# Chapter 6

# Generalized Hukuhara-Dini Semiderivative of Interval-valued Functions and its Application in Interval Optimization Problems

## 6.1 Introduction

The classical concept of derivative (Fréchet, Gâteaux, Clarke, Hadamard, and directional) has been proved one of the most useful tools in mathematics, as a large variety of problems has been described and solved by means of this topic. In solving classical optimization problems the use of derivatives is inevitable. Besides treating the classical smooth problems, the mathematicians got many impulses in the last decades from other sciences (mainly from economics, engineering, etc.) in order to treat nonsmooth, nondifferentiable optimization problems. The analysis of such problems has definitely required some generalizations of the derivative as Dini semiderivative. Since the early 1960's much effort has gone into the development of a generalized kind of differentiation that can be useful in the analysis of optimization problems. The subject has grown very rapidly since then. Our aim in this chapter is to collect and to state the most relevant results concerning Dini semiderivatives.

## 6.2 Motivation

From the literature on the analysis of IVFs, one can notice that the study of traditional generalized semiderivative (upper and lower Dini semiderivative, upper and lower Hadamard semiderivative) for IVFs have not been developed so far. However, the basic properties of generalized semiderivative might be beneficial for characterizing and capturing the optimal solutions of IOPs with nonsmooth IVFs. In this chapter, we generalize these semiderivative for IVFs.

## 6.3 Contributions

In this chapter, the notions of upper and lower $gH$-Dini semiderivative, upper and lower $gH$-Hadamard semiderivative for IVFs are proposed. The upper $gH$-Dini semiderivative and upper $gH$-Hadamard semiderivative of a $gH$-Lipschitz IVF are observed to be a positive homogeneous IVF. It is found that every $gH$-Lipschitz continuous IVF is upper $gH$-Dini semidifferentiable and upper $gH$-Hadamard semidifferentiable IVFs. Further, for a convex and $gH$-Lipschitz IVF, it is shown that the upper $gH$-Din semiderivative and upper $gH$-Hadamard semiderivative coincide with the $gH$-directional derivative. It is also observed that the $gH$-continuity of IVF is necessary condition for the existence of upper and lower $gH$-Dini semiderivative.

With the help of the studied semiderivative, we derived a few results on characterizing efficient solutions of an IOP.

Primary contributions of this chapter are as follows:

(i) For a convex and $gH$-Lipschitz continuous IVF, it is observed that the upper $gH$-Dini semiderivative coincides with $gH$-directional derivative, $gH$-Hadamard semiderivative, and upper $gH$-Clarke derivative.

(ii) For finite comparable IVF, we prove that the upper $gH$-Dini semiderivative of the maximum of all finite comparable IVFs is the maximum of their upper $gH$-Dini semiderivative.

(iii) For a convex IVF, it is proved that at any point if $gH$-Dini semiderivative does not dominate zero, then the point is an efficient solution to the IOP.

## 6.4 Dini Semiderivative of Interval-valued Functions

In this section, we define upper and lower Dini semiderivative, upper and lower Hadamard semiderivative for IVFs and derive some results related to these semiderivative.

**Definition 6.1** (Upper $gH$-Dini semiderivative)**.** Let $\mathbf{F}$ be an IVF defined on $\mathcal{S}$. For $\bar{x} \in \mathcal{S}$ and $v \in \mathcal{X}$, if the limit superior

$$\limsup_{\lambda \to 0+} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x} + \lambda v) \ominus_{gH} \mathbf{F}(\bar{x})\big) = \lim_{\delta \to 0} \left( \sup_{\lambda \in (0,\delta)} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x} + \lambda v) \ominus_{gH} \mathbf{F}(\bar{x})\big) \right)$$

exists, then the limit superior value is called *upper gH-Dini semiderivative* of $\mathbf{F}$ at $\bar{x}$ in the direction $v$, and it is denoted by $\overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(v)$. If this limit superior exists for all $v \in \mathcal{X}$, then $\mathbf{F}$ is said to be *upper gH-Dini semidifferentiable* at $\bar{x}$.

**Definition 6.2** (Lower $gH$-Dini semiderivative)**.** Let $\mathbf{F}$ be an IVF defined on $\mathcal{S}$. For $\bar{x} \in \mathcal{S}$ and $v \in \mathcal{X}$, if the limit inferior

$$\liminf_{\lambda \to 0+} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x} + \lambda v) \ominus_{gH} \mathbf{F}(\bar{x})\big) = \lim_{\delta \to 0} \left( \inf_{\lambda \in (0,\delta)} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x} + \lambda v) \ominus_{gH} \mathbf{F}(\bar{x})\big) \right)$$

exists, then the limit inferior value is called *lower gH-Dini semiderivative* of $\mathbf{F}$ at $\bar{x}$ in the direction $v$, and it is denoted by $\underline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(v)$. If this limit inferior exists for all $v \in \mathcal{X}$, then $\mathbf{F}$ is said to be *lower gH-Dini semidifferentiable* at $\bar{x}$.

If $\mathbf{F}$ has both upper and lower $gH$-Dini semiderivatives at $\bar{x}$ and they are equal, then $\mathbf{F}$ is called $gH$-Dini semidifferentiable at $\bar{x}$.

*Remark* 6.3. It is clear that $\mathbf{F}$ is lower $gH$-Dini semidifferentiable at $\bar{x}$ if and only if $(-1) \odot \mathbf{F}$ is upper $gH$-Dini semidifferentiable at $\bar{x}$.

$$\underline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(v) = (-1) \odot \overline{\mathbf{G}}_{\mathscr{D}}(\bar{x})(v), \text{ where } \mathbf{F} = (-1) \odot \mathbf{G}.$$

That is why we deal only with the upper $gH$-Dini semidifferentiability in this study.

**Definition 6.4** (Upper $gH$-Hadamard semiderivative of IVF)**.** Let $\mathbf{F}$ be an IVF on a nonempty subset $\mathcal{S}$ of $\mathcal{X}$. For $\bar{x} \in \mathcal{S}$ and $v \in \mathcal{X}$, if the limit superior

$$\limsup_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) = \lim_{\delta \to 0} \left( \sup_{\lambda \in (0,\delta),\ h \in \overline{\mathcal{B}}(v,\delta) \cap \mathcal{S}} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})\big) \right)$$

exists, then the limit superior value, denoted by $\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v)$, is called *upper gH-Hadamard semiderivative* of $\mathbf{F}$ at $\bar{x}$ in the direction $v$. If this limit superior exists for all $v \in \mathcal{X}$, then $\mathbf{F}$ is said to be *upper gH-Hadamard semidifferentiable* at $\bar{x}$.

