b}] \quad \text{or} \quad \mathbf{C} \ominus_{gH} \mathbf{B} = [\overline{c} - \overline{b}, \underline{c} - \underline{b}].$$

Then, one of the following holds true:

(a) $(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$

(b) $(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\underline{a} - \underline{c} + \overline{c} - \overline{b}, \overline{a} - \overline{c} + \underline{c} - \underline{b}]$

(c) $(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\overline{a} - \overline{c} + \underline{c} - \underline{b}, \underline{a} - \underline{c} + \overline{c} - \overline{b}]$

(d) $(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]$.

- **Case** 1. Let $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$ and $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} = |\underline{a} - \underline{b}|$. Then, $\underline{a} - \underline{b} \leq 0$.

  (a) If $(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$, then

  $$\|(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B})\|_{I(\mathbb{R})} = |\underline{a} - \underline{b}| = \|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})},$$

  which is a contradiction to the inequality (A.11).

(b) Let $(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\underline{a} - \underline{c} + \overline{c} - \overline{b}, \overline{a} - \overline{c} + \underline{c} - \underline{b}]$ which has came from the fact that $\mathbf{A} \ominus_{gH} \mathbf{C} = [\underline{a} - \underline{c}, \overline{a} - \overline{c}]$ and $\mathbf{C} \ominus_{gH} \mathbf{B} = [\overline{c} - \overline{b}, \underline{c} - \underline{b}]$. Thus,

$$\underline{a} - \underline{c} \leq \overline{a} - \overline{c} \quad \text{and} \quad \overline{c} - \overline{b} \leq \underline{c} - \underline{b}. \tag{A.13}$$

From the inequality (A.11), we obtain

$$|\underline{a} - \underline{b}| > |\underline{a} - \underline{c} + \overline{c} - \overline{b}| \text{ and } |\underline{a} - \underline{b}| > |\overline{a} - \overline{c} + \underline{c} - \underline{b}|. \tag{A.14}$$

Since $\underline{a} - \underline{b} \leq 0$, irrespective of $(\underline{a} - \underline{c} + \overline{c} - \overline{b})$ is nonnegative or nonpositive, we get from the first inequality of (A.14) that

$$\underline{a} - \underline{b} = -|\underline{a} - \underline{b}| < -|\underline{a} - \underline{c} + \overline{c} - \overline{b}| \leq \underline{a} - \underline{c} + \overline{c} - \overline{b}.$$

Hence, $\underline{c} - \underline{b} < \overline{c} - \overline{b}$, which is a contradiction to the inequality (A.13).

(c) If $(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\overline{a} - \overline{c} + \underline{c} - \underline{b}, \underline{a} - \underline{c} + \overline{c} - \overline{b}]$, then proceeding similar to the **Case** 1(b), we arrive at the contradicting inequality $\underline{a} - \underline{c} < \overline{a} - \overline{c}$.

(d) If $(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]$, then

$$\|(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B})\|_{I(\mathbb{R})} = |\underline{a} - \underline{b}| = \|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})},$$

which is a contradiction to the inequality (A.11).

- **Case** 2. Let $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$ and $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} = |\overline{a} - \overline{b}|$. Then, $\overline{a} - \overline{b} \geq 0$.

(a) If $(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$, then

$$\|(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B})\|_{I(\mathbb{R})} = |\overline{a} - \overline{b}| = \|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})},$$

which is a contradiction to the inequality (A.11).

(b) If $(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\underline{a} - \underline{c} + \overline{c} - \overline{b}, \overline{a} - \overline{c} + \underline{c} - \underline{b}]$, then proceeding similar to the **Case** 1(b), we arrive at the contradicting inequality $\overline{c} - \overline{b} > \underline{c} - \underline{b}$.

(c) If $(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\overline{a} - \overline{c} + \underline{c} - \underline{b}, \underline{a} - \underline{c} + \overline{c} - \overline{b}]$, then then proceeding similar to the **Case** 1(b), we arrive at the contradicting inequality $\overline{a} - \overline{c} > \underline{a} - \underline{c}$.

(d) If $(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]$, the

$$\|(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B})\|_{I(\mathbb{R})} = |\underline{a} - \underline{b}| = \|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})},$$

which is a contradiction to the inequality (A.11).

- **Case** 3. Let $\mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]$ and $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} = |\overline{a} - \overline{b}|$.
  All the four subcases for this case are similar to the **Case** 2.

- **Case** 4. Let $\mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]$ and $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} = |\underline{a} - \underline{b}|$.
  All the four subcases for this case are similar to the **Case** 1.

We notice that in all the possible subcases of the above four possible cases we arrive at a contradiction to the inequality (A.11). Therefore, for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in I(\mathbb{R})$,

$$\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} \leq \|(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B})\|_{I(\mathbb{R})}.$$

(iii) As $\|\mathbf{B} \ominus_{gH} \mathbf{A}\|_{I(\mathbb{R})} = \max\{|\underline{b} - \underline{a}|, |\overline{b} - \overline{a}|\}$, we break the proof in two cases.

- **Case** 1. If $(L =) \|\mathbf{B} \ominus_{gH} \mathbf{A}\|_{I(\mathbb{R})} = |\underline{b} - \underline{a}|$, then

$$|\underline{b} - \underline{a}| \geq |\overline{b} - \overline{a}| \implies |\underline{b} - \underline{a}| \geq \overline{b} - \overline{a} \implies \overline{b} \leq \overline{a} + L. \quad \text{(A.15)}$$

Since $\underline{b} - \underline{a} \leq |\underline{b} - \underline{a}|$, then

$$\underline{b} \leq \underline{a} + L. \quad \text{(A.16)}$$

From (A.15) and (A.16), we have

$$\mathbf{B} \preceq \mathbf{A} \oplus [L, L].$$

- **Case** 2. If $(L =) \|\mathbf{B} \ominus_{gH} \mathbf{A}\|_{I(\mathbb{R})} = |\overline{b} - \overline{a}|$, then

$$|\underline{b} - \underline{a}| \leq |\overline{b} - \overline{a}| \implies \underline{b} - \underline{a} \leq |\overline{b} - \overline{a}| \implies \underline{b} \leq \underline{a} + L. \quad \text{(A.17)}$$

Since $\overline{b} - \overline{a} \leq |\overline{b} - \overline{a}|$,

$$\overline{b} \leq \overline{a} + L. \quad \text{(A.18)}$$

From (A.17) and (A.18), we obtain

$$\mathbf{B} \preceq \mathbf{A} \oplus [L, L], \text{ where } L = \|\mathbf{B} \ominus_{gH} \mathbf{A}\|_{I(\mathbb{R})}.$$

(iv) If possible, let there exist $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ in $I(\mathbb{R})$ such that

$$\|(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D})\|_{I(\mathbb{R})} > \|\mathbf{A} \ominus_{gH} \mathbf{C}\|_{I(\mathbb{R})} \oplus \|\mathbf{B} \ominus_{gH} \mathbf{D}\|_{I(\mathbb{R})}. \quad \text{(A.19)}$$

