Chapter 5

Generalized Hukuhara Hadamard Dervative of Interval-valued Functions and its Application in Interval Optimization

5.1 Introduction

In conventional nonsmooth optimization theory, one of the mostly used idea of derivative is Hadamard derivative which is generalization of Gâteaux and Fréchet on Banach space. This derivative is applied to characterize optimal solutions. An explicit expression of the derivative of an extremum with respect to parameters can be obtained with the help of Hadamard derivative. So, it works well for most differentiable optimization problems including convex or concave problems.

In modern optimization theory, the concept of Hadamard derivative of a function is of fundamental importance. It serves as a basis for deriving first-order necessary, and occasionally sufficient, optimality conditions and for designing numerical algorithms. In recent years much attention has been attracted to investigation of Hadamard differentiability of functions appearing in minimax calculus, nonsmooth analysis, and stochastic programming.

5.2 Motivation

Despite of many attempts to develop calculus for IVFs, the existing ideas are not adequate to retain two most important features of classical differential calculus—linearity of the derivative with respect to the direction and the chain rule. Although some optimality conditions for IOPs are proposed by using gH-directional and gH-Gâteaux derivatives, but these derivatives are not sufficient to preserve the continuity of IVFs (see Example 2.6) and chain rule for the composition of IVFs (see Example 5.2). Even though gH-Hadamard semiderivative preserves the continuity of functions and chain rule but it is not sufficient for linearity of the derivative with respect to the direction. With the help of the derivative of lower and upper functions, some articles [85, 91] reported KKT condition to characterize efficient solutions of constraint IOPs. However, the derivative used in [85, 91] are very restrictive because this derivative is very difficult to calculate even for very simple IVF (see Example 5 of [13]). However, the KKT condition of constraint IOPs by Hadamard derivative for IVFs, do not depend on the existence of the Hadamard derivative of lower and upper functions. Also, Hadamard derivative retains the linearity of the derivative with respect to direction, the existence of continuity as well as the chain rule of derivative.

5.3 Contributions

In this chapter, the concept of gH-Hadamard derivative of IVFs is studied. It is proved that if an IVF is gH-Hadamard differentiable, then IVF is gH-continuous. By using the proposed concept of gH-Hadamard derivative, it is observed that a gH-Fréchet differentiable IVF is gH-Hadamard differentiable and vise-versa. Further, the convexity of IVFs is characterized with the help of gH-Hadamard derivative. Besides, with the help of gH-Hadamard derivative, a necessary and sufficient condition for characterizing the efficient solutions to IOPs is derived. Further, for constraint IOPs, the extended KKT necessary and sufficient condition to characterize the efficient solutions is studied.

Original contributions of this chapter are listed below:

(i) For an IVF which is defined on finite dimensional normed linear space, it is proved that gH-Fréchet implies gH-Hadamard derivative and vise-versa.

(ii) For unconstrained IOP, a necessary and sufficient condition is derived to characterize the efficient solutions.

(iii) For an constraint IOP, an extended Karush-Kuhn-Tucker condition is derived to obtain the efficient solutions.

5.4 Hadamard Derivative of Interval-valued Functions

It is noteworthy that existence of gH-directional and gH-Gâteaux derivatives do not imply the gH-continuity of an IVF. For instance, see Example 2.6. In this section, we present a stronger concept of a derivative, namely gH-Hadamard derivative for an IVF from which gH-continuity is implied.

Definition 5.1. (*gH-Hadamard derivative of IVF*). Let $\mathbf{F}$ be an IVF on a nonempty subset $\mathcal{S}$ of $\mathcal{X}$. For $\bar{x} \in \mathcal{S}$ and $v \in \mathcal{X}$, if the limit

$$\mathbf{F}_{\mathscr{H}}(\bar{x})(v) = \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})\right)$$

exists and $\mathbf{F}_{\mathscr{H}}(\bar{x})$ is a linear IVF from $\mathcal{X}$ to $I(\mathbb{R})$, then $\mathbf{F}_{\mathscr{H}}(\bar{x})(v)$ is called *gH-Hadamard derivative* of $\mathbf{F}$ at $\bar{x}$ in the direction v. If this limit exists for all $v \in \mathcal{X}$, then $\mathbf{F}$ is said to be *gH-Hadamard differentiable* at $\bar{x}$.

Remark 5.2. The limit $\mathbf{F}_{\mathscr{H}}(\bar{x})(v)$ exists if for all sequences $\{\lambda_n\}$ and $\{h_n\}$ with $\lambda_n > 0$ for all n such that $\lim_{n\to\infty} \lambda_n = 0$, $\lim_{n\to\infty} h_n = v$,

$\lim_{n\to\infty} \frac{1}{\lambda_n} \odot \left(\mathbf{F}(\bar{x} + \lambda_n h_n) \ominus_{gH} \mathbf{F}(\bar{x})\right)$ exists and the limit value is a linear IVF on $\mathcal{S}$.

Example 5.1. *Let $\mathcal{S} = \mathcal{X} = \mathbb{R}^n$ and consider the IVF $\boldsymbol{F}(x) = \|x\|^2 \odot \boldsymbol{C}$ for all $x \in \mathbb{R}^n$, where $\boldsymbol{C} \in I(\mathbb{R})$. Then we calculate the gH-Hadamard derivative at $\bar{x} = 0$ for $\boldsymbol{F}$.*

For any $\bar{x} \in \mathcal{S}$ and $v \in \mathcal{X}$, we see that

$$\begin{aligned}
\lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})\right) &= \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\|\bar{x} + \lambda h\|^2 \odot \boldsymbol{C} \ominus_{gH} \|\bar{x}\|^2 \odot \boldsymbol{C}\right) \\
&= \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\left(2\bar{x}^\top(\lambda h) + \|\lambda h\|^2\right) \odot \boldsymbol{C}\right) \\
&= 2\bar{x}^\top v \odot \boldsymbol{C}, \text{ by } gH\text{-continuity of } \bar{x}^\top h \odot \boldsymbol{C}.
\end{aligned}$$

Hence, $\boldsymbol{F}_{\mathscr{H}}(\bar{x})(v) = 2\bar{x}^\top v \odot \boldsymbol{C}$ and $\boldsymbol{F}_{\mathscr{H}}(\bar{x})$ is a linear IVF from $\mathcal{X}$ to $I(\mathbb{R})$. Therefore, $\boldsymbol{F}$ is gH-Hadamard differentiable at $\bar{x}$ with $\boldsymbol{F}_{\mathscr{H}}(\bar{x})(v) = 2\bar{x}^\top v \odot \boldsymbol{C}$.