**Definition 6.5** (Lower $gH$-Hadamard semiderivative of IVF)**.** Let $\mathbf{F}$ be an IVF on a nonempty subset $\mathcal{S}$ of $\mathcal{X}$. For $\bar{x} \in \mathcal{S}$ and $v \in \mathcal{X}$, if the limit inferior

$$\liminf_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) = \lim_{\delta \to 0} \left( \inf_{\lambda \in (0,\delta),\ h \in \overline{\mathcal{B}}(v,\delta) \cap \mathcal{S}} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x})\big) \right)$$

exists, then the limit inferior value, denoted by $\underline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v)$, is called *lower $gH$-Hadamard semiderivative* of $\mathbf{F}$ at $\bar{x}$ in the direction $v$. If this limit inferior exists for all $v \in \mathcal{X}$, then $\mathbf{F}$ is said to be *lower $gH$-Hadamard semidifferentiable* at $\bar{x}$.

*Remark* 6.6. It is clear that $\mathbf{F}$ is lower $gH$-Hadamard semidifferentiable at $\bar{x}$ if and only if $(-1) \odot \mathbf{F}$ is upper $gH$-Hadamard semidifferentiable at $\bar{x}$ and

$$\underline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v) = (-1) \odot \overline{\mathbf{G}}_{\mathscr{H}'}(\bar{x})(v), \text{ where } \mathbf{F} = (-1) \odot \mathbf{G}.$$

That is why we deal only with the upper $gH$-Hadamard semidifferentiability in this study.

**Example 6.1.** *In this example, we calculate the upper $gH$-Dini semiderivative and upper $gH$-Hadamard semiderivative at $\bar{x} = 0$ for the IVF $\boldsymbol{F}(x) = |x| \odot \boldsymbol{C}$, where $\boldsymbol{0} \preceq \boldsymbol{C} \in I(\mathbb{R})$, $\mathcal{X}$ is the Euclidean space $\mathbb{R}$, and $\mathcal{S} = \mathcal{X}$.*

*For any $v \in \mathcal{X}$, we see that*

$$\begin{aligned}
& \limsup_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) \\
\preceq\ & \limsup_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (|\bar{x}| \odot \boldsymbol{C} \oplus \lambda |h| \odot \boldsymbol{C} \ominus_{gH} |\bar{x}| \odot \boldsymbol{C}) \quad \text{by Lemma 1.7} \\
=\ & |v| \odot \boldsymbol{C}.
\end{aligned}$$

*Further,*

$$\begin{aligned}
&\limsup_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda}\odot\left(|\bar{x}+\lambda h|\odot \boldsymbol{C}\ominus_{gH}|\bar{x}|\odot \boldsymbol{C}\right)\\
\succeq\ &\limsup_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda}\odot\left(|\lambda h|\odot \boldsymbol{C}\right)\\
=\ &|v|\odot \boldsymbol{C}.
\end{aligned}$$

*Hence,* $\overline{\boldsymbol{F}}_{\mathscr{H}'}(\bar{x})(v)=|v|\odot\boldsymbol{C}$.

*In similar way, we get* $\overline{\boldsymbol{F}}_{\mathscr{D}}(\bar{x})(v)=|v|\odot\boldsymbol{C}$.

**Lemma 6.7.** *If* $\underline{f}$ *and* $\overline{f}$ *are upper Hadamard and upper Dini semidifferentiable at* $\bar{x}\in\mathcal{S}\subseteq\mathcal{X}$*, then the IVF* $\boldsymbol{F}$ *is upper gH-Hadamard and upper gH-Dini differentiable at* $\bar{x}\in\mathcal{S}$*, respectively.*

*Proof.* Since $\underline{f}$ and $\overline{f}$ are upper Hadamard semidifferentiable at $\bar{x}$. Therefore, both of the following limits

$$\limsup_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda}\phi_1(\lambda,h) \text{ and } \limsup_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda}\phi_2(\lambda,h)$$

exist, where $\phi_1(\lambda,h)=\underline{f}(\bar{x}+\lambda h)-\underline{f}(\bar{x})$ and $\phi_2(\lambda,h)=\overline{f}(\bar{x}+\lambda h)-\overline{f}(\bar{x})$. Thus,

$$\begin{aligned}
&\limsup_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda}\left(\phi_1(\lambda,h)+\phi_2(\lambda,h)\right) \text{ and } \limsup_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda}|\phi_1(\lambda,h)-\phi_2(\lambda,h)| \text{ exist}\\
\implies\ &\limsup_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{2\lambda}\Big(\phi_1(\lambda,h)+\phi_2(\lambda,h)-|\phi_1(\lambda,h)-\phi_2(\lambda,h)|\Big) \text{ and}\\
&\limsup_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{2\lambda}\Big(\phi_1(\lambda,h)+\phi_2(\lambda,h)+|\phi_1(\lambda,h)-\phi_2(\lambda,h)|\Big) \text{ exist}\\
\implies\ &\limsup_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda}\left(\min\{\phi_1(\lambda,h),\phi_2(\lambda,h)\}\right) \text{ and}\\
&\qquad\qquad\limsup_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda}\left(\max\{\phi_1(\lambda,h),\phi_2(\lambda,h)\}\right) \text{ exist}
\end{aligned}$$

$$\implies \limsup_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) \text{ exists.}$$

Hence, $\mathbf{F}$ is upper $gH$-Hadamard semidifferentiable IVF at $\bar{x} \in \mathcal{S}$.

In similar way, we can show that $\mathbf{F}$ is upper $gH$-Dini semidifferentiable IVF at $\bar{x} \in \mathcal{S}$.