According to the definition of gH-difference of two intervals,

$$\text{either}\quad \mathbf{A}\ominus_{gH}\mathbf{B}=[\underline{a}-\underline{b},\overline{a}-\overline{b}]\quad\text{or}\quad \mathbf{A}\ominus_{gH}\mathbf{B}=[\overline{a}-\overline{b},\underline{a}-\underline{b}], \tag{A.20}$$

$$\text{either}\quad \mathbf{C}\ominus_{gH}\mathbf{D}=[\underline{c}-\underline{d},\overline{c}-\overline{d}]\quad\text{or}\quad \mathbf{C}\ominus_{gH}\mathbf{D}=[\overline{c}-\overline{d},\underline{c}-\underline{d}],$$

$$\text{either}\quad \mathbf{A}\ominus_{gH}\mathbf{C}=[\underline{a}-\underline{c},\overline{a}-\overline{c}]\quad\text{or}\quad \mathbf{A}\ominus_{gH}\mathbf{B}=[\overline{a}-\overline{c},\underline{a}-\underline{c}], \tag{A.21}$$

and

$$\text{either}\quad \mathbf{B}\ominus_{gH}\mathbf{D}=[\underline{b}-\underline{d},\overline{b}-\overline{d}]\quad\text{or}\quad \mathbf{B}\ominus_{gH}\mathbf{D}=[\overline{b}-\overline{d},\underline{b}-\underline{d}].$$

Then, one of the following holds true:

(a) $(\mathbf{A}\ominus_{gH}\mathbf{B})\ominus_{gH}(\mathbf{C}\ominus_{gH}\mathbf{D})=[\underline{a}-\underline{b}-\underline{c}+\underline{d},\ \overline{a}-\overline{b}-\overline{c}+\overline{d}]$

(b) $(\mathbf{A}\ominus_{gH}\mathbf{B})\ominus_{gH}(\mathbf{C}\ominus_{gH}\mathbf{D})=[\underline{a}-\underline{b}-\overline{c}+\overline{d},\ \overline{a}-\overline{b}-\underline{c}+\underline{d}]$

(c) $(\mathbf{A}\ominus_{gH}\mathbf{B})\ominus_{gH}(\mathbf{C}\ominus_{gH}\mathbf{D})=[\overline{a}-\overline{b}-\overline{c}+\overline{d},\ \underline{a}-\underline{b}-\underline{c}+\underline{d}]$

(d) $(\mathbf{A}\ominus_{gH}\mathbf{B})\ominus_{gH}(\mathbf{C}\ominus_{gH}\mathbf{D})=[\overline{a}-\overline{b}-\underline{c}+\underline{d},\ \underline{a}-\underline{b}-\overline{c}+\overline{d}]$

- **Case** 1. Let $(\mathbf{A}\ominus_{gH}\mathbf{B})\ominus_{gH}(\mathbf{C}\ominus_{gH}\mathbf{D})=[\underline{a}-\underline{b}-\underline{c}+\underline{d},\ \overline{a}-\overline{b}-\overline{c}+\overline{d}]$.

  (a) If $\|(\mathbf{A}\ominus_{gH}\mathbf{B})\ominus_{gH}(\mathbf{C}\ominus_{gH}\mathbf{D})\|_{I(\mathbb{R})}=|\underline{a}-\underline{b}-\underline{c}+\underline{d}|$, then from equation (A.19), we have

  $$|\underline{a}-\underline{b}-\underline{c}+\underline{d}|>|\underline{a}-\underline{c}|+|\underline{b}-\underline{d}|>|\underline{a}-\underline{b}-\underline{c}+\underline{d}|,$$

  which is impossible.

  (b) If $\|(\mathbf{A}\ominus_{gH}\mathbf{B})\ominus_{gH}(\mathbf{C}\ominus_{gH}\mathbf{D})\|_{I(\mathbb{R})}=|\overline{a}-\overline{b}-\overline{c}+\overline{d}|$, then from equation (A.19), we have

  $$|\overline{a}-\overline{b}-\overline{c}+\overline{d}|>|\overline{a}-\overline{c}|+|\overline{b}-\overline{d}|>|\overline{a}-\overline{b}-\overline{c}+\overline{d}|,$$

which is again impossible.

- **Case** 2. Let $(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D}) = [\overline{a} - \overline{b} - \overline{c} + \overline{d},\ \underline{a} - \underline{b} - \underline{c} + \underline{d}]$. For this case, two subcases are similar to the **Case** 1 will lead to impossibilities.

- **Case** 3. Let $(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D}) = [\underline{a} - \underline{b} - \overline{c} + \overline{d},\ \overline{a} - \overline{b} - \underline{c} + \underline{d}]$. Then,

$$\underline{a} - \underline{b} \leq \overline{a} - \overline{b} \text{ and } \overline{c} + \overline{d} \leq \underline{c} + \underline{d}. \tag{A.22}$$

  (a) If $\|(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D})\|_{I(\mathbb{R})} = |\overline{a} - \overline{b} - \underline{c} + \underline{d}|$, then $\overline{a} - \overline{b} - \underline{c} + \underline{d} \geq 0$. From equation (A.19), we have

$$|\overline{a} - \overline{b} - \underline{c} + \underline{d}| > |\overline{a} - \overline{c}| + |\overline{b} - \overline{d}| \implies \overline{c} + \overline{d} > \underline{c} + \underline{d},$$

  which is contradictory to (A.22).

  (b) If $\|(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D})\|_{I(\mathbb{R})} = |\underline{a} - \underline{b} - \overline{c} + \overline{d}|$, then $\underline{a} - \underline{b} - \overline{c} + \overline{d} < 0$. From equation (A.19), we have

$$-(\underline{a}-\underline{b}-\overline{c}+\overline{d}) = |\underline{a}-\underline{b}-\overline{c}+\overline{d}| > |\underline{a}-\underline{c}|+|\underline{b}-\underline{d}| \implies \overline{c}+\overline{d} > \underline{c}+\underline{d},$$

  which is again contradictory to (A.22).

- **Case** 4. Let $(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D}) = [\overline{a} - \overline{b} - \underline{c} + \underline{d},\ \underline{a} - \underline{b} - \overline{c} + \overline{d}]$. All the two subcases for this case are similar to **Case** 3.

Hence, (A.19) is wrong, and thus the result follows.

□

A.4 Proof of the Lemma 1.7

Proof. Let $\mathbf{C} = [\underline{c},\ \overline{c}]$.

(i) If $\mathbf{C} \succeq \mathbf{0}$, then

$$\begin{aligned} & \underline{c} \geq 0 \text{ and } \overline{c} \geq 0 \\ \Longrightarrow\ & |x|\underline{c} + |y|\underline{c} \geq |x+y|\underline{c} \text{ and } |x|\overline{c} + |y|\overline{c} \geq |x+y|\overline{c} \\ \Longrightarrow\ & |x+y| \odot \mathbf{C} \preceq |x| \odot \mathbf{C} \oplus |y| \odot \mathbf{C}. \end{aligned}$$

(ii) If $\mathbf{C} \preceq \mathbf{0}$, then

$$\begin{aligned} & \underline{c} \leq 0 \text{ and } \overline{c} \leq 0 \\ \Longrightarrow\ & |x|\underline{c} + |y|\underline{c} \leq |x+y|\underline{c} \text{ and } |x|\overline{c} + |y|\overline{c} \leq |x+y|\overline{c} \\ \Longrightarrow\ & |x+y| \odot \mathbf{C} \succeq |x| \odot \mathbf{C} \oplus |y| \odot \mathbf{C}. \end{aligned}$$

(iii) If $\mathbf{C} \not\prec \mathbf{0}$, then

$$\overline{c} \geq 0 \implies |x|\overline{c} + |y|\overline{c} \geq |x+y|\overline{c} \implies |x+y| \odot \mathbf{C} \not\succ |x| \odot \mathbf{C} \oplus |y| \odot \mathbf{C}.$$

□

A.5 Proof of the Lemma 1.10

Proof. (i) If

$$\mathbf{F}(x) \not\prec \mathbf{0} \text{ for all } x \in \mathcal{S}, \tag{A.23}$$

then due to linearity of $\mathbf{F}$, we have

$$\mathbf{F}(x) = (-1) \odot \mathbf{F}(-x) \not\succ \mathbf{0} \text{ for all } x \in \mathcal{S} \tag{A.24}$$

since $\mathbf{F}(-x) \not\prec \mathbf{0}$ by (A.23). From (A.23) and (A.24), it is clear that $\mathbf{0}$ and $\mathbf{F}(x)$ are not comparable.