Note 13. *By definitions of gH-Hadamard semiderivative (Definition 4.4.1) and gH-Hadamard derivative (Definition 5.1), it is clear that if* $\boldsymbol{F}_{\mathscr{H}}(\bar{x})(h)$ *exists, then* $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(h)$ *exists and equals to* $\boldsymbol{F}_{\mathscr{H}}(\bar{x})(h)$*. However, the converse is not true. For instance, let us consider an IVF* $\boldsymbol{F}: \mathbb{R}^n \to I(\mathbb{R})$*, is defined by*

$$\boldsymbol{F}(x) = \|x\| \odot \boldsymbol{C}, \ x \in \mathbb{R}^n.$$

For any $v \in \mathbb{R}^n$ *and* $\bar{x} = 0$*, we see that*

$$\lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) = \lim_{\substack{\lambda \to 0+ \\ h \to v}} \left(\left(\frac{1}{\lambda} \odot \lambda \right) \odot (\|h\| \odot \boldsymbol{C}) \right) = \|v\| \odot \boldsymbol{C}.$$

Hence, $\boldsymbol{F}$ *is gH-Hadamard semidifferentiable at* $\bar{x}$ *with* $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) = \|v\| \odot \boldsymbol{C}$*. However, the limit value is not a linear IVF on* $\mathcal{X}$*. Therefore,* $\boldsymbol{F}_{\mathscr{H}}(\bar{x})(h)$ *does not exist.*

Theorem 5.3. *Let* $\mathcal{X} = \mathbb{R}^n$*,* $\mathcal{S}$ *be a nonempty subset of* $\mathcal{X}$*,* $\boldsymbol{F}$ *be an IVF on* $\mathcal{S}$ *and* $\bar{x} \in \mathcal{S}$*. Then the following statements are equivalent:*

(i) $\boldsymbol{F}$ *is gH-Fréchet differentiable at* $\bar{x}$*.*

(ii) $\boldsymbol{F}$ *is gH-Hadamard differentiable at* $\bar{x}$*.*

Proof. (i) $\implies$ (ii). Since $\mathbf{F}$ is gH-Fréchet differentiable at $\bar{x} \in \mathcal{S}$, there exists a gH-continuous and linear IVF $\mathbf{G}$ such that

$$\lim_{\lambda \to 0+} \frac{\|\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{G}(\lambda h)\|_{I(\mathbb{R})}}{\|\lambda h\|} = 0, \quad \text{for all } h \in \mathcal{X} \backslash \{\hat{0}\}$$

$$\text{or,} \ \lim_{\lambda \to 0+} \frac{1}{\lambda} \|\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{G}(\lambda h)\|_{I(\mathbb{R})} = 0, \quad \text{for all } h \in \mathcal{X} \backslash \{\hat{0}\}. \quad (5.1)$$

Since $\mathbf{G}$ is linear, and thus $\mathbf{G}(\lambda\ h) = \lambda \odot \mathbf{G}(h)$, the equation (5.1) gives

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda \odot \mathbf{G}(h)) = \mathbf{0}, \quad \text{for all } h \in \mathcal{X} \backslash \{\hat{0}\}$$
$$\text{or, } \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) = \mathbf{G}(h), \quad \text{for all } h \in \mathcal{X} \backslash \{\hat{0}\}.$$

Since $\mathbf{G}$ is gH-continuous, we have

$$\lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) = \mathbf{G}(v).$$

Hence, $\mathbf{F}$ is gH-Hadamard differentiable at $\bar{x}$.

(ii) $\implies$ (i). As $\mathbf{F}$ is gH-Hadamard differentiable at $\bar{x} \in \mathcal{S}$, $\mathbf{F}_{\mathscr{H}}(\bar{x})(v)$ exists for all v and $\mathbf{F}_{\mathscr{H}}(\bar{x})$ is a linear IVF. Let

$$Q(h) = \frac{1}{\|h\|} \odot (\mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{H}}(\bar{x})(h)), \ h \neq \hat{0}.$$

Consider a sequence $\{h_n\}$ converging to 0. As $\mathcal{W} = \{h/\|h\| : h \in \mathcal{X}, h \neq \hat{0}\}$ is a compact set, there exists a subsequences $\{h_{n_k}\}$ and a point $\bar{v} \in \mathcal{W}$ such that $w_{n_k} = \frac{h_{n_k}}{\|h_{n_k}\|} \to \bar{v} \in \mathcal{W}$.

Note that the sequence $\{t_{n_k}\}$, defined by $t_{n_k} = \|h_{n_k}\|$, converges to 0. Since $\mathbf{F}_{\mathscr{H}}(\bar{x})(\bar{v})$ exists and

$\mathbf{F}_{\mathscr{H}}(\bar{x})(w_{n_k}) \to \mathbf{F}_{\mathscr{H}}(\bar{x})(\bar{v})$ as $k \to \infty$, we have

$$Q(h_{n_k}) = \frac{1}{t_{n_k}} \odot \big(\mathbf{F}(\bar{x} + t_{n_k} w_{n_k}) \ominus_{gH} \mathbf{F}(\bar{x})\big) \ominus_{gH} \mathbf{F}_{\mathscr{H}}(\bar{x})(w_{n_k}) \to \mathbf{0} \text{ as } k \to \infty.$$

This implies that $\lim_{k \to \infty} \|Q(h_{n_k})\|_{I(\mathbb{R})} = 0$.

As $\{h_n\}$ is an arbitrarily chosen sequence that converges to 0, $\lim_{\|h\| \to 0} \|Q(h)\|_{I(\mathbb{R})} = 0$. Hence, $\mathbf{F}$ is gH-Fréchet differentiable at $\bar{x}$. □

Remark 5.4.1. *If $\mathcal{X}$ is infinite dimensional, then Theorem 5.3 is not true. For instance, see Example 1 of [90]. According to this example, there exists a degenerate IVF $\boldsymbol{F}$ which is gH-Hadamard differentiable at $\bar{x}$ but not gH-Fréchet differentiable at $\bar{x}$.*

Theorem 5.4. *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X} = \mathbb{R}^n$. If the function $\boldsymbol{F}: \mathcal{S} \to I(\mathbb{R})$ has a gH-Hadamard derivative at $\bar{x} \in \mathcal{S}$, then the function $\boldsymbol{F}$ is gH-continuous at $\bar{x}$.*

Proof. Since **F** is gH-Hadamard differentiable at $\bar{x} \in \mathcal{S}$, **F** is gH-Fréchet differentiable at $\bar{x}$ by Theorem 5.3. Also, from Theorem 2.10, the function **F** is gH-continuous at $\bar{x}$. □

Remark 5.4.2. *The converse of Theorem 5.4 is not true. For instance, consider the gH-continuous IVF $\boldsymbol{F}(x) = \|x\| \odot \boldsymbol{C}$ for all $x \in \mathbb{R}^n$. Therefore, for any $v \in \mathbb{R}^n$ and $\bar{x} = 0$, we see that*

$$\lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) = \|v\| \odot \boldsymbol{C}.$$

Hence, the limit value is not a linear IVF on $\mathcal{S}$. Therefore, $\boldsymbol{F}_{\mathscr{H}}(\bar{x})(h)$ does not exist.

Note 14. *By the definitions of gH-directional (Definition 2.4.1), gH-Gâteaux (Definition 2.5.3) and gH-Hadamard (Definition 5.1) derivatives of IVF $\boldsymbol{F}$, it is clear that if $\boldsymbol{F}_{\mathscr{H}}(\bar{x})(h)$ exists, then $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h)$ and $\boldsymbol{F}_{\mathscr{G}}(\bar{x})(h)$ exist and they are equal to $\boldsymbol{F}_{\mathscr{H}}(\bar{x})(h)$. However, the converse is not true. For instance, consider the IVF $\boldsymbol{F}: \mathbb{R}^2 \to I(\mathbb{R})$ defined by*

$$\boldsymbol{F}(x, y) = \begin{cases} \left(\frac{x^6}{(y - x^2)^2 + x^8}\right) \odot [3,\ 9], & \text{if } (x, y) \neq (0, 0), \\ \boldsymbol{0}, & \text{otherwise.} \end{cases}$$

For $\bar{x} = (0, 0)$ *and arbitrary* $h = (h_1, h_2) \in \mathbb{R}^2$*, we have*

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) = \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\left(\frac{\lambda^6 h_1^6}{(\lambda h_2 - \lambda^2 h_1^2)^2 + \lambda^8 h_1^8} \right) \odot [3,\ 9] \right) = \boldsymbol{0}.$$