□

*Remark* 6.8. Let $\mathbf{F}$ be upper $gH$-Dini semidiffrentiable IVF at a point $\bar{x}$ in $\mathcal{S}$. Then, $\mathbf{F}$ is not necessarily upper $gH$-Hadamard semidifferentiable at $\bar{x} \in \mathcal{S}$. For instance, take $\mathcal{X}$ as the Euclidean space $\mathbb{R}^2$, $\mathcal{S} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0, x_2 \geq 0\}$ and the IVF $\mathbf{F} : \mathcal{S} \to I(\mathbb{R})$, which is defined by

$$\mathbf{F}(x_1, x_2) = \begin{cases} x_1^2 \left(1 + \frac{1}{x_2}\right) \odot [3,\ 8] & \text{if } x = (x_1, x_2) \neq (0, 0) \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then, at $\bar{x} = (0, 0)$ and $v = (v_1, v_2) \in \mathcal{X}$ such that for sufficiently small $\lambda > 0$ so that $\bar{x} + \lambda v \in \mathcal{S}$, we have

$$\limsup_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda v) \ominus_{gH} \mathbf{F}(\bar{x})) = \begin{cases} \frac{v_1^2}{v_2} \odot [3,\ 8] & \text{if } v_2 \neq 0 \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Hence, $\mathbf{F}$ is a upper $gH$-Dini semidifferentiable at $\bar{x}$ in every direction $v \in \mathcal{X}$. Again, for $x = (x_1, x_2) \in \mathcal{S}$ and $h = (h_1, h_2) \neq (0, 0) \in \mathcal{X}$, we have Along $h_2 = mh_1^2$, where $m$ is any real number,

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) = \frac{1}{m} \odot [3,\ 8].$$

Hence, $\frac{1}{m} \to \infty$ as $m \to 0$. Consequently, for $v = (0, 0)$,

$$\limsup_{\substack{\lambda \to 0+ \\ v \to 0}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda v) \ominus_{gH} \mathbf{F}(\bar{x})) \quad \text{does not exist.}$$

This implies that $\mathbf{F}$ has no upper $gH$-Hadamard semiderivative at $\bar{x} \in \mathcal{S}$.

The following theorem extends the well-known result from [42] for Lipschitz continuous functions to $gH$-Lipschitz continuous IVFs with the help of Lemma 6.7.

**Theorem 6.9.** *Let $\mathcal{S} \subset \mathcal{X}$ and $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be a gH-Lipschitz continuous IVF at an interior point $\bar{x}$ with a Lipschitz constant $K'$. Then, $\boldsymbol{F}$ is upper gH-Hadamard semidifferentiable at $\bar{x}$ and*

$$\|\overline{\boldsymbol{F}}_{\mathscr{H}'}(\bar{x})(v)\|_{I(\mathbb{R})} \leq K'\|v\| \text{ for all } v \in \mathcal{X}.$$

*Proof.* Since $\mathbf{F}$ is $gH$-Lipschitz continuous at $\bar{x} \in \mathcal{S}$, for any $v \in \mathcal{X}$, we get for $\lambda > 0$ that

$$\left\| \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) \right\|_{I(\mathbb{R})} \leq \frac{1}{\lambda} K' \|\bar{x} + \lambda h - \bar{x}\| = K'\|h\|, \tag{6.1}$$

if $\lambda$ are sufficiently close to 0. From inequality (6.1), we have

$$\left| \frac{1}{\lambda} \left( \underline{f}(\bar{x} + \lambda h) - \underline{f}(\bar{x}) \right) \right| \leq K' \left( \|v\| + \|v - h\| \right)$$

and

$$\left| \frac{1}{\lambda} \left( \overline{f}(\bar{x} + \lambda h) - \overline{f}(x) \right) \right| \leq K' \left( \|v\| + \|v - h\| \right),$$

as $\lambda \to 0+$ and $h \to v$.

Hence, the limit superior $\overline{\underline{f}}_{\mathscr{H}'}(\bar{x})(v)$ and $\overline{\overline{f}}_{\mathscr{H}'}(\bar{x})(v)$ exist at $\bar{x}$. By Lemma 6.7, the limit superior $\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v)$ exists.

Furthermore, by $gH$-Lipschitz continuity of $\mathbf{F}$ on $\mathcal{S}$, we have the following for all $v \in \mathcal{X}$:

$$\begin{aligned}
\|\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v)\|_{I(\mathbb{R})} &= \left\| \limsup_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) \right\|_{I(\mathbb{R})} \\
&\leq \limsup_{\substack{\lambda \to 0+ \\ h \to v}} \left\| \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) \right\|_{I(\mathbb{R})} \text{ by Lemma 3.1} \\
&\leq K'\|v\| \text{ by (6.1)}.
\end{aligned}$$

□

**Theorem 6.10.** *Let $\mathcal{S} \subset \mathcal{X}$ and $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be a $gH$-Lipschitz continuous IVF at an interior point $\bar{x}$ with a Lipschitz constant $K'$. Then, $\boldsymbol{F}$ is upper $gH$-Dini semidifferentiable at $\bar{x}$ and*

$$\|\overline{\boldsymbol{F}}_{\mathscr{D}}(\bar{x})(v)\|_{I(\mathbb{R})} \leq K'\|v\| \text{ for all } v \in \mathcal{X}.$$

*Proof.* Since $\mathbf{F}$ is $gH$-Lipschitz continuous at $\bar{x} \in \mathcal{S}$, for any $v \in \mathcal{X}$, we get for $\lambda > 0$ that

$$\left\| \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda v) \ominus_{gH} \mathbf{F}(\bar{x})) \right\|_{I(\mathbb{R})} \leq \frac{1}{\lambda} K'\|\bar{x} + \lambda v - \bar{x}\| = K'\|v\|, \tag{6.2}$$

if $\lambda$ are sufficiently close to 0. From inequality (6.2), we have

$$\left| \frac{1}{\lambda} \left( \underline{f}(\bar{x} + \lambda v) - \underline{f}(\bar{x}) \right) \right| \leq K'\|v\| \text{ and } \left| \frac{1}{\lambda} \left( \overline{f}(\bar{x} + \lambda v) - \overline{f}(x) \right) \right| \leq K'\|v\|,$$

as $\lambda \to 0+$.

Hence, the limit superior $\overline{\underline{f}}_{\mathscr{D}}(\bar{x})(v)$ and $\overline{\overline{f}}_{\mathscr{D}}(\bar{x})(v)$ exist at $\bar{x}$. By Lemma 6.7, the limit superior $\overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(v)$ exists.