(ii) If $\mathbf{F}(x) \preceq \mathbf{0}$ for all $x \in \mathcal{S}$, then due to linearity of $\mathbf{F}$, we have $\mathbf{F}(x) = (-1) \odot \mathbf{F}(-x) \succeq \mathbf{0}$ for all $x \in \mathcal{S}$.

Hence, $\mathbf{F}(x) = \mathbf{0}$.

□

B. Appendix B

B.1 Proof of the Lemma 2.7

Proof. First we show that

$$\mathbf{F}(\lambda(x_1, x_2)) = \lambda \odot \mathbf{F}(x_1, x_2) \text{ for all } \lambda \in \mathbb{R}.$$

- **Case** 1. Let $\lambda < 0$. For this case, there are following four subcases.

(a) If $x_1 < 0$ and $x_2 < 0$, then $\lambda x_1 > 0$ and $\lambda x_2 > 0$. Therefore,

$$\begin{aligned}
\mathbf{F}(\lambda(x_1, x_2)) &= (\lambda x_1) \odot [\underline{a}, \overline{a}] \oplus (\lambda x_2) \odot [\underline{b}, \overline{b}] \\
&= [\lambda x_1 \underline{a} + \lambda x_2 \underline{b}, \lambda x_1 \overline{a} + \lambda x_2 \overline{b}] \\
&= \lambda \odot ([x_1 \overline{a} + x_2 \overline{b}, x_1 \underline{a} + x_2 \underline{b}]) \\
&= \lambda \odot ([x_1 \overline{a}, x_1 \underline{a}] \oplus [x_2 \overline{b}, x_2 \underline{b}]) \\
&= \lambda \odot (x_1 \odot [\underline{a}, \overline{a}] \oplus x_2 \odot [\underline{b}, \overline{b}]) \\
&= \lambda \odot \mathbf{F}(x_1, x_2).
\end{aligned}$$

(b) If $x_1 < 0$ and $x_2 \geq 0$, then $\lambda x_1 > 0$ and $\lambda x_2 \leq 0$. Thus,

$$\begin{aligned}
\mathbf{F}(\lambda(x_1, x_2)) &= (\lambda x_1) \odot [\underline{a}, \overline{a}] \oplus (\lambda x_2) \odot [\underline{b}, \overline{b}] \\
&= [\lambda x_1 \underline{a} + \lambda x_2 \overline{b}, \lambda x_1 \overline{a} + \lambda x_2 \underline{b}] \\
&= \lambda \odot ([x_1 \overline{a} + x_2 \underline{b}, x_1 \underline{a} + x_2 \overline{b}]) \\
&= \lambda \odot ([x_1 \overline{a}, x_1 \underline{a}] \oplus [x_2 \underline{b}, x_2 \overline{b}]) \\
&= \lambda \odot (x_1 \odot [\underline{a}, \overline{a}] \oplus x_2 \odot [\underline{b}, \overline{b}])
\end{aligned}$$

$$= \lambda \odot \mathbf{F}(x_1, x_2).$$

(c) If $x_1 \geq 0$ and $x_2 < 0$, then $\lambda x_1 \leq 0$ and $\lambda x_2 > 0$. Therefore,

$$\begin{aligned}
\mathbf{F}(\lambda(x_1, x_2)) &= (\lambda x_1) \odot [\underline{a}, \overline{a}] \oplus (\lambda x_2) \odot [\underline{b}, \overline{b}] \\
&= [\lambda x_1 \overline{a} + \lambda x_2 \underline{b}, \lambda x_1 \underline{a} + \lambda x_2 \overline{b}] \\
&= \lambda \odot ([x_1 \underline{a} + x_2 \overline{b}, x_1 \overline{a} + x_2 \underline{b}]) \\
&= \lambda \odot ([x_1 \underline{a}, x_1 \overline{a}] \oplus [x_2 \overline{b}, x_2 \overline{b}]) \\
&= \lambda \odot (x_1 \odot [\underline{a}, \overline{a}] \oplus x_2 \odot [\underline{b}, \overline{b}]) \\
&= \lambda \odot \mathbf{F}(x_1, x_2).
\end{aligned}$$

(d) If $x_1 \geq 0$ and $x_2 \geq 0$, then $\lambda x_1 \leq 0$ and $\lambda x_2 \leq 0$. So,

$$\begin{aligned}
\mathbf{F}(\lambda(x_1, x_2)) &= (\lambda x_1) \odot [\underline{a}, \overline{a}] \oplus (\lambda x_2) \odot [\underline{b}, \overline{b}] \\
&= [\lambda x_1 \overline{a} + \lambda x_2 \overline{b}, \lambda x_1 \underline{a} + \lambda x_2 \underline{b}] \\
&= \lambda \odot ([x_1 \underline{a} + x_2 \underline{b}, x_1 \overline{a} + x_2 \overline{b}]) \\
&= \lambda \odot ([x_1 \underline{a}, x_1 \overline{a}] \oplus [x_2 \underline{b}, x_2 \overline{b}]) \\
&= \lambda \odot (x_1 \odot [\underline{a}, \overline{a}] \oplus x_2 \odot [\underline{b}, \overline{b}]) \\
&= \lambda \odot \mathbf{F}(x_1, x_2).
\end{aligned}$$

From all the subcases of Case 1, we have

$$\mathbf{F}(\lambda(x_1, x_2)) = \lambda \odot \mathbf{F}(x_1, x_2) \text{ for every } \lambda < 0. \tag{B.1}$$

- **Case** 2. Let $\lambda \geq 0$.

(a) If $x_1 \geq 0$ and $x_2 \geq 0$, then $\lambda x_1 \geq 0$ and $\lambda x_2 \geq 0$. Therefore,

$$\begin{aligned}
\mathbf{F}(\lambda(x_1, x_2)) &= (\lambda x_1) \odot [\underline{a}, \overline{a}] \oplus (\lambda x_2) \odot [\underline{b}, \overline{b}] \\
&= [\lambda x_1 \underline{a} + \lambda x_2 \underline{b}, \lambda x_1 \overline{a} + \lambda x_2 \overline{b}] \\
&= \lambda \odot ([x_1 \underline{a} + x_2 \underline{b}, x_1 \overline{a} + x_2 \overline{b}]) \\
&= \lambda \odot (x_1 \odot [\underline{a}, \overline{a}] \oplus x_2 \odot [\underline{b}, \overline{b}]) \\
&= \lambda \odot \mathbf{F}(x_1, x_2).
\end{aligned}$$

(b) If $x_1 \geq 0$ and $x_2 < 0$, then $\lambda x_1 \geq 0$ and $\lambda x_2 \leq 0$. Hence,

$$\begin{aligned}
\mathbf{F}(\lambda(x_1, x_2)) &= (\lambda x_1) \odot [\underline{a}, \overline{a}] \oplus (\lambda x_2) \odot [\underline{b}, \overline{b}] \\
&= [\lambda x_1 \underline{a} + \lambda x_2 \overline{b}, \lambda x_1 \overline{a} + \lambda x_2 \underline{b}] \\
&= \lambda \odot ([x_1 \underline{a} + x_2 \overline{b}, x_1 \overline{a} + x_2 \underline{b}]) \\
&= \lambda \odot (x_1 \odot [\underline{a}, \overline{a}] \oplus x_2 \odot [\underline{b}, \overline{b}]) \\
&= \lambda \odot \mathbf{F}(x_1, x_2).
\end{aligned}$$

