Chapter 7

Conclusions and future scopes

A compendium of the main conclusions that can be made from this thesis is given in this chapter. It also includes a summary of the directions for future research scopes.

7.1 General conclusions

The principal contributions of this thesis are as follows.

- Several inequalities of interval analysis which are helpful to develop the theories of interval-valued functions and interval optimization problems are proved.
- Rigorous analysis on directional derivative, Gâteaux derivative, Fréchet Derivative, Clarke derivative, Hadamard semiderivative, Hadamard derivative, and Dini semiderivative for IVFs with several examples have been studied. Detailed explanation about the relations among all these derivatives with several results and examples have been shown. Further, a necessary and sufficient condition for the existence of these derivatives for IVFs is explained.

- Detail explanation of monotonicity, boundedness, and Lipschitz continuity, pseudoconvexity, and quasiconvexity for IVFs are explained. Further, a rigorous analysis on linearity, sublinearity, supremum and infemum, limit superior and limit inferior for IVFs have been studied.

- Rigorous information about the optimality conditions of unconstraint IOPs with the help of all proposed derivatives have been found. Also, by using Hadamard derivative for IVFs, a necessary and sufficient Karush-Kuhn-Tucker condition for constraint IOPs are studied.

7.2 Contributions of the thesis

This thesis appertains to the theories on analysis of IVFs and characterization to the solutions of IOPs through the idea of gereralized derivatives of IVFs. Chapter wise contributions of this dissertation are highlighted below.

Chapter 2 describes the notions of directional derivative, Gâteaux derivative and Fréchet derivative for IVFs. For an IVF, the existence of Fréchet derivative is shown to imply the existence of Gâteaux derivative, and the existence of Gâteaux derivative is observed to indicate the presence of directional derivative. It is proved that the existence of Gâteaux derivative implies the existence of Fréchet derivative for a Lipschitz continuous IVF. The concepts of linear and monotonic IVFs are studied, and it is observed that for a convex IVF on a linear space, the directional derivative exists at any point for every direction. Further, it is shown that the proposed derivatives are useful to check the convexity of an IVF and to characterize efficient points of an optimization problem with the interval-valued objective function. It is observed that at an efficient point of an IVF, none of its directional derivatives dominates zero, and the Gâteaux derivative must contain zero. The entire study is supported by suitable illustrative examples.

Chapter 3 explaines the notions of Clarke derivative, pseudoconvex and quasiconvex IVFs. To describe the properties of Clarke derivative, the concepts of limit superior, limit inferior, and sublinear IVFs are studied. The upper Clarke derivative function of a Lipschitz continuous IVF is observed to be sublinear IVF. It is found that every Lipschitz continuous IVF is upper Clarke differentiable. Further, for a convex and Lipschitz continuous IVF, it is shown that the upper Clarke derivative coincides with the directional derivative. With the help of the studied pseudoconvex, quasiconvex, and Lipschitz IVFs, a few results on characterizing efficient solutions to an interval optimization problem with upper Clarke and Fréchet differentiable IVF on a starshaped constraint set are obtained. Importantly, it is shown that at an efficient point of an IOP on a starshaped set, the upper Clarke derivative of the objective function does not dominate zero. The entire study is supported by suitable illustrative examples.

Chapter 4 describes the notion of Hadamard semiderivative for IVFs. For directionally differentiable IVFs, it is proved that Lipschitz continuity implies the existence of Hadamard semiderivative for IVFs. Further, it is observed that continuity of IVF is a necessary condition for the existence of Hadamard semidifferentiability. It is found that the composition of a Hadamard semidifferentiable and a Hadamard semidifferentiable IVF is a Hadamard semidifferentiable IVF. Further, for finitely many comparable IVFs, it is shown that the Hadamard semiderivative of their maximum is the maximum of their Hadamard semiderivative. Proposed semiderivative is observed to be useful to check the convexity of an IVF. To characterize the efficient points of IOPs, it is observed that at an efficient point, the Hadamard semiderivative of the objective function does not strictly dominate zero. Further, it is found that if Hadamard semiderivative does not strictly dominate zero, the point is an efficient point of IOP. For constraint IOPs, the Karush-Kuhn-Tucker sufficient condition to obtain efficient solutions is derived. The entire study is supported by suitable illustrative examples.