Furthermore, by $gH$-Lipschitz continuity of $\mathbf{F}$ on $\mathcal{S}$, we have the following for all

$v \in \mathcal{X}$:

$$\begin{aligned}\|\overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(v)\|_{I(\mathbb{R})} &= \left\| \limsup_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda v) \ominus_{gH} \mathbf{F}(\bar{x})) \right\|_{I(\mathbb{R})} \\ &\leq \limsup_{\lambda \to 0+} \left\| \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda v) \ominus_{gH} \mathbf{F}(\bar{x})) \right\|_{I(\mathbb{R})} \text{ by Lemma 3.1} \\ &\leq K'\|v\| \text{ by (6.2)}.\end{aligned}$$

□

**Theorem 6.11.** *Let $\boldsymbol{F} : \mathcal{X} \to I(\mathbb{R})$ be gH-Lipschitz continuous IVF at some $\bar{x} \in \mathcal{X}$. Then,*

(i) *for all $v \in \mathbb{R}^n$,*

$$\overline{\boldsymbol{F}}_{\mathscr{C}}(\bar{x})(v) \not\prec \overline{\boldsymbol{F}}_{\mathscr{H}'}(\bar{x})(v) \text{ and } \underline{\boldsymbol{F}}_{\mathscr{C}}(\bar{x})(v) \not\succ \underline{\boldsymbol{F}}_{\mathscr{H}'}(\bar{x})(v),$$

(ii) *the IVFs $\overline{\boldsymbol{F}}_{\mathscr{H}'}(\bar{x}), \overline{\boldsymbol{F}}_{\mathscr{C}}(\bar{x}) : \mathcal{X} \to I(\mathbb{R})$ satisfy*

$$\overline{\boldsymbol{F}}_{\mathscr{H}'}(\bar{x})(\alpha v) = \alpha \odot \overline{\boldsymbol{F}}_{\mathscr{H}'}(\bar{x})(v) \text{ and } \overline{\boldsymbol{F}}_{\mathscr{C}}(\bar{x})(\alpha v) = \alpha \odot \overline{\boldsymbol{F}}_{\mathscr{C}}(\bar{x})(v)$$

*for all $\alpha \geq 0$ and all $v \in \mathbb{R}^n$.*

*Proof.* (i). Since $\mathbf{F}$ is $gH$-Lipschitz continuous at $\bar{x} \in \mathcal{X}$, then $\overline{\mathbf{F}}_{\mathscr{C}}(\bar{x})(v)$ and $\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v)$ exist. Also, $\mathbf{F}_{\mathscr{H}'}(\bar{x})(0) = \mathbf{0}$ since $\mathbf{F}_{\mathscr{D}}(\bar{x})(0) = \mathbf{0}$ and $\mathbf{F}$ is $gH$-lipschitz continuous.

Due to existence of $\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v)$, there exist sequences $\{\lambda_n\}, \lambda_n > 0$ and $\{h_n\}, h_n \neq v$ such that $\lambda_n \to 0+, h_n \to v$ as $n \to \infty$ and

$$\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v) = \limsup_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) = \lim_{n \to \infty} \frac{1}{\lambda_n} \odot \big(\mathbf{F}(\bar{x} + \lambda_n h_n) \ominus_{gH} \mathbf{F}(\bar{x})\big).$$

By (iii) of Lemma 1.5, we have

$$\begin{aligned}
&\frac{1}{\lambda_n} \odot \big(\mathbf{F}(\bar{x} + \lambda_n h_n) \ominus_{gH} \mathbf{F}(\bar{x})\big) \\
\nsucc\ &\frac{1}{\lambda_n} \odot \Big(\mathbf{F}(\bar{x} + \lambda_n (h_n - v) + \lambda_n v) \ominus_{gH} \mathbf{F}(\bar{x} + \lambda_n (h_n - v))\Big) \\
&\qquad\qquad \oplus \frac{1}{\lambda_n} \odot \Big(\mathbf{F}(\bar{x} + \lambda_n (h_n - v)) \ominus_{gH} \mathbf{F}(\bar{x})\Big). \qquad (6.3)
\end{aligned}$$

Since $h_n - v \to 0$ and $\bar{x} + \lambda_n(h_n - v) \to \bar{x}$ as $n \to \infty$, then

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \odot \big(\mathbf{F}(\bar{x} + \lambda_n (h_n - v)) \ominus_{gH} \mathbf{F}(\bar{x})\big) = \mathbf{F}_{\mathscr{H}'}(\bar{x})(0) = \mathbf{0}, \qquad (6.4)$$

and

$$\limsup_{n\to\infty} \frac{1}{\lambda_n} \odot \Big(\mathbf{F}(\bar{x} + \lambda_n (h_n - v) + \lambda_n v) \ominus_{gH} \mathbf{F}(\bar{x} + \lambda_n (h_n - v))\Big) \preceq \overline{\mathbf{F}}_{\mathscr{C}}(\bar{x})(v). \quad (6.5)$$

From (6.3), (6.4) and (6.5), we obtain

$$\overline{\mathbf{F}}_{\mathscr{C}}(\bar{x})(v) \nprec \overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v).$$

Let $\mathbf{G} = (-1) \odot \mathbf{F}$. Then, $\mathbf{G}$ is also $gH$-Lipschitz continuous at $\bar{x}$ and

$$\begin{aligned}
\overline{\mathbf{G}}_{\mathscr{C}}(\bar{x})(v) \nprec \overline{\mathbf{G}}_{\mathscr{H}'}(\bar{x})(v) \quad &\Longrightarrow \quad (-1) \odot \overline{\mathbf{G}}_{\mathscr{C}}(\bar{x})(v) \nsucc (-1) \odot \overline{\mathbf{G}}_{\mathscr{H}'}(\bar{x})(v) \\
&\Longrightarrow \quad \underline{\mathbf{F}}_{\mathscr{C}}(\bar{x})(v) \nsucc \underline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v).
\end{aligned}$$

(ii). For an arbitrary $v \in \mathcal{S}$ and $\delta \geq 0$, we have

$$\begin{aligned}
\limsup_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda\delta h) \ominus_{gH} \mathbf{F}(\bar{x})) &= \delta \odot (\limsup_{\substack{\lambda\to 0+\\ h\to v}} \tfrac{1}{\lambda\delta} \odot (\mathbf{F}(\bar{x} + \lambda\delta h) \ominus_{gH} \mathbf{F}(\bar{x}))) \\
&= \delta \odot \mathbf{F}_{\mathscr{C}}(\bar{x})(v).
\end{aligned}$$

Thus, $\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(\delta v) = \delta \odot \overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v)$.