Hence, $\boldsymbol{F}$ *is gH-directional and gH-Gâteaux differentiable at* $\bar{x}$ *with* $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h) = \boldsymbol{F}_{\mathscr{G}}(\bar{x})(h) = \boldsymbol{0}$*.*

Let $\lambda_n = \frac{1}{n}$ *and* $h_n = (\frac{1}{n}, \frac{1}{n^3})$ *for* $n \in \mathbb{N}$*. Then, for* $\bar{x} = (0, 0)$*, we have*

$$\lim_{n \to \infty} \frac{1}{\lambda_n} \odot (\boldsymbol{F}(\bar{x} + \lambda_n h_n) \ominus_{gH} \boldsymbol{F}(\bar{x})) = \lim_{n \to \infty} n^5 \odot [3, 9]. \tag{5.2}$$

Hence, $\boldsymbol{F}_{\mathscr{H}}(\bar{x})(0)$ *does not exist.*

Theorem 5.5. *Let* $\mathcal{S}$ *be a nonempty convex subset of* $\mathbb{R}^n$ *and the IVF* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *has gH-Hadamard derivative at every* $\bar{x} \in \mathcal{S}$*. If the function* $\boldsymbol{F}$ *is convex on* $\mathcal{S}$*, then*

$$\boldsymbol{F}(v) \ominus_{gH} \boldsymbol{F}(\bar{x}) \not\prec \boldsymbol{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}), \quad \textit{for all } v \in \mathcal{S}.$$

Proof. Since $\mathbf{F}$ is convex on $\mathcal{S}$, for any $\bar{x},\ h \in \mathcal{S}$ and $\lambda,\ \lambda' \in (0, 1]$ with $\lambda + \lambda' = 1$, we have

$$\begin{aligned}
&\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) = \mathbf{F}(\lambda h + \lambda' \bar{x}) \preceq \lambda \odot \mathbf{F}(h) \oplus \lambda' \odot \mathbf{F}(\bar{x}) \\
\Longrightarrow\ &\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \preceq (\lambda \odot \mathbf{F}(h) \oplus \lambda' \odot \mathbf{F}(\bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \\
\Longrightarrow\ &\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \preceq \lambda \odot (\mathbf{F}(h) \ominus_{gH} \mathbf{F}(\bar{x})) \\
\Longrightarrow\ &\frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})) \preceq \mathbf{F}(h) \ominus_{gH} \mathbf{F}(\bar{x}).
\end{aligned}$$

From Theorem 5.4, $\mathbf{F}$ is gH-continuous. Thus, as $\lambda \to 0+$ and $h \to v$, we obtain

$$\mathbf{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}) \preceq \mathbf{F}(v) \ominus_{gH} \mathbf{F}(\bar{x}), \quad \text{for all } v \in \mathcal{S}. \tag{5.3}$$

If possible, let

$$\mathbf{F}(v') \ominus_{gH} \mathbf{F}(\bar{x}') \prec \mathbf{F}_{\mathscr{H}}(\bar{x}')(v' - \bar{x}') \text{ for some } v' \in \mathcal{X}.$$

Then,

$$\mathbf{F}(v') \ominus_{gH} \mathbf{F}(\bar{x}') \prec \mathbf{F}_{\mathscr{H}}(\bar{x}')(v' - \bar{x}'),$$

which contradicts (5.3). Hence,

$$\mathbf{F}(v) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}), \quad \text{for all } v \in \mathcal{S}.$$

□

Remark 5.4.3. *The converse of Theorem 5.5 is not true. For example, let us consider the IVF* $\boldsymbol{F}: \mathbb{R} \to I(\mathbb{R})$ *defined by*

$$\boldsymbol{F}(x) = [-4x^2, 6x^2].$$

At $\bar{x} = 0 \in \mathbb{R}$*, for arbitrary* $v \in \mathbb{R}$*, we have*

$$\boldsymbol{F}_{\mathscr{H}}(\bar{x})(v) = \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) = \boldsymbol{0}.$$

Hence, $\boldsymbol{F}(v) \ominus_{gH} \boldsymbol{F}(\bar{x}) \nprec \boldsymbol{F}_{\mathscr{H}}(\bar{x})(v - \bar{x})$ *for all* $v \in \mathbb{R}$*. However,* $\underline{f}$ *is not convex on* $\mathbb{R}$*. Thus, from Lemma 1.8,* $\boldsymbol{F}$ *is not convex on* $\mathbb{R}$*.*

Remark 5.4.4. *For a convex IVF* $\boldsymbol{F}$ *on* $\mathcal{S} \subset \mathbb{R}^n$*, the inequality '*$\boldsymbol{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}) \ominus_{gH} \boldsymbol{F}_{\mathscr{H}}(v)(v - \bar{x}) \preceq \boldsymbol{0}$ *for all* $\bar{x}, v \in \mathcal{S}$*' is not true. For instance, consider the convex IVF* $\boldsymbol{F}: \mathbb{R} \to I(\mathbb{R})$ *defined by*

$$\boldsymbol{F}(x) = [x^2, 3x^2].$$

At $\bar{x} \in \mathbb{R}$*, for arbitrary* $v \in \mathbb{R}$*, we have* $\boldsymbol{F}_{\mathscr{H}}(\bar{x})(v-\bar{x}) = 2\bar{x}(v-\bar{x}) \odot [1,3]$*. For* $\bar{x} = 1$ *and* $v = 2$*, we obtain* $\boldsymbol{F}_{\mathscr{H}}(\bar{x})(v-\bar{x}) \ominus \boldsymbol{F}_{\mathscr{H}}(v)(v-\bar{x}) = [-10, 2] \npreceq \boldsymbol{0}$*.*

Theorem 5.6. *Let* $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ *be an IVF and* $\bar{x} \in \mathbb{R}^n$*. Then, for a given direction* $v \in \mathbb{R}^n$*, the following statements are equivalent:*

(i) $\boldsymbol{F}$ *is gH-Hadamard differentiable at* $\bar{x}$*;*

(ii) *There exists a linear IVF* $\boldsymbol{L} : \mathbb{R}^n \to I(\mathbb{R})$ *such that for any path* $f : \mathbb{R} \to \mathbb{R}^n$ *with* $f(0) = \bar{x}$ *for which* $f_{\mathscr{D}}(0)(1)$ *exists, we have*

$$(\boldsymbol{F} \circ f)_{\mathscr{D}}(0)(1) = \boldsymbol{L}(\bar{x})(v), \quad \textit{where } v = f_{\mathscr{D}}(0)(1).$$

Proof. (i) $\implies$ (ii). Let $\{\delta_n\}$ be a sequence of positive real numbers with $\delta_n \to 0^+$ and $h_n = \frac{1}{\delta_n}(f(\delta_n) - f(0))$ for all $n \in \mathbb{N}$. Since $f_{\mathscr{D}}(0)(1)$ exists, we have

$$\lim_{n\to\infty} h_n = \lim_{n\to\infty} \frac{1}{\delta_n} \odot (f(\delta_n) - f(0)) = f_{\mathscr{D}}(0)(1) = v. \tag{5.4}$$

If $\mathbf{F}$ is gH-Hadamard differentiable at $\bar{x}$, then

$$\begin{aligned}
& \mathbf{F}_{\mathscr{H}}(\bar{x})(v) \\
= \ & \lim_{n\to\infty} \frac{1}{\delta_n} \odot (\mathbf{F}(\bar{x} + \delta_n h_n) \ominus_{gH} \mathbf{F}(\bar{x})) \\
= \ & \lim_{n\to\infty} \frac{1}{\delta_n} \odot (\mathbf{F}(f(\delta_n)) \ominus_{gH} \mathbf{F}(f(0))), \text{ since } f(0) = \bar{x} \text{ and } h_n = \frac{1}{\delta_n}(f(\delta_n) - f(0)) \\
= \ & \lim_{n\to\infty} \frac{1}{\delta_n} \odot ((\mathbf{F} \circ f)(\delta_n) \ominus_{gH} (\mathbf{F} \circ f)(0)).
\end{aligned}$$

Hence, $(\mathbf{F} \circ f)_{\mathscr{D}}(0)(1) = \mathbf{F}_{\mathscr{H}}(\bar{x})(v)$. Due to the linearity of $\mathbf{F}_{\mathscr{H}}(\bar{x})(v)$ on $\mathbb{R}^n$, by taking $\mathbf{L}(\bar{x})(v) = \mathbf{F}_{\mathscr{H}}(\bar{x})(v)$, we get the desired result.