Chapter 5 explaines the study of Hadamard derivative for IVFs. For an IVF, it is shown that the existence of Hadamard derivative implies the existence of Fréchet derivative and vise-versa. Further, it is proved that the existence of Hadamard derivative implies the existence of continuity of IVFs. It is found that the composition of Hadamard differentiable real-valued function and Hadamard differentiable IVF is Hadamard differentiable. Further, for finite comparable IVF, it is proved that the Hadamard derivative of the maximum of all finite comparable IVFs is the maximum of their Hadamard derivative. The proposed derivative is observed to be useful to check the convexity of an IVF and to characterize efficient points of an optimization problem with IVF. For a convex IVF, it is observed that if at a point the Hadamard derivative does not dominate to zero, then the point is an efficient point. Further, it is proved that at an efficient point, the Hadamard derivative does not dominate zero and also contains zero. For constraint IOPs, an extended Karush-Kuhn-Tucker condition by using the proposed derivative is derived. The entire study is supported by suitable illustrative examples.

Chapter 6 describes the notions of upper and lower gH-Dini semiderivative, upper and lower gH-Hadamard semiderivative for IVFs. The upper gH-Dini semiderivative and upper gH-Hadamard semiderivative of a gH-Lipschitz IVF are observed to be a positive homogeneous IVF. It is found that every gH-Lipschitz continuous IVF is upper gH-Dini semidifferentiable and upper gH-Hadamard semidifferentiable

IVFs. Further, for a convex and gH-Lipschitz IVF, it is shown that the upper gH-Din semiderivative and upper gH-Hadamard semiderivative coincide with the gH-directional derivative. It is also observed that the gH-continuity of IVF is necessary condition for existence of upper and lower gH-Dini semiderivative. With the help of the studied semiderivative, we derived a few results on characterizing efficient solutions of an IOP.

7.3 Future scopes of studies

The concerned research community is engaged to discover new theories and simultaneously trying to improve existing theories. From the analysis of the work presented in this thesis, there are several scopes for the extension. The opportunities for future research are mentioned below.

- In optimization theory, it is shown that how to solve the optimization problems by using subgradient, Michel-Penot derivative, and other derivatives. The proposed concepts of IVF may be effective to generalized Michel-Penot and other derivatives for IVFs, which are helpful to solve IOPs. Also, we shall attempt to propose a subgradient technique to find out the complete set of efficient points of IOPs by using derived concepts of this thesis.

- Another promising direction of future research can be the analysis of the fuzzy-valued functions (FVFs) as the alpha-cuts of fuzzy numbers are compact intervals. So, in the future, one can attempt to extend the proposed idea of all these derivatives for fuzzy-valued functions.

- In this thesis, we have derived the optimality conditions of constrained IOPs only for Hadamard diffrentiable IVFs. One may find the optimality conditions of constrained IOPs for other proposed derivatives.

- A noisy and uncertain environment in the control system or a differential equation appears due to the incomplete information of demand for a product and changes in the climate. In the future, we will try to solve a control problem in a noisy or uncertain environment with the help of proposed derivatives of IVFs.

- In many classification problems, the data set may not be precise and thus involves uncertainty. This may be due to errors in measurement, implementation. The standard Support Vector Machines (SVM) formulation is not applicable for such data as these quantities are interval-valued. Thus, we shall try to formulate the SVM problem for the interval-valued data set and try to solve SVM problems by using the proposed derivatives of this thesis.

7.4 Applications of derived concepts in some other fields

- **Application on Control Theory**

  The applications of the proposed derivatives in control systems and differential equations in a noisy or uncertain environment can be shown by solving the following problem (7.1). Noisy environments eventually appear due to the incomplete information (e.g., demand for a product) or unpredictable changes (e.g., changes in the climate) in the system. The general control problem in a noisy or uncertain environment that one may consider to study is the following:

$$\left.\begin{aligned} \min \quad & \mathbf{J}(x,u,t) \\ \text{subject to} \quad & \frac{dx}{dt} = \mathbf{F}(x,u,t), \\ & x(0) = x_0, \end{aligned}\right\} \tag{7.1}$$

where $x : [0,\infty) \to \mathbb{R}^n$ and $u : [0,\infty) \to \mathbb{R}^m$ are state and control variables, respectively, and $\mathbf{J}$ and $\mathbf{F}$ are two gH-Fréchet differentiable or gH-Hadamard differentiable interval-valued functions with respect to the the control variable u. In such a control problem, one may attempt to show the applications of derived derivatives to find the optimal control of the system.