In similar way we can easily check that $\overline{\mathbf{F}}_{\mathscr{C}}(\bar{x})(\delta v) = \delta \odot \overline{\mathbf{F}}_{\mathscr{C}}(\bar{x})(v)$. □

For convex and $gH$-Lipschitz continuous IVFs, $gH$-directional derivative and upper $gH$-Clarke derivative coincide as the next theorem states.

**Theorem 6.12.** *Let $\boldsymbol{F} : \mathcal{X} \to I(\mathbb{R})$ be convex IVF on a convex set $\mathcal{X}$ and $gH$-Lipschitz continuous at some $\bar{x} \in \mathcal{X}$. Then, the upper $gH$-Hadamard semiderivative and the upper $gH$-Dini semiderivative of $\boldsymbol{F}$ at $\bar{x}$ in the direction $v \in \mathcal{X}$ are equals.*

*Proof.* Since $\mathbf{F}$ is $gH$-Lipschitz continuous at $\bar{x}$, from Theorem 6.10 and 6.9, we get that $\mathbf{F}$ is upper $gH$-Dini semidifferentiable and upper $gH$-Hadamard semidifferentiable at any $\bar{x}$ in every direction $v \in \mathcal{X}$. Thus, by Definitions 6.1 and 6.4, we observe that

$$\overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(v) \preceq \overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v) \text{ for all } v. \tag{6.6}$$

For the reverse of inequality (6.6) , we write

$$\begin{aligned}\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v) &= \limsup_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x}))\\ &= \lim_{\substack{\epsilon\to 0+\\ \delta\to 0+}} \sup_{\|h-v\|<\delta} \sup_{0<\lambda<\epsilon} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x}))\,.\end{aligned}$$

Due to convexity of $\mathbf{F}$ on $\mathcal{X}$ and Lemma 3.1 of [28], we have the following equality

$$\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v) = \lim_{\substack{\epsilon\to 0+\\ \delta\to 0+}} \sup_{\|h-v\|<\delta} \frac{1}{\epsilon} \odot (\mathbf{F}(\bar{x}+\epsilon h) \ominus_{gH} \mathbf{F}(\bar{x}))\,. \tag{6.7}$$

For an arbitrary $\alpha > 0$ and from (6.7), we obtain

$$\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v) = \lim_{\epsilon\to 0+} \sup_{\|h-v\|<\epsilon\alpha} \frac{1}{\epsilon} \odot (\mathbf{F}(\bar{x}+\epsilon h) \ominus_{gH} \mathbf{F}(\bar{x}))\,.$$

Since $\mathbf{F}$ is $gH$-Lipschitz continuous at $\bar{x}$ and $\|h-v\| < \epsilon\alpha$ for sufficiently small $\epsilon > 0$, then

$$\begin{aligned}
& \left\| \tfrac{1}{\epsilon} \odot (\mathbf{F}(\bar{x}+\epsilon h) \ominus_{gH} \mathbf{F}(\bar{x})) \ominus_{gH} \tfrac{1}{\epsilon} \odot (\mathbf{F}(\bar{x}+\epsilon v) \ominus_{gH} \mathbf{F}(\bar{x})) \right\|_{I(\mathbb{R})} \\
\leq\ & \left\| \tfrac{1}{\epsilon} \odot (\mathbf{F}(\bar{x}+\epsilon h) \ominus_{gH} \mathbf{F}(\bar{x}+\epsilon v)) \right\|_{I(\mathbb{R})} \text{ by (iv) of Lemma 1.6} \\
\leq\ & \tfrac{1}{\epsilon} K' \|h-v\|, \text{ where } K' \text{ is the Lipschitz constant of } \mathbf{F} \text{ at } \bar{x} \in \mathcal{X} \\
\leq\ & K'\alpha\epsilon.
\end{aligned}$$

Then, by (iii) of Lemma 1.6, we have

$$\begin{aligned}
\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v) &\preceq \limsup_{\epsilon \to 0+} \frac{1}{\epsilon} \odot (\mathbf{F}(\bar{x}+\epsilon v) \ominus_{gH} \mathbf{F}(\bar{x})) \oplus [K'\alpha\epsilon, K'\alpha\epsilon] \\
&= \overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(v) \oplus [K'\alpha\epsilon, K'\alpha\epsilon].
\end{aligned}$$

Due to arbitrariness of $\alpha > 0$, we get

$$\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v) \preceq \overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(v) \text{ for all } v. \tag{6.8}$$

From (6.6) and (6.8), we obtain

$$\overline{\mathbf{F}}_{\mathscr{C}}(\bar{x})(v) = \overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(v).$$

□

**Theorem 6.13.** *Let $\mathbf{F} : \mathcal{X} \to I(\mathbb{R})$ be convex IVF on a convex set $\mathcal{X}$ and $gH$-Lipschitz continuous at some $\bar{x} \in \mathcal{X}$. Then, the upper $gH$-Hadamard semiderivative and the $gH$-directional derivative of $\mathbf{F}$ at $\bar{x}$ in the direction $v \in \mathcal{X}$ are equals.*

*Proof.* Since $\mathbf{F}$ is a convex IVF on $\mathcal{X}$, we get by Theorem 3.1 of [28] that $\mathbf{F}$ is

$gH$-directionally differentiable at $\bar{x}$ in every direction $v \in \mathcal{X}$. Also, as $\mathbf{F}$ is $gH$-Lipschitz continuous at $\bar{x}$, from Theorem 6.9, we get that $\mathbf{F}$ is upper $gH$-Hadamard semidifferentiable at any $\bar{x}$ in every direction $v \in \mathcal{X}$. Thus, by Definitions 2.4.1 and 6.4, we observe that

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(v) \preceq \overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v) \text{ for all } v. \tag{6.9}$$