(ii) $\implies$ (i). If possible, assume that $\mathbf{F}$ is not gH-Hadamard differentiable at $\bar{x}$. Then, there exist sequences $h_n \to v$ and $\delta_n \to 0^+$ such that

$$\begin{aligned} &\text{either, } \lim_{n\to\infty} \frac{1}{\delta_n} \odot (\mathbf{F}(\bar{x} + \delta_n h_n) \ominus_{gH} \mathbf{F}(\bar{x})) \text{ does not exist} \\ &\text{or, limit value is not linear IVF on } \mathbb{R}^n. \end{aligned} \tag{5.5}$$

Since $h_n \to v$ and $\delta_n \to 0^+$, for every $\epsilon > 0$ there exist a natural number N and a real number a such that

$$\|h_n\| \leq a, \quad \|h_n - v\| < \epsilon, \text{ and } \quad \delta_n < \epsilon/a \text{ for all } n > N. \tag{5.6}$$

By using the sequences $\{h_n\}$ and $\{\delta_n\}$, we construct a function $f : \mathbb{R} \to \mathbb{R}^n$ as follows:

$$f(\delta) = \begin{cases} \bar{x} + \delta v, & \text{if } \delta \leq 0, \\ \bar{x} + \delta h_n, & \text{if } \delta_n \leq \delta < \delta_{n-1}, n \geq 2, \\ \bar{x} + \delta h_1, & \text{if } \delta \geq \delta_1. \end{cases}$$

Thus the function f yields $f(0) = \bar{x}$ and $f_{\mathscr{D}}(0)(1) = v$ (for details, see p. 92 in [23]). By hypothesis,

$(\mathbf{F} \circ f)_{\mathscr{D}}(0)(1)$ exists and equals to $\mathbf{L}(\bar{x})(v)$, where $v = f_{\mathscr{D}}(0)(1)$. From the construction of f, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{\delta_n} \odot ((\mathbf{F} \circ f)(\delta_n) \ominus_{gH} (\mathbf{F} \circ f)(0)) = \mathbf{L}(\bar{x})(v) \\ \text{or, } &\lim_{n\to\infty} \frac{1}{\delta_n} \odot (\mathbf{F}(f(\delta_n)) \ominus_{gH} \mathbf{F}(f(0))) = \mathbf{L}(\bar{x})(v) \\ \text{or, } &\lim_{n\to\infty} \frac{1}{\delta_n} \odot (\mathbf{F}(\bar{x} + \delta_n h_n) \ominus_{gH} \mathbf{F}(\bar{x})) = \mathbf{L}(\bar{x})(v), \end{aligned}$$

which contradicts to (5.5). Therefore, $\mathbf{F}$ is gH-Hadamard differentiable at $\bar{x}$.

□

Theorem 5.7 (Chain rule)**.** *Let $H : \mathbb{R}^m \to \mathbb{R}^n$ be a vector-valued function and $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ be an IVF. Assume that for a point $\bar{x} \in \mathbb{R}^m$ and direction $v \in \mathbb{R}^m$,*

(a) *$H_{\mathscr{D}}(\bar{x})(v)$ exists for all $v \in \mathbb{R}^m$, and*

(b) *$\boldsymbol{F}_{\mathscr{H}}(\bar{y})(z)$ exists, where $\bar{y} = H(\bar{x})$ and $z = H_{\mathscr{D}}(\bar{x})(v)$.*

Then,

(i) *$(\boldsymbol{F} \circ H)_{\mathscr{D}}(\bar{x})(v)$ exists and $(\boldsymbol{F} \circ H)_{\mathscr{D}}(\bar{x})(v) = \boldsymbol{F}_{\mathscr{H}}(\bar{y})(z)$*

(ii) *if $H_{\mathscr{H}}(\bar{x})(v)$ exists, then $(\boldsymbol{F} \circ H)_{\mathscr{H}}(\bar{x})(v)$ exists and*

$$(\boldsymbol{F} \circ H)_{\mathscr{H}}(\bar{x})(v) = \boldsymbol{F}_{\mathscr{H}}(\bar{y})(\bar{z}), \quad \text{where } \bar{y} = H(\bar{x}),\ \bar{z} = H_{\mathscr{H}}(\bar{x})(v).$$

Proof. (i) For $\delta > 0$, define

$$\mathbf{Q}(\delta) = \frac{1}{\delta} \odot \big(\mathbf{F}(H(\bar{x}+\delta v)) \ominus_{gH} \mathbf{F}(H(\bar{x}))\big) \quad \text{and} \quad \theta(\delta) = \frac{1}{\delta}\big(H(\bar{x}+\delta v) - H(\bar{x})\big). \tag{5.7}$$

Then,

$$\mathbf{Q}(\delta) = \frac{1}{\delta} \odot \big(\mathbf{F}(H(\bar{x}) + \delta\theta(\delta)) \ominus_{gH} \mathbf{F}(H(\bar{x}))\big). \tag{5.8}$$

Since $\theta(\delta) \to H_{\mathscr{D}}(\bar{x})(v)$ as $\delta \to 0+$, from (5.7), (5.8) and the hypothesis (b), we have

$$\begin{aligned} & \mathbf{F}_{\mathscr{H}}(\bar{y})(z) \\ = \ & \lim_{\delta \to 0+} \frac{1}{\delta} \odot (\mathbf{F}(H(\bar{x} + \delta v)) \ominus_{gH} \mathbf{F}(H(\bar{x}))), \text{ where } \bar{y} = H(\bar{x}), z = H_{\mathscr{D}}(\bar{x})(v) \\ = \ & \lim_{\delta \to 0+} \frac{1}{\delta} \odot ((\mathbf{F} \circ H)(\bar{x} + \delta v) \ominus_{gH} (\mathbf{F} \circ H)(\bar{x}))) \\ = \ & (\mathbf{F} \circ H)_{\mathscr{D}}(\bar{x})(v). \end{aligned}$$

(ii) For $\delta > 0$ and $h \in \mathbb{R}^m$, define

$$\mathbf{Q}'(\delta, h) = \frac{1}{\delta} \odot (\mathbf{F}(H(\bar{x} + \delta h)) \ominus_{gH} \mathbf{F}(H(\bar{x}))) \text{ and } \Phi(\delta, h) = \frac{1}{\delta} (H(\bar{x} + \delta h) - H(\bar{x})) . \tag{5.9}$$

Then,

$$\mathbf{Q}'(\delta, h) = \frac{1}{\delta} \odot (\mathbf{F}(H(\bar{x}) + \delta\Phi(\delta, h)) \ominus_{gH} \mathbf{F}(H(\bar{x}))) . \tag{5.10}$$