(c) If $x_1 < 0$ and $x_2 \geq 0$, then $\lambda x_1 \leq 0$ and $\lambda x_2 \geq 0$. Thus,

$$\begin{aligned}
\mathbf{F}(\lambda(x_1, x_2)) &= (\lambda x_1) \odot [\underline{a}, \overline{a}] \oplus (\lambda x_2) \odot [\underline{b}, \overline{b}] \\
&= [\lambda x_1 \overline{a} + \lambda x_2 \underline{b}, \lambda x_1 \underline{a} + \lambda x_2 \overline{b}] \\
&= \lambda \odot ([x_1 \overline{a} + x_2 \underline{b}, x_1 \underline{a} + x_2 \overline{b}]) \\
&= \lambda \odot (x_1 \odot [\underline{a}, \overline{a}] \oplus x_2 \odot [\underline{b}, \overline{b}]) \\
&= \lambda \odot \mathbf{F}(x_1, x_2).
\end{aligned}$$

(d) If $x_1 < 0$ and $x_2 < 0$, then $\lambda x_1 \leq 0$ and $\lambda x_2 \leq 0$. Therefore,

$$\begin{aligned}\mathbf{F}(\lambda(x_1, x_2)) &= (\lambda x_1) \odot [\underline{a}, \overline{a}] \oplus (\lambda x_2) \odot [\underline{b}, \overline{b}] \\ &= [\lambda x_1 \overline{a} + \lambda x_2 \overline{b}, \lambda x_1 \underline{a} + \lambda x_2 \underline{b}] \\ &= \lambda \odot ([x_1 \overline{a} + x_2 \overline{b}, x_1 \underline{a} + x_2 \underline{b}]) \\ &= \lambda \odot (x_1 \odot [\underline{a}, \overline{a}] \oplus x_2 \odot [\underline{b}, \overline{b}]) \\ &= \lambda \odot \mathbf{F}(x_1, x_2).\end{aligned}$$

Hence, from all the subcases of Case 2, we have

$$\mathbf{F}(\lambda(x_1, x_2)) = \lambda \odot \mathbf{F}(x_1, x_2) \quad \text{for every} \quad \lambda \geq 0. \tag{B.2}$$

Next, we show that

1. when x_1 and x_2 have the same sign, and y_1 and y_2 have the same sign,

$$\mathbf{F}((x_1, y_1) + (x_2, y_2)) = \mathbf{F}(x_1, y_1) \oplus \mathbf{F}(x_2, y_2),$$

2. when x_1 and x_2 have different signs, and y_1 and y_2 have the same sign,

$$\mathbf{F}((x_1, y_1) + (x_2, y_2)) \quad \text{and} \quad \mathbf{F}(x_1, y_1) \oplus \mathbf{F}(x_2, y_2) \text{ are not comparable,}$$

3. when x_1 and x_2 have the same sign, and y_1 and y_2 have different signs,

$$\mathbf{F}((x_1, y_1) + (x_2, y_2)) \quad \text{and} \quad \mathbf{F}(x_1, y_1) \oplus \mathbf{F}(x_2, y_2) \text{ are not comparable, and}$$

4. when x_1 and x_2 have different signs, and y_1 and y_2 have different signs, then

$$\mathbf{F}((x_1, y_1) + (x_2, y_2)) \text{ and } \mathbf{F}(x_1, y_1) \oplus \mathbf{F}(x_2, y_2) \text{ are not comparable.}$$

- **Case** 1. Let x_1 and x_2 have the same sign, and y_1 and y_2 have the same sign. A straightforward calculation proves that

$$\mathbf{F}((x_1, y_1)+(x_2, y_2)) = (x_1+x_2)\odot[\underline{a}, \overline{a}]\oplus(y_1+y_2)\odot[\underline{b}, \overline{b}] = \mathbf{F}(x_1, y_1)\oplus\mathbf{F}(x_2, y_2).$$

- **Case** 2. Suppose that x_1 and x_2 have different signs, and y_1 and y_2 have the same sign. Since y_1 and y_2 have the same sign, evidently,

$$(y_1 + y_2) \odot [\underline{b}, \overline{b}] = y_1 \odot [\underline{b}, \overline{b}] \oplus y_2 \odot [\underline{b}, \overline{b}].$$

Thus, to prove that

$$\mathbf{F}((x_1, y_1) + (x_2, y_2)) \text{ and } \mathbf{F}(x_1, y_1) \oplus \mathbf{F}(x_2, y_2) \text{ are not comparable}$$

it is sufficient to prove that when x_1 and x_2 have different signs,

$$(x_1 + x_2) \odot [\underline{a}, \overline{a}] \text{ and } x_1 \odot [\underline{a}, \overline{a}] \oplus x_2[\underline{a}, \overline{a}] \text{ are not comparable.}$$

(a) For $x_1 > 0$ and $x_2 < 0$ with $x_1 + x_2 < 0$, we have

$$(x_1 + x_2) \odot [\underline{a}, \overline{a}] = [(x_1 + x_2)\overline{a}, (x_1 + x_2)\underline{a}]$$

and

$$x_1 \odot [\underline{a}, \overline{a}] \oplus x_2[\underline{a}, \overline{a}] \;=\; [x_1\underline{a}, x_1\overline{a}] \oplus [x_2\overline{a}, x_2\underline{a}]$$

$$= [x_1\underline{a} + x_2\overline{a}, x_1\overline{a} + x_2\underline{a}].$$

If possible let $(x_1+x_2)\odot[\underline{a},\overline{a}]$ and $x_1\odot[\underline{a},\overline{a}]\oplus x_2[\underline{a},\overline{a}]$ be comparable. Then,

$$\text{either } (x_1 + x_2)\overline{a} > x_1\underline{a} + x_2\overline{a} \text{ and } (x_1 + x_2)\underline{a} > x_1\overline{a} + x_2\underline{a}, \tag{B.3}$$

$$\text{or } (x_1 + x_2)\overline{a} < x_1\underline{a} + x_2\overline{a} \text{ and } (x_1 + x_2)\underline{a} < x_1\overline{a} + x_2\underline{a}. \tag{B.4}$$

If $(x_1 + x_2)\overline{a} > x_1\underline{a} + x_2\overline{a}$, then

$$\begin{aligned} & x_1\overline{a} + x_2\overline{a} > x_1\underline{a} + x_2\overline{a} \\ \text{or,}\quad & x_1\overline{a} > x_1\underline{a} \\ \text{or,}\quad & x_1\overline{a} + x_2\underline{a} > x_1\underline{a} + x_2\underline{a} \\ \text{or,}\quad & x_1\overline{a} + x_2\underline{a} > (x_1 + x_2)\underline{a}, \end{aligned}$$

which is a contradiction to the second inequality of (B.3).

If $(x_1 + x_2)\overline{a} < x_1\underline{a} + x_2\overline{a}$, then

$$\begin{aligned} & x_1\overline{a} < x_1\underline{a} \\ \text{or,}\quad & x_1\overline{a} + x_2\underline{a} < x_1\underline{a} + x_2\underline{a} \\ \text{or,}\quad & x_1\overline{a} + x_2\underline{a} < (x_1 + x_2)\underline{a}, \end{aligned}$$

which is a contradiction to the second inequality of (B.4).