Since $\mathbf{F}$ is convex and $gH$-Lipschitz continuous on $\mathcal{X}$, then by (iii) of Lemma 1.6 and Theorem 6.12, we have

$$\begin{aligned}
\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v) &\preceq \limsup_{\epsilon \to 0+} \frac{1}{\epsilon} \odot (\mathbf{F}(\bar{x}+\epsilon v) \ominus_{gH} \mathbf{G}(\bar{x})) \oplus [K'\alpha\epsilon, K'\alpha\epsilon] \\
&= \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \odot (\mathbf{F}(\bar{x}+\epsilon v) \ominus_{gH} \mathbf{F}(\bar{x})) \oplus [K'\alpha\epsilon, K'\alpha\epsilon] \text{ due to Lemma 3.1 of [28]} \\
&= \mathbf{F}_{\mathscr{D}}(\bar{x})(v) \oplus [K'\alpha\epsilon, K'\alpha\epsilon],
\end{aligned}$$

where $K'$ is Lipschitz constant and $\alpha$ is arbitrary positive real number.
Due to arbitrariness of $\alpha > 0$, we get

$$\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v) \preceq \mathbf{F}_{\mathscr{D}}(\bar{x})(v) \text{ for all } v. \tag{6.10}$$

From (6.9) and (6.10), we obtain

$$\overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v) = \mathbf{F}_{\mathscr{D}}(\bar{x})(v).$$

□

**Theorem 6.14.** *Let $\boldsymbol{F} : \mathcal{X} \to I(\mathbb{R})$ be convex IVF on a convex set $\mathcal{X}$ and $gH$-Lipschitz continuous at some $\bar{x} \in \mathcal{X}$. Then, at $\bar{x}$ in the direction $v \in \mathcal{X}$,*

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(v) = \boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) = \overline{\boldsymbol{F}}_{\mathscr{C}}(\bar{x})(v) = \overline{\boldsymbol{F}}_{\mathscr{H}'}(\bar{x})(v) = \overline{\boldsymbol{F}}_{\mathscr{D}}(\bar{x})(v).$$

*Proof.* By using Theorems 6.12, 6.13 and Theorem 3.2 of [28], we get required result. □

*Remark* 6.15. If IVF $\mathbf{F}$ is defined by $\mathbf{F}(x) = |x| \odot \mathbf{C}$, where $\mathbf{0} \preceq \mathbf{C} \in I(\mathbb{R})$, $\mathcal{X}$ is the Euclidean space $\mathbb{R}$, and $\mathcal{S} = \mathcal{X}$, then we can easily check that

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(v) = \mathbf{F}_{\mathscr{H}'}(\bar{x})(v) = \overline{\mathbf{F}}_{\mathscr{C}}(\bar{x})(v) = \overline{\mathbf{F}}_{\mathscr{H}'}(\bar{x})(v) = \overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(v) = |v| \odot \mathbf{C}.$$

**Theorem 6.16.** *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$ and IVF $\boldsymbol{F}$ be defined on $\mathcal{S}$. Let $\boldsymbol{F}$ be lower and upper gH-Dini semidifferentiable at $\bar{x} \in \mathcal{S}$. Then $\boldsymbol{F}$ is gH-continuous at $\bar{x}$.*

*Proof.* Let us suppose that $\mathbf{F}$ is not $gH$-continuous at $\bar{x}$. Then, there exists $\epsilon > 0$ and a sequence $x_n \to \bar{x}$ such that

$$\|\mathbf{F}(x_n) \ominus_{gH} \mathbf{F}(\bar{x})\|_{I(\mathbb{R})} > \epsilon.$$

It follows that

$$\left\| \frac{\mathbf{F}(x_n) \ominus_{gH} \mathbf{F}(\bar{x})}{x_n - \bar{x}} \right\|_{I(\mathbb{R})} \to +\infty$$

for $x_n \to \bar{x}$. This implies that either upper $gH$-Dini semiderivative or lower $gH$-Dini semiderivative is not finite. Which is a contraction to $\mathbf{F}$ is not lower and upper $gH$-Dini semiderivative at $\bar{x}$. Hence, $\mathbf{F}$ is $gH$-continuous at $\bar{x}$. □

**Theorem 6.17.** *Let $I$ be a finite set of indices and $\boldsymbol{F}_i : \mathcal{X} \to I(\mathbb{R})$ be family of comparable gH-continuous IVFs such that $\overline{\boldsymbol{F}}_{\mathscr{D}}(\bar{x})(h)$ exists and for all $x \in \mathcal{X}$, $\boldsymbol{F}(x) = \max_{i \in I} \boldsymbol{F}_i(x)$. Then,*

$$\overline{\boldsymbol{F}}_{\mathscr{D}}(\bar{x})(h) = \max_{i \in \mathcal{A}(\bar{x})} \overline{\boldsymbol{F}}_{i_{\mathscr{D}}}(\bar{x})(h), \text{ where } \mathcal{A}(\bar{x}) = \{i : \boldsymbol{F}_i(\bar{x}) = \boldsymbol{F}(\bar{x})\}.$$

*Proof.* Let $\bar{x} \in \mathcal{X}$ and $h \in \mathcal{X}$ such that $\bar{x} + \lambda h \in \mathcal{X}$ for $\lambda > 0$. Then,

$$\begin{aligned}
&\mathbf{F}_i(\bar{x} + \lambda h) \preceq \mathbf{F}(\bar{x} + \lambda h) \text{ for all } i \in I \\
\text{or, } &\mathbf{F}_i(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x}) \preceq \mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } i \in I \\
\text{or, } &\mathbf{F}_i(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}_i(\bar{x}) \preceq \mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for each } i \in \mathcal{A}(\bar{x}) \\
\text{or, } &\limsup_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{F}_i(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}_i(\bar{x})) \preceq \limsup_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) \\
\text{or, } &\max_{i \in \mathcal{A}(\bar{x})} \overline{\mathbf{F}}_{i_{\mathscr{D}}}(\bar{x})(h) \preceq \overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(h). \qquad (6.11)
\end{aligned}$$