- **Application to Support Vector Machine**

Support Vector Machines (SVMs) are generally used in solving classification problems. Here we consider to describe a binary classification problem. For a given data set $D = \{(x_i, y_i) | x_i \in \mathbb{R}^n,\ y_i \in \{-1,1\}, i = 1,2,\cdots,m\}$, the problem of classifying data using SVMs is equivalent to the following optimization problem:

$$\left.\begin{aligned} \min_{w,b} \quad & F(w,b) = \tfrac{1}{2}\|w\|^2 \\ \text{subject to} \quad & y_i(w^T x_i + b) \geq 1,\ i = 1,2,\ldots,m, \end{aligned}\right\} \tag{7.2}$$

where $w \in \mathbb{R}^n$ is the weight vector and $b \in \mathbb{R}$ is the bias. The constraints represent the condition that the data points lie on either side of the separating hyperplanes $w^T x + b = \pm 1$.

In many classification problems, the data set may not be precise and thus involves uncertainty. This may be due to errors in measurement, implementation, etc. For example, let us assume that we want to predict whether there will be rain tomorrow or not. The data we may require the wind speed, humidity levels, temperature, etc. These variables usually have values in intervals like 10–13 km/hr wind speed, 40–50% humidity, 30–35^oC temperature, etc. The conventional SVM formulation is not applicable for such data as the pertaining quantities are interval-valued. Thus, we formulate the SVM problem for the interval-valued data set

$$\{(\boldsymbol{X}_i, y_i) \mid \boldsymbol{X}_i \in I(\mathbb{R})^n,\ y_i \in \{-1, 1\}, i = 1, 2, \cdots, m\}$$

by

$$\left.\begin{aligned} &\min_{w,b}\ F(w, b) = \tfrac{1}{2}\|w\|^2, \\ &\text{subject to } \mathbf{G}_i(w, b) = [1, 1] \ominus_{gH} y_i \odot \left(w^\top \odot \boldsymbol{X}_i \oplus b\right) \preceq \mathbf{0}, \quad i = 1, 2, \ldots, m. \end{aligned}\right\} \tag{7.3}$$

We note that the functions F and $\mathbf{G}_i$ are gH-Hadamard differentiable and convex. At $\bar{x} = (\bar{w}, \bar{b})$, in the direction $v = (w, b)$, we have

$$\mathbf{F}_{\mathscr{H}}(\bar{x})(v) = w \odot [\bar{w}, \bar{w}] \text{ and } \mathbf{G}_{i\mathscr{H}}(\bar{x})(d) = -\left(w \odot (y_i \odot \boldsymbol{X}_i) \oplus by_i\right).$$

According to Theorem 5.23, for an efficient point $(\bar{w}, \bar{b})$ of (7.3), there exist nonnegative scalars $u_1, u_2, \ldots, u_m$ such that

$$\mathbf{0} \in \left(w \odot [\bar{w}, \bar{w}] \oplus \sum_{i=1}^{m} u_i \odot -\left(w \odot (y_i \odot \boldsymbol{X}_i) \oplus by_i\right)\right), \tag{7.4}$$

$$\text{and} \quad \mathbf{0} = u_i \odot \mathbf{G}_i(w^*, b^*), \quad i = 1, 2, \ldots, m. \tag{7.5}$$

The condition (7.4) can be simplified as

$$\mathbf{0} \in \left([w^*, w^*] \oplus \sum_{i=1}^{m}(-u_i y_i) \odot \boldsymbol{X}_i\right) \text{ and } \sum_{i=1}^{m} u_i y_i = 0.$$

The data points $\boldsymbol{X}_i$ for which $u_i \neq 0$ are called support vectors. By (7.5), corresponding to any $u_i > 0$, we have $\mathbf{G}_i(w^*, b^*) = \mathbf{0}$. Thus, corresponding to w^*, the value of the bias b^* is such a quantity that $\mathbf{G}_i(w^*, b^*) = \mathbf{0}$ for all of those $i \in \{1, 2, \ldots, m\}$ for which $u_i > 0$.