Hence, none of (B.3) and (B.4) is true.

Thus, $(x_1 + x_2) \odot [\underline{a}, \overline{a}]$ and $x_1 \odot [\underline{a}, \overline{a}] \oplus x_2[\underline{a}, \overline{a}]$ are not comparable.

(b) For $x_2 > 0$ and $x_1 < 0$ with $x_1 + x_2 < 0$, the proof is similar to the Case 2a.

(c) For $x_1 < 0$ and $x_2 > 0$ with $x_1 + x_2 > 0$, we have

$$(x_1 + x_2) \odot [\underline{a}, \overline{a}] = [(x_1 + x_2)\underline{a}, (x_1 + x_2)\overline{a}]$$

and

$$\begin{aligned} x_1 \odot [\underline{a}, \overline{a}] \oplus x_2 \odot [\underline{a}, \overline{a}] &= [x_1\overline{a}, x_1\underline{a}] \oplus [x_2\underline{a} + x_2\overline{a}] \\ &= [x_1\overline{a} + x_2\underline{a}, x_1\underline{a} + x_2\overline{a}]. \end{aligned}$$

If possible let $(x_1+x_2)\odot[\underline{a}, \overline{a}]$ and $x_1\odot[\underline{a}, \overline{a}]\oplus x_2[\underline{a}, \overline{a}]$ be comparable. Then,

$$\text{either } (x_1 + x_2)\underline{a} > x_1\overline{a} + x_2\underline{a} \text{ and } (x_1 + x_2)\overline{a} > x_1\underline{a} + x_2\overline{a}, , \tag{B.5}$$

$$\text{or } (x_1 + x_2)\underline{a} < x_1\overline{a} + x_2\underline{a} \text{ and } (x_1 + x_2)\overline{a} < x_1\underline{a} + x_2\overline{a} \tag{B.6}$$

If $(x_1 + x_2)\underline{a} > x_1\overline{a} + x_2\underline{a}$, then

$$\begin{aligned} & x_1\underline{a} > x_1\overline{a} \\ \text{or,} \quad & x_1\underline{a} + x_2\overline{a} > x_1\overline{a} + x_2\overline{a} \\ \text{or,} \quad & x_1\underline{a} + x_2\overline{a} > (x_1 + x_2)\overline{a}, \end{aligned}$$

which is a contradiction to the second inequality of (B.5).

If $(x_1 + x_2)\underline{a} < x_1\overline{a} + x_2\underline{a}$, then

$$\begin{aligned} & x_1\underline{a} < x_1\overline{a} \\ \text{or,} \quad & x_1\underline{a} + x_2\overline{a} < x_1\overline{a} + x_2\overline{a} \\ \text{or,} \quad & x_1\underline{a} + x_2\overline{a} < (x_1 + x_2)\overline{a}, \end{aligned}$$

which is a contradiction to the second inequality of (B.6).

(d) For $x_2 < 0$ and $x_1 > 0$ with $x_1 + x_2 > 0$, the proof is similar to the Case 2c.

- **Case** 3. Suppose that x_1 and x_2 have the same sign and y_1 and y_2 have different signs. By interchanging the role of x_1 and x_2 with y_1 and y_2, we note that this case is identical to the Case 2. Hence,

$$\mathbf{F}((x_1, y_1) + (x_2, y_2)) \ \text{ and } \ \mathbf{F}(x_1, y_1) \oplus \mathbf{F}(x_2, y_2) \text{ are not comparable}$$

- **Case** 4. Suppose that x_1 and x_2 have different signs, and y_1 and y_2 have different signs. For this case, only in the following two subcases, we prove that $\mathbf{F}((x_1, y_1) + (x_2, y_2))$ and $\mathbf{F}(x_1, y_1) \oplus \mathbf{F}(x_2, y_2)$ are not comparable. The same conclusion can be proved analogously for all other possible subcases.

(a) Let $x_1 > 0$ and $x_2 < 0$ with $x_1 + x_2 > 0$, and $y_1 < 0$ and $y_2 > 0$ with $y_1 + y_2 < 0$. Then, we have

$$\begin{aligned} & (x_1 + x_2) \odot [\underline{a}, \overline{a}] \oplus (y_1 + y_2) \odot [\underline{b}, \overline{b}] \\ = \quad & [(x_1 + x_2)\underline{a} + (y_1 + y_2)\overline{b}, (x_1 + x_2)\overline{a} + (y_1 + y_2)\underline{b}] \end{aligned}$$

and

$$\begin{aligned} & x_1 \odot [\underline{a}, \overline{a}] \oplus y_1 \odot [\underline{b}, \overline{b}] \oplus x_2 \odot [\underline{a}, \overline{a}] \oplus y_2 \odot [\underline{b}, \overline{b}] \\ = & [x_1\underline{a} + y_1\overline{b} + x_2\overline{a} + y_2\underline{b}, x_1\overline{a} + y_1\underline{b} + x_2\underline{a} + y_2\overline{b}]. \end{aligned}$$

If possible let $(x_1 + x_2) \odot [\underline{a}, \overline{a}] \oplus (y_1 + y_2) \odot [\underline{b}, \overline{b}]$ and $x_1 \odot [\underline{a}, \overline{a}] \oplus y_1 \odot [\underline{b}, \overline{b}] \oplus x_2 \odot [\underline{a}, \overline{a}] \oplus y_2 \odot [\underline{b}, \overline{b}]$ be comparable. Then,

$$\text{either } \left\{ \begin{aligned} (x_1 + x_2)\underline{a} + (y_1 + y_2)\overline{b} < x_1\underline{a} + y_1\overline{b} + x_2\overline{a} + y_2\underline{b} \\ \text{and } (x_1 + x_2)\overline{a} + (y_1 + y_2)\underline{b} < x_1\overline{a} + y_1\underline{b} + x_2\underline{a} + y_2\overline{b} \end{aligned} \right\} \tag{B.7}$$

$$\text{or } \left\{ \begin{aligned} & (x_1 + x_2)\underline{a} + (y_1 + y_2)\overline{b} > x_1\underline{a} + y_1\overline{b} + x_2\overline{a} + y_2\underline{b} \\ & \text{and } (x_1 + x_2)\overline{a} + (y_1 + y_2)\underline{b} > x_1\overline{a} + y_1\underline{b} + x_2\underline{a} + y_2\overline{b}. \end{aligned} \right\} \tag{B.8}$$

If the first inequality of (B.7) holds, i.e., $(x_1 + x_2)\underline{a} + (y_1 + y_2)\overline{b} < x_1\underline{a} + y_1\overline{b} + x_2\overline{a} + y_2\underline{b}$, then

$$\begin{aligned} & x_2\underline{a} + y_2\overline{b} < y_2\underline{b} + x_2\overline{a} \\ \text{or,} \quad & x_1\overline{a} + x_2\underline{a} + y_1\underline{b} + y_2\overline{b} < (x_1 + x_2)\overline{a} + (y_1 + y_2)\underline{b}, \end{aligned}$$

which is a contradiction to the second inequality of (B.7).

If the second inequality of (B.8) holds, i.e., $(x_1+x_2)\overline{a}+(y_1+y_2)\underline{b} > x_1\overline{a} + y_1\underline{b} + x_2\underline{a} + y_2\overline{b}$, then

$$\begin{aligned} & x_2\overline{a} + y_2\underline{b} > x_2\underline{a} + y_2\overline{b} \\ \text{or,} \quad & x_1\underline{a} + y_1\overline{b} + x_2\overline{a} + y_2\underline{b} > (x_1 + x_2)\underline{a} + (y_1 + y_2)\overline{b}, \end{aligned}$$

which is a contradiction to the first inequality of (B.8).