To prove the reverse inequality, we claim that there exists a neighbourhood $\mathcal{N}(\bar{x})$ such that $\mathcal{A}(x) \subset \mathcal{A}(\bar{x})$ for all $x \in \mathcal{N}(\bar{x})$. Assume contrarily that there exists a sequence $\{x_k\}$ in $\mathcal{X}$ with $x_k \to \bar{x}$ such that $\mathcal{A}(x_k) \not\subset \mathcal{A}(\bar{x})$. We can choose $i_k \in \mathcal{A}(x_k)$ but $i_k \notin \mathcal{A}(\bar{x})$. Since $\mathcal{A}(x_k)$ is closed, $i_k \to \bar{i} \in \mathcal{A}(x_k)$. By $gH$-continuity of $\mathbf{F}$ we have

$$\mathbf{F}_{\bar{i}}(x_k) = \mathbf{F}(x_k) \implies \mathbf{F}_{\bar{i}}(\bar{x}) = \mathbf{F}(\bar{x}),$$

which is a contradiction to $i_k \notin \mathcal{A}(\bar{x})$. Thus, $\mathcal{A}(x) \subset \mathcal{A}(\bar{x})$ for all $x \in \mathcal{N}(\bar{x})$.

Let us choose a sequence $\{\lambda_k\}, \lambda_k \to 0$ such $\bar{x} + \lambda_k h \in \mathcal{N}(\bar{x})$ for all $h \in \mathcal{X}$. Then,

$$\begin{aligned}
&\mathbf{F}_i(\bar{x}) \preceq \mathbf{F}(\bar{x}) \text{ for all } i \in I \\
\text{or, } &\mathbf{F}(\bar{x} + \lambda_k h) \ominus_{gH} \mathbf{F}(\bar{x}) \preceq \mathbf{F}(\bar{x} + \lambda_k h) \ominus_{gH} \mathbf{F}_i(\bar{x}) \text{ for all } i \in \mathcal{A}(\bar{x}) \\
\text{or, } &\mathbf{F}(\bar{x} + \lambda_k h) \ominus_{gH} \mathbf{F}(\bar{x}) \preceq \mathbf{F}_i(\bar{x} + \lambda_k h) \ominus_{gH} \mathbf{F}_i(\bar{x}) \text{ for all } i \in \mathcal{A}(\bar{x} + \lambda_k h) \\
\text{or, } &\limsup_{k \to \infty} \frac{1}{\lambda_k} \odot (\mathbf{F}(\bar{x} + \lambda_k h) \ominus_{gH} \mathbf{F}(\bar{x})) \preceq \limsup_{k \to \infty} \frac{1}{\lambda_k} \odot (\mathbf{F}_i(\bar{x} + \lambda_k h) \ominus_{gH} \mathbf{F}_i(\bar{x})) \\
\text{or, } &\overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(h) \preceq \max_{i \in \mathcal{A}(\bar{x})} \overline{\mathbf{F}}_{i_{\mathscr{D}}}(\bar{x})(h). \qquad (6.12)
\end{aligned}$$

From (6.11) and (6.12) , we obtain

$$\overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(h) = \max \overline{\mathbf{F}}_{i_{\mathscr{D}}}(\bar{x})(h) \text{ for all } i \in \mathcal{A}(\bar{x}).$$

□

## 6.5 Characterization of Efficient Solutions

In this section, we present the characterization of efficient solutions for IOPs based on the properties of upper $gH$-Dini semidifferentiable IVFs.

**Theorem 6.18** (Necessary condition for efficient points)**.** *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$, $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF, and $\bar{x} \in \mathcal{S}$ be an efficient point of the IOP (1.5). If the function $\boldsymbol{F}$ has a upper $gH$-Dini semiderivative at $\bar{x}$ in the direction $h - \bar{x}$ for any $x \in \mathcal{S}$, then*

$$\overline{\boldsymbol{F}}_{\mathscr{D}}(\bar{x})(v - \bar{x}) \nprec \boldsymbol{0} \text{ for all } h \in \mathcal{S}. \tag{6.13}$$

*Proof.* Let $\bar{x} \in \mathcal{S}$ be an efficient point of the IVF $\mathbf{F}$. For any point $x \in \mathcal{X}$, the upper $gH$-Dini semiderivative of $\mathbf{F}$ at $\bar{x}$ in the direction $h - \bar{x}$ is given by

$$\overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(h - \bar{x}) = \limsup_{\lambda \to 0+} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})\big). \tag{6.14}$$

Since the point $\bar{x}$ is an efficient point of the function $\mathbf{F}$, for any $h \in \mathcal{X}$ and $\lambda > 0$ with $\bar{x} + \lambda(h - \bar{x}) \in \mathcal{S}$, we get

$$\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \nprec \mathbf{F}(\bar{x})$$

$$\text{or, } \mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0}.$$

This implies that

$$\max\left\{\underline{f}(\bar{x}+\lambda(h-\bar{x})-\underline{f}(\bar{x}),\overline{f}(\bar{x}+\lambda(h-\bar{x})-\overline{f}(\bar{x})\right\}\geq 0.$$

Since $\lambda > 0$, from above inequality, we get

$$\limsup_{\lambda\to 0+}\frac{1}{\lambda}\max\left\{\underline{f}(\bar{x}+\lambda(h-\bar{x})-\underline{f}(\bar{x}),\overline{f}(\bar{x}+\lambda(h-\bar{x})-\overline{f}(\bar{x})\right\}\geq 0$$

$$\text{or,}\quad \max\left\{\overline{\underline{f}}_{\mathscr{D}}(\bar{x})(h),\overline{\overline{f}}_{\mathscr{D}}(\bar{x})(h)\right\}\geq 0. \tag{6.15}$$

From (6.14) and (6.15), we have

$$\overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(h-\bar{x})\nprec \mathbf{0} \text{ for all } h\in\mathcal{S}.$$

□

**Note 15.** *One may think that in Theorem 6.18, instead of using the "not better strict dominance "relation of compact intervals (Definition 1.4.3), we may use "not strict dominance "relation of compact intervals (Definition 1.4.2). However, this assumption is not sufficient. For instance, consider $\mathcal{X}=\mathbb{R}$, $\mathcal{S}=[-1,7]$, and the IVF $\boldsymbol{F}:\mathcal{S}\to I(\mathbb{R})$ that is defined by*

$$\boldsymbol{F}(x)=[x^2-2x+1,x^2+6].$$

*For any $x,h\in\mathcal{S}$ such that $x+\lambda h\in\mathcal{S}$, we have the following for any $v\in\mathcal{X}$:*

$$\limsup_{\lambda\to 0+}\frac{1}{\lambda}\odot\left(\boldsymbol{F}(x+\lambda h)\ominus_{gH}\boldsymbol{F}(x)\right)=2h\odot[x-1,x].$$