Since $\Phi(\delta, h) \to H_{\mathscr{H}}(\bar{x})(v)$ as $\delta \to 0+$ and $h \to v$, from (5.9), (5.10) and the hypothesis (b), we have

$$\begin{aligned} & \mathbf{F}_{\mathscr{H}}(\bar{y})(\bar{k}) \\ = & \lim_{\substack{\delta\to 0+ \\ h\to v}} \frac{1}{\delta} \odot \big(\mathbf{F}(H(\bar{x} + \delta h)) \ominus_{gH} \mathbf{F}(H(\bar{x}))\big), \text{ where } \bar{y} = H(\bar{x}),\ \bar{z} = H_{\mathscr{H}}(\bar{x})(v) \\ = & \lim_{\substack{\delta\to 0+ \\ h\to v}} \frac{1}{\delta} \odot \big((\mathbf{F} \circ H)(\bar{x} + \delta h) \ominus_{gH} (\mathbf{F} \circ H)(\bar{x})\big) \\ = & (\mathbf{F} \circ H)_{\mathscr{H}}(\bar{x})(v). \end{aligned}$$

□

The weaker assumption—the existence of $G_{\mathscr{D}}(\bar{x})(v)$ and $\mathbf{F}_{\mathscr{D}}(\bar{y})(k)$ with $\bar{y} = G(\bar{x}), k = G_{\mathscr{D}}(\bar{x})(v)$—is not sufficient to prove Theorem 5.7. For the proof of this theorem, we require a strong assumption (b) of Theorem 5.7. This is illustrated by the following example that the composition $\mathbf{F} \circ G$, of a gH-Gâteaux differentiable IVF $\mathbf{F}$ and a Gâteaux differentiable vector-valued function G, is not gH-Gâteaux differentiable and even not gH-directional differentiable in any direction $v \neq 0$.

Example 5.2. *Consider the IVF* $\boldsymbol{F} : \mathbb{R}^2 \to I(\mathbb{R})$ *defined by*

$$\boldsymbol{F}(x, y) = \begin{cases} \left(\frac{x^6}{(y-x^2)^2+x^8}\right) \odot [2,\ 6], & \textit{if } (x, y) \neq (0, 0), \\ \boldsymbol{0}, & \textit{otherwise,} \end{cases}$$

and the vector-valued function $G : \mathbb{R} \to \mathbb{R}^2$ *by* $G(x) = (x, x^2)$ *for all* $x \in \mathbb{R}$.

It is clear that G *is Gâteaux differentiable function at* $\bar{x} = 0$ *in every direction. Note that* $\bar{y} = G(\bar{x}) = (0, 0)$ *and for any* $h \in \mathbb{R}^2$, *we have*

$$\begin{aligned} \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{y} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{y})) &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\left(\frac{\lambda^6 h_1^6}{(\lambda h_2 - \lambda^2 h_1^2)^2 + \lambda^8 h_1^8} \right) \odot [2,\ 6] \right) \\ &= \boldsymbol{0}. \end{aligned}$$

Then, due to the linearity and gH*-continuity of the limit value,* $\boldsymbol{F}$ *is also* gH*-Gâteaux differentiable IVF at* $\bar{y} = G(\bar{x})$.

The composition of $\boldsymbol{F}$ *and* G *is*

$$\boldsymbol{H}(x) = (\boldsymbol{F} \circ G)(x) = \begin{cases} \left(\frac{1}{x^2}\right) \odot [2,\ 6], & \text{if } (x, y) \neq (0, 0), \\ \boldsymbol{0}, & \text{otherwise.} \end{cases}$$

Since for $h \neq 0$,

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{H}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{H}(\bar{x})) = \lim_{\lambda \to 0+} \frac{1}{\lambda^3 h} \odot [2, 6]$$

does not exist, $\boldsymbol{H} = \boldsymbol{F} \circ G$ *is not* gH*-directional differentiable IVF at* $G(\bar{x}) = 0$ *in any direction* $h \neq 0$.

Theorem 5.8. *Let* I *be a finite set of indices and* $\boldsymbol{F}_i : \mathcal{X} \to I(\mathbb{R})$ *be a family of IVFs such that* $\boldsymbol{F}_{i_{\mathscr{H}}}(\bar{x})(h)$

exists for all $h \in \mathcal{X}$. *For each* $x \in \mathcal{X}$, *let the intervals in* $\{\boldsymbol{F}_i(x) : i \in I\}$ *be comparable. If* $\boldsymbol{F}(x) = \max_{i \in I} \boldsymbol{F}_i(x)$ *for all* $x \in \mathcal{X}$, *then,*

$$\boldsymbol{F}_{\mathscr{H}}(\bar{x})(h) = \max_{i \in \mathcal{A}(\bar{x})} \boldsymbol{F}_{i_{\mathscr{H}}}(\bar{x})(h), \text{ where } \mathcal{A}(\bar{x}) = \{i \in I : \boldsymbol{F}_i(\bar{x}) = \boldsymbol{F}(\bar{x})\}.$$

Proof. Let $\bar{x} \in \mathcal{X}$ and $d \in \mathcal{X}$ be such that $\bar{x} + \lambda d \in \mathcal{X}$ for $\lambda > 0$. Then,

$$\begin{aligned}
&\mathbf{F}_i(\bar{x} + \delta d) \ \preceq \ \mathbf{F}(\bar{x} + \delta d), \quad \text{for all } i \in I \\
\text{or, } &\mathbf{F}_i(\bar{x} + \delta d) \ominus_{gH} \mathbf{F}(\bar{x}) \ \preceq \ \mathbf{F}(\bar{x} + \delta d) \ominus_{gH} \mathbf{F}(\bar{x}), \quad \text{for all } i \in I \\
\text{or, } &\mathbf{F}_i(\bar{x} + \delta d) \ominus_{gH} \mathbf{F}_i(\bar{x}) \ \preceq \ \mathbf{F}(\bar{x} + \delta d) \ominus_{gH} \mathbf{F}(\bar{x}), \quad \text{for each } i \in \mathcal{A}(\bar{x}) \\
\text{or, } &\lim_{\substack{\delta \to 0+ \\ d \to h}} \frac{1}{\delta} \odot (\mathbf{F}_i(\bar{x} + \delta d) \ominus_{gH} \mathbf{F}_i(\bar{x})) \ \preceq \ \lim_{\substack{\delta \to 0+ \\ d \to h}} \frac{1}{\delta} \odot (\mathbf{F}(\bar{x} + \delta d) \ominus_{gH} \mathbf{F}(\bar{x})) \\
\text{or, } &\max_{i \in \mathcal{A}(\bar{x})} \mathbf{F}_{i_{\mathscr{H}}}(\bar{x})(h) \ \preceq \ \mathbf{F}_{\mathscr{H}}(\bar{x})(h). \qquad (5.11)
\end{aligned}$$

To prove the reverse inequality, we claim that there exists a neighbourhood $\mathcal{N}(\bar{x})$ such that $\mathcal{A}(x) \subset \mathcal{A}(\bar{x})$ for all $x \in \mathcal{N}(\bar{x})$. Assume on contrary that there exists a sequence $\{x_k\}$ in $\mathcal{X}$ with $x_k \to \bar{x}$ such that $\mathcal{A}(x_k) \not\subset \mathcal{A}(\bar{x})$. We can choose $i_k \in \mathcal{A}(x_k)$ but $i_k \notin \mathcal{A}(\bar{x})$. Since $\mathcal{A}(x_k)$ is closed, $i_k \to \bar{i} \in \mathcal{A}(x_k)$. By gH-continuity of $\mathbf{F}$, we have

$$\mathbf{F}_{\bar{i}}(x_k) = \mathbf{F}(x_k) \implies \mathbf{F}_{\bar{i}}(\bar{x}) = \mathbf{F}(\bar{x}),$$

which contradicts to $i_k \notin \mathcal{A}(\bar{x})$. Thus, $\mathcal{A}(x) \subset \mathcal{A}(\bar{x})$ for all $x \in \mathcal{N}(\bar{x})$.