Thus, neither (B.7) nor (B.8) is true, and hence $(x_1+x_2)\odot[\underline{a},\overline{a}]\oplus(y_1+y_2)\odot[\underline{b},\overline{b}]$ and $x_1\odot[\underline{a},\overline{a}]\oplus y_1\odot[\underline{b},\overline{b}]\oplus x_2\odot[\underline{a},\overline{a}]\oplus y_2\odot[\underline{b},\overline{b}]$ are not comparable.

(b) Let $x_1>0$ and $x_2<0$ with $x_1+x_2<0$, and $y_1<0$ and $y_2>0$ with $y_1+y_2<0$. Then, we have

$$\begin{aligned} & (x_1+x_2)\odot[\underline{a},\overline{a}]\oplus(y_1+y_2)\odot[\underline{b},\overline{b}] \\ = \; & [(x_1+x_2)\overline{a}+(y_1+y_2)\overline{b},(x_1+x_2)\underline{a}+(y_1+y_2)\underline{b}] \end{aligned}$$

and

$$\begin{aligned} & x_1\odot[\underline{a},\overline{a}]\oplus y_1\odot[\underline{b},\overline{b}]\oplus x_2\odot[\underline{a},\overline{a}]\oplus y_2\odot[\underline{b},\overline{b}] \\ = \; & [x_1\underline{a}+y_1\overline{b}+x_2\overline{a}+y_2\underline{b},x_1\overline{a}+y_1\underline{b}+x_2\underline{a}+y_2\overline{b}]. \end{aligned}$$

If possible let $(x_1+x_2)\odot[\underline{a},\overline{a}]\oplus(y_1+y_2)\odot[\underline{b},\overline{b}]$ and $x_1\odot[\underline{a},\overline{a}]\oplus y_1\odot[\underline{b},\overline{b}]\oplus x_2\odot[\underline{a},\overline{a}]\oplus y_2\odot[\underline{b},\overline{b}]$ be comparable. Then,

$$\text{either}\left\{\begin{aligned} & (x_1+x_2)\overline{a}+(y_1+y_2)\overline{b}<x_1\underline{a}+y_1\overline{b}+x_2\overline{a}+y_2\underline{b} \\ \text{and } & (x_1+x_2)\underline{a}+(y_1+y_2)\underline{b}<x_1\overline{a}+y_1\underline{b}+x_2\underline{a}+y_2\overline{b} \end{aligned}\right\} \tag{B.9}$$

$$\text{or}\left\{\begin{aligned} & (x_1+x_2)\overline{a}+(y_1+y_2)\overline{b}>x_1\underline{a}+y_1\overline{b}+x_2\overline{a}+y_2\underline{b} \\ \text{and } & (x_1+x_2)\underline{a}+(y_1+y_2)\underline{b}>x_1\overline{a}+y_1\underline{b}+x_2\underline{a}+y_2\overline{b}. \end{aligned}\right\} \tag{B.10}$$

If the first inequality of (B.9) holds, i.e., $(x_1+x_2)\overline{a}+(y_1+y_2)\overline{b} < x_1\underline{a}+y_1\overline{b}+x_2\overline{a}+y_2\underline{b}$, then

$$\begin{aligned} & x_1\overline{a}+y_2\overline{b} < x_1\underline{a}+y_2\underline{b} \\ \text{or,} \quad & x_1\overline{a}+y_1\underline{b}+x_2\underline{a}+y_2\overline{b} < (x_1+x_2)\underline{a}+(y_1+y_2)\underline{b}, \end{aligned}$$

which is a contradiction to the second inequality of (B.9).

If the second inequality of (B.10) holds,i.e., $(x_1+x_2)\underline{a}+(y_1+y_2)\underline{b} > x_1\overline{a}+y_1\underline{b}+x_2\underline{a}+y_2\overline{b}$, then

$$\begin{aligned} & x_1\underline{a}+y_2\underline{b} > x_1\overline{a}+y_2\overline{b} \\ \text{or,} \quad & x_1\underline{a}+y_1\overline{b}+x_2\overline{a}+y_2\underline{b} > (x_1+x_2)\overline{a}+(y_1+y_2)\overline{b}, \end{aligned}$$

which is a contradiction to the first inequality of (B.10).

Thus, neither (B.9) nor (B.10) is true, and hence $(x_1+x_2)\odot[\underline{a},\overline{a}]\oplus(y_1+y_2)\odot[\underline{b},\overline{b}]$ and $x_1\odot[\underline{a},\overline{a}]\oplus y_1\odot[\underline{b},\overline{b}]\oplus x_2\odot[\underline{a},\overline{a}]\oplus y_2\odot[\underline{b},\overline{b}]$ are not comparable.

From (B.1), (B.2) and four cases after (B.2), we see that **F** is a linear IVF. $\square$

C. Appendix C

C.1 Proof of the Lemma 3.1

Proof. (i) Since

$$\limsup_{x\to\bar{x}} \left(\underline{f}(x)+\underline{g}(x)\right) \leq \limsup_{x\to\bar{x}} \underline{f}(x) + \limsup_{x\to\bar{x}} \underline{g}(x) \text{ and}$$

$$\limsup_{x\to\bar{x}} \left(\overline{f}(x)+\overline{g}(x)\right) \leq \limsup_{x\to\bar{x}} \overline{f}(x) + \limsup_{x\to\bar{x}} \overline{g}(x),$$

then

$$\begin{aligned} & \left[\limsup_{x\to\bar{x}}(\underline{f}(x)+\underline{g}(x)),\ \limsup_{x\to\bar{x}}\left(\overline{f}(x)+\overline{g}(x)\right)\right] \\ \preceq\ & \left[\limsup_{x\to\bar{x}} \underline{f}(x), \limsup_{x\to\bar{x}} \overline{f}(x)\right] \oplus \left[\limsup_{x\to\bar{x}} \underline{g}(x), \limsup_{x\to\bar{x}} \overline{g}(x)\right], \end{aligned}$$

which implies $\limsup_{x\to\bar{x}} (\mathbf{F}(x) \oplus \mathbf{G}(x)) \preceq \limsup_{x\to\bar{x}} \mathbf{F}(x) \oplus \limsup_{x\to\bar{x}} \mathbf{G}(x)$.