*Hence,* $\overline{\boldsymbol{F}}_{\mathscr{D}}(\bar{x})(h) = 2h \odot [\bar{x}-2, \bar{x}]$*. Note that* $\bar{x} = 0$ *is an efficient point of IOP (1.5) because*

$$\boldsymbol{F}(y) \nprec \boldsymbol{F}(\bar{x}) \text{ for all } y \in \mathcal{S}.$$

*However,* $\overline{\boldsymbol{F}}_{\mathscr{D}}(\bar{x})(h) \prec \boldsymbol{0}$ *for all* $h > 0$.

**Theorem 6.19** (Sufficient condition for efficient points)**.** *Let* $\mathcal{S}$ *be a nonempty convex subset of* $\mathcal{X}$ *and* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *be a convex IVF. If the function* $\boldsymbol{F}$ *has a upper gH-Dini semderivative at* $\bar{x} \in \mathcal{S}$ *in the direction* $h - \bar{x}$ *with*

$$\overline{\boldsymbol{F}}_{\mathscr{D}}(\bar{x})(h-\bar{x}) \nprec \boldsymbol{0} \text{ for all } h \in \mathcal{X}, \tag{6.16}$$

*then* $\bar{x}$ *must be an efficient point of the IOP (1.5).*

*Proof.* Suppose at $\bar{x} \in \mathcal{S}$, for each direction $h - \bar{x}$, we have

$$\overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(h-\bar{x}) \nprec \mathbf{0} \text{ for all } h \in \mathcal{X}.$$

If possible, let $\bar{x}$ be not an efficient point of $\mathbf{F}$. Then, there exists at least one $y \in \mathcal{S}$ such that

$$\mathbf{F}(y) \prec \mathbf{F}(\bar{x}).$$

Therefore, for any $\lambda \in (0, 1]$ we have

$$\begin{aligned} &\lambda \odot \mathbf{F}(y) \prec \lambda \odot \mathbf{F}(\bar{x}) \\ \text{or, } &\lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(\bar{x}) \prec \lambda \odot \mathbf{F}(\bar{x}) \oplus \lambda' \odot \mathbf{F}(\bar{x}), \text{ where } \lambda' = 1 - \lambda \\ \text{or, } &\lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(\bar{x}) \prec (\lambda + \lambda') \odot \mathbf{F}(\bar{x}) = \mathbf{F}(\bar{x}). \end{aligned}$$

Due to the convexity of $\mathbf{F}$ on $\mathcal{S}$, we have

$$\begin{aligned}
&\mathbf{F}(\bar{x} + \lambda(y - \bar{x})) = \mathbf{F}(\lambda y + \lambda' \bar{x}) \preceq \lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(\bar{x}) \prec \mathbf{F}(\bar{x}) \\
&\text{or, } \mathbf{F}(\bar{x} + \lambda(y - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \prec \mathbf{0} \\
&\text{or, } \limsup_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda(y - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})) \preceq \mathbf{0} \\
&\text{or, } \overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(y - \bar{x}) \preceq \mathbf{0}. \qquad (6.17)
\end{aligned}$$

This contradicts the assumption that $\overline{\mathbf{F}}_{\mathscr{D}}(\bar{x})(h - \bar{x}) \nprec \mathbf{0}$ for all $h \in \mathcal{X}$. Hence, $\bar{x}$ is the efficient point of the IOP (1.5). □

**Note 16.** *Converse of Theorem 6.19 is not true. For example, consider* $\mathcal{X} = \mathbb{R}$, $\mathcal{S} = [-1, 2]$, *and the IVF* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *that is defined by*

$$\boldsymbol{F}(x) = [x^2 - 4x + 4, 2x^2 + 80].$$

*Since* $\underline{f}$ *and* $\overline{f}$ *are convex and Lipschitz continuous on* $\mathcal{S}$, $\boldsymbol{F}$ *is convex and gH-Lipschitz continuous IVF on* $\mathcal{S}$ *by Lemma 1.8 and Lemma 3.3. Also, from Theorem 6.10,* $\boldsymbol{F}$ *has upper gH-Dini semiderivative at* $\bar{x} = 0 \in \mathcal{S}$ *in every direction* $h \in \mathcal{X}$. *Since*

$$\begin{aligned}
&\limsup_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) \\
= &\limsup_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\left[(\lambda h)^2 - 4(\lambda h) + 4, 2(\lambda h)^2 + 80\right] \ominus_{gH} [4, 80]\right) \\
= &\ h \odot [-4, 0],
\end{aligned}$$

*then* $\overline{\boldsymbol{F}}_{\mathscr{D}}(\bar{x})(h) = h \odot [-4, 0]$ *for all* $h \in \mathcal{X}$. *Hence you can easily check that* $\bar{x} = 0$ *is an efficient solution of the IOP (1.5).*

*However, for all* $h > 0$ *we have* $\overline{\boldsymbol{F}}_{\mathscr{D}}(\bar{x})(h) \prec \boldsymbol{0}$.

## 6.6 Concluding Remarks

In this chapter, the concept of upper and lower Dini semiderivative, upper and lower Hadamard semiderivative for IVFs have been proposed. The upper Dini semiderivative and upper Hadamard semiderivative of a Lipschitz continuous IVF are observed to be a positive homogeneous IVF. It has been found that every Lipschitz continuous IVF is upper Dini semi differentiable and upper Hadamard semidifferentiable IVFs. Further, for a convex and Lipschitz IVF, it has been shown that the upper Dini semiderivative and upper Hadamard semiderivative of IVF coincide with the directional derivative of IVF. It has also been observed that the continuity of IVF is necessary condition for existence of upper and lower Dini semiderivative of IVF. With the help of the studied semiderivative, we have been derived a few results on characterizing efficient solutions of an IOP.

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