Let us choose a sequence $\{\delta_k\}, \delta_k \to 0$ such that $\bar{x} + \delta_k d \in \mathcal{N}(\bar{x})$ for all $d \in \mathcal{X}$. Then,

$$\begin{aligned}
&\mathbf{F}_i(\bar{x}) \ \preceq \ \mathbf{F}(\bar{x}), \quad \text{for all } i \in I \\
\text{or, } &\mathbf{F}(\bar{x} + \delta_k d) \ominus_{gH} \mathbf{F}(\bar{x}) \preceq \mathbf{F}(\bar{x} + \delta_k d) \ominus_{gH} \mathbf{F}_i(\bar{x}), \quad \text{for all } i \in \mathcal{A}(\bar{x}) \\
\text{or, } &\mathbf{F}(\bar{x} + \delta_k d) \ominus_{gH} \mathbf{F}(\bar{x}) \preceq \mathbf{F}_i(\bar{x} + \delta_k d) \ominus_{gH} \mathbf{F}_i(\bar{x}), \quad \text{for all } i \in \mathcal{A}(\bar{x} + \delta_k d) \\
\text{or, } &\lim_{\substack{k \to \infty \\ d \to h}} \frac{1}{\delta_k} \odot (\mathbf{F}(\bar{x} + \delta_k d) \ominus_{gH} \mathbf{F}(\bar{x})) \preceq \lim_{\substack{k \to \infty \\ d \to h}} \frac{1}{\delta_k} \odot (\mathbf{F}_i(\bar{x} + \delta_k d) \ominus_{gH} \mathbf{F}_i(\bar{x})) \\
\text{or, } &\mathbf{F}_{\mathscr{H}}(\bar{x})(h) \preceq \max_{i \in \mathcal{A}(\bar{x})} \mathbf{F}_{i_{\mathscr{H}}}(\bar{x})(h). \qquad (5.12)
\end{aligned}$$

From (5.11) and (5.12) , we obtain

$$\mathbf{F}_{\mathscr{H}}(\bar{x})(h) = \max \mathbf{F}_{i_{\mathscr{H}}}(\bar{x})(h) \text{ for all } i \in \mathcal{A}(\bar{x}).$$

□

5.5 Characterization of Efficient Solutions

In this section, we present some characterizations of efficient solutions for IOPs with the help of the properties of gH-Hadamard differentiable IVFs.

Theorem 5.9. (Sufficient condition for efficient points). *Let $\mathcal{S}$ be a nonempty convex subset of $\mathcal{X}$ and $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be a convex IVF. If the function $\boldsymbol{F}$ has a gH-Hadamard derivative at $\bar{x} \in \mathcal{S}$ in the direction $v - \bar{x}$ with*

$$\boldsymbol{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}) \nprec \boldsymbol{0}, \quad \text{for all } v \in \mathcal{X}, \tag{5.13}$$

then $\bar{x}$ must be an efficient point of the IOP (1.5).

Proof. Assume that $\bar{x}$ is not an efficient point of $\mathbf{F}$. Then, there exists at least one $y \in \mathcal{S}$ such that for any $\lambda \in (0, 1]$, we have

$$\begin{aligned}
&\lambda \odot \mathbf{F}(y) \prec \lambda \odot \mathbf{F}(\bar{x}),\\
\text{or, } &\lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(\bar{x}) \prec \lambda \odot \mathbf{F}(\bar{x}) \oplus \lambda' \odot \mathbf{F}(\bar{x}), \text{ where } \lambda' = 1 - \lambda,\\
\text{or, } &\lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(\bar{x}) \prec (\lambda + \lambda') \odot \mathbf{F}(\bar{x}) = \mathbf{F}(\bar{x}).
\end{aligned}$$

Due to the convexity of $\mathbf{F}$ on $\mathcal{S}$, we have

$$\mathbf{F}(\bar{x} + \lambda(y - \bar{x})) = \mathbf{F}(\lambda y + \lambda'\bar{x}) \preceq \lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(\bar{x}) \prec \mathbf{F}(\bar{x}),$$

$$\text{or, } \mathbf{F}(\bar{x} + \lambda(y - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \prec \mathbf{0},$$

$$\text{or, } \mathbf{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}) \preceq \mathbf{0}. \tag{5.14}$$

Now we have the following two possibilities.

- **Case** I: If $\mathbf{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}) = \mathbf{0}$, then $\mathbf{F}_{\mathscr{D}}(\bar{x})(v - \bar{x}) = \mathbf{0}$ and

$$\underline{f}_{\mathscr{D}}(\bar{x})(v - \bar{x}) = 0 \text{ and } \overline{f}_{\mathscr{D}}(\bar{x})(v - \bar{x}) = 0. \tag{5.15}$$

  Due to Lemma 1.8, $\underline{f}$ and $\overline{f}$ are convex on $\mathcal{S}$. From (5.15), we observe that $\bar{x}$ is a minimum point of $\underline{f}$ and $\overline{f}$. Consequently, $\bar{x}$ is an efficient point of $\mathbf{F}$. This contradicts to our assumption that $\bar{x}$ is not efficient point of $\mathbf{F}$.

- **Case** II: If $\mathbf{F}_{\mathscr{H}}(\bar{x})(v-\bar{x}) \prec \mathbf{0}$, then this contradicts the assumption that $\mathbf{F}_{\mathscr{H}}(\bar{x})(v-\bar{x}) \nprec \mathbf{0}$ for all $v \in \mathcal{X}$.

Hence, $\bar{x}$ is the efficient point of the IOP (1.5). □

Remark 5.10. The relation (5.13) can be seen as a variational inequality for interval-valued functions. For details as variational inequalities, we refer [**?**]. The converse of Theorem 5.9 is not true. For example, consider $\mathcal{X} = \mathbb{R}$, $\mathcal{S} = [-1, 2]$, and the convex IVF $\mathbf{F} : \mathcal{S} \to I(\mathbb{R})$ defined by

$$\mathbf{F}(x) = [4x^2 - 4x + 1, 2x^2 + 75].$$

At $\bar{x} = 0$ and for $v \in \mathcal{X}$, $\mathbf{F}_{\mathscr{H}}(\bar{x})(v) = v \odot [-4, 0]$ for all $v \in \mathcal{X}$.

From Figure 5.1, it is clear that $\bar{x} = 0$ is an efficient solution of the IOP (1.5). However, for all $v > 0$ we have $\mathbf{F}_{\mathscr{H}}(\bar{x})(v) \prec \mathbf{0}$.