(ii) Since $\underline{f}$ and $\overline{f}$ are real-valued functions, for any $\lambda \geq 0$, we have

$$\limsup_{x\to\bar{x}} \left(\lambda \underline{f}(x)\right) = \lambda \limsup_{x\to\bar{x}} \underline{f}(x) \text{ and } \limsup_{x\to\bar{x}} \left(\lambda \overline{f}(x)\right) = \lambda \limsup_{x\to\bar{x}} \overline{f}(x). \quad \text{(C.1)}$$

Hence, for any $\lambda \geq 0$,

$$\begin{aligned} \limsup_{x\to\bar{x}} (\lambda \odot \mathbf{F}(x)) &= \left[\limsup_{x\to\bar{x}} \left(\lambda \underline{f}(x)\right),\ \limsup_{x\to\bar{x}} \left(\lambda \overline{f}(x)\right)\right] \\ &= \lambda \odot \limsup_{x\to\bar{x}} \mathbf{F}(x) \text{ by (C.1)}. \end{aligned}$$

(iii) Let f be a real-valued function. Then, $\left|\limsup_{x\to\bar{x}} f(x)\right| \leq \limsup_{x\to\bar{x}} |f(x)|$. By the definition of norm on $I(\mathbb{R})$,

$$\begin{aligned}\left\|\limsup_{x\to\bar{x}} \mathbf{F}(x)\right\|_{I(\mathbb{R})} &= \max\left\{\left|\limsup_{x\to\bar{x}} \underline{f}(x)\right|, \ \left|\limsup_{x\to\bar{x}} \overline{f}(x)\right|\right\}\\ &\leq \limsup_{x\to\bar{x}} \ \|\mathbf{F}(x)\|_{I(\mathbb{R})}.\end{aligned}$$

□

C.2 Proof of the Lemma 3.2

Proof. Since $\underline{f}$ and $\overline{f}$ are upper Clarke differentiable at $\bar{x}$. Therefore, both of the following limits

$$\limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \frac{1}{\lambda} l_1(\lambda) \text{ and } \limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \frac{1}{\lambda} l_2(\lambda)$$

exist, where $l_1(\lambda) = \underline{f}(x+\lambda h) - \underline{f}(x)$ and $l_2(\lambda) = \overline{f}(x+\lambda h) - \overline{f}(x)$. Thus,

$$\begin{aligned}
& \limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \frac{1}{\lambda}\left(l_1(\lambda) + l_2(\lambda)\right) \text{ and } \limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \frac{1}{\lambda}|l_1(\lambda) - l_2(\lambda)| \text{ exist}\\
\implies & \limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \frac{1}{2\lambda}\Big(l_1(\lambda) + l_2(\lambda) - |l_1(\lambda) - l_2(\lambda)|\Big) \text{ and}\\
& \limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \frac{1}{2\lambda}\Big(l_1(\lambda) + l_2(\lambda) + |l_1(\lambda) - l_2(\lambda)|\Big) \text{ exist}\\
\implies & \limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \frac{1}{\lambda}\left(\min\left\{l_1(\lambda), l_2(\lambda)\right\}\right) \text{ and } \limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \frac{1}{\lambda}\left(\max\left\{l_1(\lambda), l_2(\lambda)\right\}\right) \text{ exist}\\
\implies & \limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \frac{1}{\lambda} \odot \Big[\min\left\{l_1(\lambda), l_2(\lambda)\right\}, \max\left\{l_1(\lambda), l_2(\lambda)\right\}\Big] \text{ exists}\\
\implies & \limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \frac{1}{\lambda} \odot \left(\mathbf{F}(x+\lambda h) \ominus_{gH} \mathbf{F}(x)\right) \text{ exists.}
\end{aligned}$$

Hence, $\mathbf{F}$ is upper gH-Clarke differentiable IVF at $\bar{x} \in \mathcal{S}$. □

C.3 Proof of the Lemma 3.3

Proof. (i) Let $\mathbf{F}$ be gH-continuous at $\bar{x} \in \mathcal{S}$. Thus, for any $d \in \mathbb{R}^n$ such that $\bar{x} + d \in \mathcal{S}$,

$$\lim_{\|d\| \to 0} (\mathbf{F}(\bar{x} + d) \ominus_{gH} \mathbf{F}(\bar{x})) = \mathbf{0},$$

which implies

$$\lim_{\|d\| \to 0} (\underline{f}(\bar{x} + d) - \underline{f}(\bar{x})) \to 0 \text{ and } \lim_{\|d\| \to 0} (\overline{f}(\bar{x} + d) - \overline{f}(\bar{x})) \to 0,$$

i.e., $\underline{f}$ and $\overline{f}$ are continuous at $\bar{x} \in \mathcal{S}$.

Conversely, let the functions $\underline{f}$ and $\overline{f}$ be continuous at $\bar{x} \in \mathcal{S}$. If possible, let $\mathbf{F}$ be not gH-continuous at $\bar{x}$. Then, as $\|d\| \to 0$, $(\mathbf{F}(\bar{x} + d) \ominus_{gH} \mathbf{F}(\bar{x})) \not\to \mathbf{0}$. Therefore, as $\|d\| \to 0$ at least one of the functions $(\underline{f}(\bar{x} + d) - \underline{f}(\bar{x}))$ and $(\overline{f}(\bar{x} + d) - \overline{f}(\bar{x}))$ does not tend to 0. So it is clear that at least one of the functions $\underline{f}$ and $\overline{f}$ is not continuous at $\bar{x}$. This contradicts the assumption that the functions $\underline{f}$ and $\overline{f}$ both are continuous at $\bar{x}$. Hence, $\mathbf{F}$ is gH-continuous at $\bar{x}$.

(ii) Let $\mathbf{F}$ be gH-Lipschitz continuous on $\mathcal{S}$. Thus, there exists $K > 0$ such that for any $x,\ y \in \mathcal{X}$ we have

$$\begin{aligned} & \|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq K\|x - y\| \\ \implies & \left|\underline{f}(x) - \underline{f}(y)\right| \leq K\|x - y\| \text{ and } \left|\overline{f}(x) - \overline{f}(y)\right| \leq K\|x - y\|. \end{aligned}$$

Hence, $\underline{f}$ and $\overline{f}$ are Lipschitz continuous on $\mathcal{S}$.

Conversely, let the functions $\underline{f}$ and $\overline{f}$ be Lipschitz continuous on $\mathcal{S}$. Thus, there exist $K_1,\ K_2 > 0$ such that for all $x,\ y \in \mathcal{S}$,

$$
\begin{aligned}
& \left|\underline{f}(x) - \underline{f}(y)\right| \leq K_1\|x-y\| \text{ and } \left|\overline{f}(x) - \overline{f}(y)\right| \leq K_2\|x-y\| \\
\implies\ & \max\left\{\left|\underline{f}(x) - \underline{f}(y)\right|,\ \left|\overline{f}(x) - \overline{f}(y)\right|\right\} \leq \bar{K}\|x-y\|, \\
& (\text{where } \bar{K} = \max\{K_1,\ K_2\}) \\
\implies\ & \|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq \bar{K}\|x-y\|.
\end{aligned}
$$

Hence, $\mathbf{F}$ is gH-Lipschitz continuous IVF on $\mathcal{S}$.

(iii) Let $\mathbf{F}$ be gH-Lipschitz continuous on $\mathcal{S}$. Then, there exists an $K > 0$ such that for all $x,\ y \in \mathcal{S}$, we have

$$\|\mathbf{F}(y) \ominus_{gH} \mathbf{F}(x)\|_{I(\mathbb{R})} \leq K\|y-x\|.$$

For $h = y - x \in \mathcal{S}$,

$$
\begin{aligned}
& \|\mathbf{F}(x+h) \ominus_{gH} \mathbf{F}(x)\|_{I(\mathbb{R})} \leq K\|h\| \\
\implies\ & \lim_{\|h\|\to 0} \|\mathbf{F}(x+h) \ominus_{gH} \mathbf{F}(x)\|_{I(\mathbb{R})} = 0 \\
\implies\ & \lim_{\|h\|\to 0} (\mathbf{F}(x+h) \ominus_{gH} \mathbf{F}(x)) = \mathbf{0}.
\end{aligned}
$$

Hence, $\mathbf{F}$ is gH-continuous at $x \in \mathcal{S}$.