As the functions F and $\mathbf{G}_i$ are gH-Hadamard differentiable and convex, by Theorems 5.23 and 5.24, the set of conditions by which we obtain the efficient solutions of the SVM IOP (7.3) are

$$\begin{cases} \mathbf{0} \in \left([w, w] \oplus \sum_{i=1}^{m}(-u_i y_i) \odot \boldsymbol{X}_i\right), \\ \sum_{i=1}^{m} u_i y_i = 0, \\ \mathbf{0} = u_i \odot \mathbf{G}_i(w, b), \ i = 1, 2, \ldots, m. \end{cases} \tag{7.6}$$

Corresponding to any of the value of w that satisfies (7.6), we define the set of possible values of the bias by

$$\bigcap_{i:\ u_i > 0} \{b \mid \mathbf{G}_i(w, b) = \mathbf{0}\}.$$

By using any solution $\bar{w}$ and $\bar{b}$ of (7.6) and (7.4), a classifying hyperplane and the SVM classifier function are given by

$$\bar{w}^\top \boldsymbol{X} + \bar{b} = \mathbf{0} \quad \text{and} \quad s^*(\boldsymbol{X}) = \operatorname{sign}\left(\bar{w}^\top \boldsymbol{X} + \bar{b}\right),$$

where sign $(\cdot)$ denotes the sign function.

Example 7.1. *Consider the interval data set*

$$\mathbf{X}_1 = \begin{bmatrix}[3,4],[1,2]\end{bmatrix}, y_1 = 1, \quad \mathbf{X}_2 = \begin{bmatrix}[4,5],[2,3]\end{bmatrix}, y_2 = 1,$$
$$\mathbf{X}_3 = \begin{bmatrix}[5,6],[1,2]\end{bmatrix}, y_3 = 1, \quad \mathbf{X}_4 = \begin{bmatrix}[0,1],[1,2]\end{bmatrix}, y_4 = -1,$$
$$\mathbf{X}_5 = \begin{bmatrix}[1,2],[2,3]\end{bmatrix}, y_5 = -1, \quad \mathbf{X}_6 = \begin{bmatrix}[0,2],[3,4]\end{bmatrix}, y_6 = -1.$$

For this data set we find a classifying hyperplane with the help of the IOP SVM (7.3).

In order to find a classifying hyperplane, we need to find a possible solution (w, b) of (7.6) along with the corresponding u_i's.

We observe that for $(u_1, u_2, u_3, u_4, u_5, u_6) = (1, 0, 0, 0, 1, 0)$ we have $\sum_{i=1}^{6} u_i y_i = 0$. For these values of u_i's, the first condition in (7.6) reduces to

$$\begin{aligned} &\mathbf{0}_v^n \in ([w, w] \oplus (-1) \odot \boldsymbol{X}_1 \oplus \boldsymbol{X}_5) \\ \text{or, } &[w, w] \in (-1) \odot ((-1) \odot \boldsymbol{X}_1 \oplus \boldsymbol{X}_5) \\ \text{or, } &w \in ([1, 3], [-2, 0]). \end{aligned} \tag{7.7}$$

Denoting $w = (w_1, w_2) \in \mathbb{R}^2$, the condition (7.7) reduces to

$$1 \leq w_1 \leq 3 \text{ and } -2 \leq w_2 \leq 0. \tag{7.8}$$

Let us choose $w_1^* = 1$ and $w_2^* = -2$. Corresponding to this $w^* = (w_1^*, w_2^*) = (1, -2)$, from (7.4) and the third condition in (7.6), the set of possible values of the bias b is given by

$$\begin{aligned} & \bigcap_{i=1,5} \{b \in \mathbb{R} \mid \mathbf{G}_i(w^*, b) = \mathbf{0}\} \\ = \ & \{b \in \mathbb{R} \mid \mathbf{G}_1(w^*, b) = \mathbf{0}\} \bigcap \{b \in \mathbb{R} \mid \mathbf{G}_5(w^*, b) = \mathbf{0}\} \\ = \ & \{b \in \mathbb{R} \mid b \in [-2, 1]\} \cap \{b \in \mathbb{R} \mid b \in [-6, -1]\} \\ = \ & \{b \in \mathbb{R} \mid -2 \leq b \leq -1\}. \end{aligned}$$

Thus corresponding to $w_1^* = 1$ and $w_2^* = -2$ the set of classifying hyperplanes is given by

$$\begin{aligned} & w_1^* x_1 + w_2^* x_2 + b = 0, \quad -2 \leq b \leq -1 \\ \text{i.e.,} \quad & x_1 - 2x_2 + b = 0, -2 \leq b \leq -1. \end{aligned}$$

For any choice of b in $[-2, -1]$, note that the value of the objective function F is identical (and it is $\frac{5}{2}$).

****