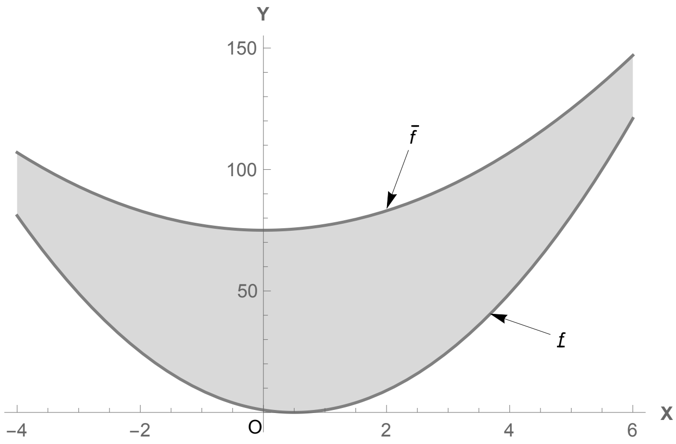

FIGURE 5.1: The IVF $\mathbf{F}$ of Remark 5.10

Theorem 5.11. (Necessary condition for efficient points). *Let $\mathcal{S}$ be a linear subspace of $\mathcal{X}$, $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF and $\bar{x} \in \mathcal{S}$ be an efficient point of the IOP (1.5). If the function $\boldsymbol{F}$ has a gH-Hadamard derivative at $\bar{x}$ in every direction $v \in \mathcal{S}$, then*

$$\boldsymbol{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}) \nprec \boldsymbol{0}, \quad \text{for all } v \in \mathcal{S}.$$

Proof. Since the point $\bar{x}$ is an efficient point of the function $\mathbf{F}$, for any $h \in \mathcal{S}$ and $\lambda > 0$, we have

$$\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0}. \tag{5.16}$$

If $\mathbf{F}_{\mathscr{H}}(\bar{x})(v-\bar{x}) \preceq \mathbf{0}$, then due to linearity of $\mathbf{F}_{\mathscr{H}}(\bar{x})$ on $\mathcal{S}$, we have $\mathbf{F}_{\mathscr{H}}(\bar{x})(v-\bar{x}) = \mathbf{0}$ by (ii) of Lemma 1.10. Therefore, $\mathbf{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}) \nprec \mathbf{0}$ for all $v \in \mathcal{S}$.

If $\mathbf{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}) \nprec \mathbf{0}$, then the result holds. □

Remark 5.12. One may think that in Theorem 5.11, instead of considering the fact that the IVF $\mathbf{F}$ is defined on a linear subspace of $\mathcal{S}$, we may take $\mathbf{F}$ being defined on any nonempty convex subset of $\mathcal{S}$. However, this assumption is not sufficient. For instance, consider $\mathcal{X} = \mathbb{R}$, $\mathcal{S} = [-1, 7]$, and the convex IVF $\mathbf{F} : \mathcal{S} \to I(\mathbb{R})$ defined by $\mathbf{F}(x) = [x^2 - 4x + 4, x^2 + 5]$. Then at $\bar{x} \in \mathcal{S}$, $\mathbf{F}_{\mathscr{H}}(\bar{x})(v) = 2v \odot [\bar{x} - 2, \bar{x}]$ for all $v \in \mathcal{X}$. Note that $\bar{x} = 0$ is an efficient point of IOP (1.5) because $\mathbf{F}(y) \nprec \mathbf{F}(\bar{x})$ for all $y \in \mathcal{S}$. However, $\mathbf{F}_{\mathscr{H}}(\bar{x})(v) \prec \mathbf{0}$ for all $v > 0$.

Theorem 5.13. *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$, $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF, and $\bar{x} \in \mathcal{S}$ be an efficient point of the IOP (1.5). If the IVF $\boldsymbol{F}$ has a gH-Hadamard derivative at $\bar{x}$ in every direction $v \in \mathcal{S}$, then there exist no $v \in \mathcal{S}$ such that $\boldsymbol{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}) < \boldsymbol{0}$.*

Proof. Since the point $\bar{x}$ is an efficient point of the function $\mathbf{F}$, for any $h \in \mathcal{S}$ and $\lambda > 0$, we have

$$\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0}.$$

This implies that

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \max\{\underline{f}(\bar{x} + \lambda(h - \bar{x})) - \underline{f}(\bar{x}), \overline{f}(\bar{x} + \lambda(h - \bar{x})) - \overline{f}(\bar{x})\} \geq 0. \tag{5.17}$$

From (5.17) and Lemma 1.2, there is no $v \in \mathcal{S}$ such that $\mathbf{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}) < \mathbf{0}$. □

Theorem 5.14. . *Let $\mathcal{S}$ be a linear subspace of $\mathcal{X}$, $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF, and $\bar{x} \in \mathcal{S}$ be an efficient point of the IOP (1.5). If the IVF $\boldsymbol{F}$ has a gH-Hadamard derivative at $\bar{x}$ in every direction $v \in \mathcal{S}$, then*

$$0 \in \boldsymbol{F}_{\mathscr{H}}(\bar{x})(v), \quad \textit{for all } v \in \mathcal{S}.$$

The converse holds if $\boldsymbol{F}$ is convex on $\mathcal{X}$.

Proof. Let $\bar{x}$ be an efficient point of IOP (1.5). Then, by Theorem 5.11, we have $\mathbf{F}_{\mathscr{H}}(\bar{x})(v) \nprec \mathbf{0}$ for all $v \in \mathcal{S}$. Due to linearity of $\mathbf{F}_{\mathscr{H}}(\bar{x})$ and $v = -h$, we obtain $\mathbf{F}_{\mathscr{H}}(\bar{x})(h) \nsucc \mathbf{0}$ for all $h \in \mathcal{S}$. Hence, $0 \in \mathbf{F}_{\mathscr{H}}(\bar{x})(v)$ for all $v \in \mathcal{S}$.

Conversely, let $\mathbf{F}$ be convex on $\mathcal{S}$ and assume that $\mathbf{F}$ has a gH-Hadamard derivative at $\bar{x}$ in every direction $w \in \mathcal{X}$. Let $0 \in \mathbf{F}_{\mathscr{H}}(\bar{x})(w)$ for all $w \in \mathcal{X}$. Then, due to linearity of $\mathbf{F}_{\mathscr{H}}(\bar{x})$ on $\mathcal{S}$, we have

$$\mathbf{F}_{\mathscr{H}}(\bar{x})(w) \nprec \mathbf{0} \text{ and } \mathbf{0} \nprec \mathbf{F}_{\mathscr{H}}(\bar{x})(w) \text{ for all } w.$$

Hence, $\bar{x}$ is efficient point of IOP (1.5) by Theorem 5.9. □

5.6 Fritz John and Karush-Kuhn-Tucker Optimality Conditions

In this section, we derive an extended KKT necessary and sufficient optimality conditions to characterize efficient solutions of IOPs.

Lemma 5.15. *Let $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ be a gH-Hadamard differentiable IVF at $\bar{x}$ in the direction $v \in \mathbb{R}^n$ with $\boldsymbol{F}_{\mathscr{H}}(\bar{x})(v) \prec \boldsymbol{0}$. Then, there exists $\delta > 0$ such that for each $\lambda \in (0, \delta)$,*

$$\boldsymbol{F}(\bar{x} + \lambda v) \prec \boldsymbol{F}(\bar{x}).$$

Proof. Since $\mathbf{F}_{\mathscr{H}}(\bar{x})(v) \prec \mathbf{0}$, there exist $\delta, \delta' > 0$ such that for all $h \in \mathbb{R}^n$, we have

$$\frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) \prec \mathbf{0}, \ \lambda \in (0, \delta) \text{ and } \|v - h\| < \delta'.$$

Due to gH-continuity of $\mathbf{F}$ at v, we get

$$\mathbf{F}(\bar{x} + \lambda v) \ominus_{gH} \mathbf{F}(\bar{x}) \prec \mathbf{0}, \ \forall \ \lambda \in (0, \delta),$$

which implies $\mathbf{F}(\bar{x} + \lambda v) \prec \mathbf{F}(\bar{x}), \ \forall \ \lambda \in (0, \delta)$. □

Definition 5.16. Let $\mathbf{F} : \mathbb{R}^n \to I(\mathbb{R})$ be a gH-Hadamard differentiable IVF at $\bar{x}$. Then, the set of descent directions at $\bar{x}$ is defined by

$$\hat{\mathbf{F}}(\bar{x}) = \{d \in \mathbb{R}^n : \mathbf{F}_{\mathscr{H}}(\bar{x})(d) \prec \mathbf{0}\}.$$

As for any d in $\hat{\mathbf{F}}(\bar{x})$, $\lambda d \in \hat{\mathbf{F}}(\bar{x})$ for all $\lambda > 0$, the set $\hat{\mathbf{F}}(\bar{x})$ is called the cone of descent direction.