□

D. Appendix D

D.1 Proof of the Lemma 4.1

Proof. Since $\underline{f}$ and $\overline{f}$ are Hadamard semidifferentiable at $\bar{x}$, both of the following limits

$$\lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} l_1(\lambda, h) \text{ and } \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} l_2(\lambda, h)$$

exist, where $l_1(\lambda, h) = \underline{f}(x + \lambda h) - \underline{f}(x)$ and $l_2(\lambda, h) = \overline{f}(x + \lambda h) - \overline{f}(x)$. Thus,

$$\begin{aligned}
& \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \left(l_1(\lambda, h) + l_2(\lambda, h)\right) \text{ and } \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} |l_1(\lambda, h) - l_2(\lambda, h)| \text{ exist} \\
\implies & \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{2\lambda} \Big(l_1(\lambda, h) + l_2(\lambda, h) - |l_1(\lambda, h) - l_2(\lambda, h)|\Big) \text{ and} \\
& \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{2\lambda} \Big(l_1(\lambda, h) + l_2(\lambda, h) + |l_1(\lambda, h) - l_2(\lambda, h)|\Big) \text{ exist} \\
\implies & \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \left(\min\left\{l_1(\lambda, h), l_2(\lambda, h)\right\}\right) \text{ and } \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \left(\max\left\{l_1(\lambda, h), l_2(\lambda, h)\right\}\right) \text{ exist} \\
\implies & \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \Big[\min\left\{l_1(\lambda, h), l_2(\lambda, h)\right\}, \max\left\{l_1(\lambda, h), l_2(\lambda, h)\right\}\Big] \text{ exists} \\
\implies & \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\mathbf{F}(x + \lambda h) \ominus_{gH} \mathbf{F}(x)\right) \text{ exists.}
\end{aligned}$$

Hence, $\mathbf{F}$ is gH-Hadamard semidifferentiable IVF at $\bar{x} \in \mathcal{S}$, and

$$\begin{aligned}
&\mathbf{F}_{\mathscr{H}'}(\bar{x})(v)\\
= &\lim_{\substack{\lambda \to 0+\\ h \to v}} \frac{1}{\lambda} \odot \left(\mathbf{F}(x + \lambda h) \ominus_{gH} \mathbf{F}(x)\right)\\
= &\lim_{\substack{\lambda \to 0+\\ h \to v}} \frac{1}{\lambda} \odot \Big[\min\left\{l_1(\lambda, h), l_2(\lambda, h)\right\}, \max\left\{l_1(\lambda, h), l_2(\lambda, h)\right\}\Big]\\
= &\left[\min\left\{\lim_{\substack{\lambda \to 0+\\ h \to v}} \frac{1}{\lambda} l_1(\lambda, h), \lim_{\substack{\lambda \to 0+\\ h \to v}} \frac{1}{\lambda} l_2(\lambda, h)\right\}, \max\left\{\lim_{\substack{\lambda \to 0+\\ h \to v}} \frac{1}{\lambda} l_1(\lambda, h), \lim_{\substack{\lambda \to 0+\\ h \to v}} \frac{1}{\lambda} l_2(\lambda, h)\right\}\right]\\
= &\left[\min\left\{\underline{f}_{\mathscr{H}'}(\bar{x})(v), \overline{f}_{\mathscr{H}'}(\bar{x})(v)\right\}, \max\left\{\underline{f}_{\mathscr{H}'}(\bar{x})(v), \overline{f}_{\mathscr{H}'}(\bar{x})(v)\right\}\right]
\end{aligned}$$

□

D.2 Proof of the Lemma 4.8

Proof. Let $\mathbf{F}$ be semiconvex on $\mathcal{S}$. Then, there exists a monotonic increasing IVF $\mathbf{E} : \mathbb{R}_+ \to I(\mathbb{R}_+)$ such that $\mathbf{E}(\delta) \to \mathbf{0}$ as $\delta \to 0+$ and

$$\mathbf{F}(\lambda_1 x_1 + \lambda_2 x_2) \preceq \lambda_1 \odot \mathbf{F}(x_1) \oplus \lambda_2 \odot \mathbf{F}(x_2) \oplus \lambda_1 \lambda_2 \|x - y\| \odot \mathbf{E}\left(\|x - y\|\right)$$

for all $x, y \in \mathcal{S}$ and $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_1 + \lambda_2 = 1$.

Let $\mathbf{E}(\delta) = [\underline{e}(\delta), \overline{e}(\delta)]$. Then, $\underline{e}$ and $\overline{e}$ are monotonic increasing real-valued function, by Remark 2.4.1, such that

$$\underline{f}(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 \underline{f}(x_1) + \lambda_2 \underline{f}(x_2) \oplus \lambda_1 \lambda_2 \|x - y\| \underline{e}\left(\|x - y\|\right)$$

and

$$\overline{f}(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 \overline{f}(x_1) + \lambda_2 \overline{f}(x_2) \oplus \lambda_1 \lambda_2 \|x - y\| \overline{e}\left(\|x - y\|\right)$$

for all $x, y \in \mathcal{S}$ and $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_1 + \lambda_2 = 1$.

Hence, $\underline{f}$ and $\overline{f}$ are semiconvex on $\mathcal{S}$. □

E. List of Publications

[1] Ghosh, D., Chauhan, R. S., Mesiar, R., & Debnath, A. K. (2019). Generalized Hukuhara Gâteaux and Fréchet derivatives of interval-valued functions and their application in optimization with interval-valued functions. *Information Sciences*, 510, 317-340.

[2] Chauhan, R. S., & Ghosh, D. (2021). An erratum to "Extended Karush-Kuhn-Tucker condition for constrained interval optimization problems and its application in support vector machines". *Information Sciences*, 559, 309-313.

[3] Chauhan, R. S., Ghosh, D., Ramik, D., Debnath, A. K. Generalized Hukuhara-Clarke derivative of interval-valued functions and its properties. *Soft computing*, arXiv preprint arXiv:2010.16182. (Accepted)

[4] Ghosh, D., Debnath, A. K., Chauhan, R. S., & Castillo, O. Generalized-Hukuhara gradient efficient-direction method to solve optimization problems with interval-valued functions and its application in Least Squares Problems. *International Journal of Fuzzy Systems, arXiv preprint arXiv:2011.10462.* (Accepted)

[5] Chauhan, R. S., Ghosh, D., Ramik, D., Debnath, A. K. Generalized Hukuhara-pseudoconvex and quasiconvex interval-valued functions and their application in optimization problems with gH-Clarke derivative. *Journal of Computational and Applied Mathematics.* (Under Review)

[6] Chauhan, R. S., Ghosh, D. Generalized Hukuhara Hadamard semidervative of interval-valued functions and its application in interval optimization. *International Jounral of Uncertain Systems.* (Under Review)

[7] Chauhan, R. S., Ghosh, D., Ansari, Q. H. Generalized Hukuhara Hadamard dervative of interval-valued functions and its application in interval optimization. *Positivity.* (Under Review)

[8] Debnath, A. K., Ghosh, D., Chauhan, R. S. Generalized Hukuhara subgradient method for optimization problem with interval-valued functions and its application in Lasso problem. *Journal of Applied Mathematics and Computing.* (Under Review)

[9] Ghosh, D., Dempe, S., Debnath, A. K., & Chauhan, R. S. Lagrange multipliers characterization of efficient solutions for interval optimization problems. *Optimization.* (Under Review)

[10] Anshika, Ghosh, D., Chauhan, R. S., and Mesiar, R. Generalized-Hukuhara subdifferential analysis and its application in nonconvex composite optimization problems with interval-valued functions. *Information Sciences.* (Under Review)

[11] Chauhan, R. S. and Ghosh, D. Generalized Hukuhara-Dini Semiderivative of Interval-valued Functions and its Application in Interval-Optimization Problems (Ready for Submit)