Definition 5.17. [26] Given a nonempty set $\mathcal{S} \subseteq \mathbb{R}^n$ and $\bar{x} \in \mathcal{S}$. At $\bar{x}$, the cone of feasible directions of $\mathcal{S}$ is defined by

$$\hat{\mathcal{S}}(\bar{x}) = \{d \in \mathbb{R}^n : d \neq 0, \ \bar{x} + \lambda d \in \mathcal{S}, \ \forall \ \lambda \in (0, \delta) \text{ and for some } \delta > 0\}.$$

Lemma 5.18. *Let $\mathcal{S} \subseteq \mathbb{R}^n$ and $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ be a gH-Hadamard differentiable IVF at $\bar{x} \in \mathcal{S}$. If $\bar{x}$ is an efficient solution of the IOP (1.5), then $\hat{\boldsymbol{F}}(\bar{x}) \cap \hat{\mathcal{S}}(\bar{x}) = \emptyset$.*

Proof. Assume contrary that $\hat{\mathbf{F}}(\bar{x}) \cap \hat{\mathcal{S}}(\bar{x}) \neq \emptyset$ and $d \in \hat{\mathbf{F}}(\bar{x}) \cap \hat{\mathcal{S}}(\bar{x})$. By Lemma 5.15 and Definition 5.17, there exist $\delta_1, \delta_2 > 0$ such that

$$\bar{x} + \lambda d \in \mathcal{S} \text{ for all } \lambda \text{ in } (0, \delta_1) \quad \text{and} \quad \mathbf{F}(\bar{x} + \lambda d) \prec \mathbf{F}(\bar{x}) \text{ for all } \lambda \text{ in } (0, \delta_2).$$

Taking $\delta = \min\{\delta_1, \delta_2\}$, we see that for all $\lambda \in (0, \delta)$,

$$\bar{x} + \lambda d \in \mathcal{S} \text{ and } \mathbf{F}(\bar{x} + \lambda d) \prec \mathbf{F}(\bar{x}).$$

This is contradictory to $\bar{x}$ being a local efficient point. Hence, $\hat{\mathbf{F}}(\bar{x}) \cap \hat{\mathcal{S}}(\bar{x}) = \emptyset$. □

Lemma 5.19. *For $i = 1, 2, \ldots, m$, let $\boldsymbol{G}_i : \mathbb{R}^n \to I(\mathbb{R})$ be IVF, X be a non-empty open set in $\mathbb{R}^n$, and $\mathcal{S} = \{x \in X : \boldsymbol{G}_i(x) \preceq \mathbf{0}$ for $i = 1, 2, \ldots, m\}$. Let $\bar{x} \in \mathcal{S}$ and $I(\bar{x}) = \{i : \boldsymbol{G}_i(\bar{x}) = \mathbf{0}\}$. For all $i \in I(\bar{x})$, assume that $\boldsymbol{G}_i$ is gH-Hadamard differentiable at $\bar{x}$ and gH-continuous for $i \notin I(\bar{x})$, define*

$$\hat{G}(\bar{x}) = \{d : \boldsymbol{G}_{i\mathscr{H}}(\bar{x})(d)_{(}\bar{x}) \prec \mathbf{0} \text{ for all } i \in I(x_0)\}.$$

Then, $\hat{G}(\bar{x}) \subseteq \hat{\mathcal{S}}(\bar{x})$, where $\hat{\mathcal{S}}(\bar{x}) = \{d \in \mathbb{R}^n : d \neq 0, \ \bar{x} + \alpha d \in S \ \forall \alpha \in (0, \delta)$ for some $\delta > 0\}$.

Proof. It is similar to proof of Lemma 3.1 in [26] for gH-Hadamard derivative, and therefore, we omit. □

With the help of Lemma 5.19, we characterize an efficient solution of a constrained IOP. It is shown that at a local efficient solution, the cones of descent direction and feasible direction have an empty intersection.

Theorem 5.20. *Let $\mathcal{S}$ be a non-empty open set in $\mathbb{R}^n$. Consider an IOP*

$$\left.\begin{aligned} &\min \ \boldsymbol{F}(x) \\ &\text{such that} \quad \boldsymbol{G}_i(x) \preceq \mathbf{0}, \quad \text{for } i = 1, 2, \ldots, m \\ &x \in \mathcal{S}, \end{aligned}\right\} \tag{5.18}$$

where $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ and $\boldsymbol{G}_i : \mathbb{R}^n \to I(\mathbb{R})$ for $i = 1, 2, \ldots, m$. For a feasible point x_0, define $I(x_0) = \{i : \boldsymbol{G}_i(\bar{x}) = 0\}$. At $\bar{x}$, let $\boldsymbol{F}$ and $\boldsymbol{G}_i$, $i \in I(\bar{x})$, be gH-Hadamard differentiable, and for $i \notin I(\bar{x})$, $\boldsymbol{G}_i$ be gH-continuous. If $\bar{x}$ is a local efficient solution of (5.18), then

$$\hat{F}(\bar{x}) \cap \hat{G}(\bar{x}) = \emptyset,$$

where $\hat{F}(\bar{x}) = \{d : \boldsymbol{F}_{\mathscr{H}}(\bar{x})(d)_{(}\bar{x}) \prec \mathbf{0}\}$ and $\hat{G}(\bar{x}) = \{d : \boldsymbol{G}_{i\mathscr{H}}(\bar{x})(d)_{(}\bar{x}) \prec \mathbf{0}$ *for each* $i \in I(\bar{x})\}$.

Proof. By Lemma 5.18 and Lemma 5.19, we obtain

$$x_0 \text{ is a local efficient solution } \implies \hat{F}(\bar{x}) \cap \hat{\mathcal{S}}(\bar{x}) = \emptyset \implies \hat{F}(\bar{x}) \cap \hat{G}(\bar{x}) = \emptyset.$$

□

Theorem 5.21. (Extended Fritz John necessary optimality condition). *Let* $\mathcal{S}$ *be a non-empty open set in* $\mathbb{R}^n$*;* $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ *and* $\boldsymbol{G}_i : \mathbb{R}^n \to I(\mathbb{R})$ *for* $i = 1, 2, \ldots, m$ *be IVFs. Consider the IOP:*

$$\left.\begin{aligned} &\min \ \boldsymbol{F}(x), \\ &\text{such that } \boldsymbol{G}_i(x) \preceq \mathbf{0}, \quad i = 1, 2, \ldots, m \\ &x \in \mathcal{S}. \end{aligned}\right\} \tag{5.19}$$

For a feasible point $\bar{x}$*, define* $I(\bar{x}) = \{i : \boldsymbol{G}_i(\bar{x}) = \boldsymbol{0}\}$*. Let* $\boldsymbol{F}$ *and* $\boldsymbol{G}_i$ *be gH-Hadamard differentiable at* $\bar{x}$ *for* $i \in I(\bar{x})$ *and gH-continuous for* $i \notin I(\bar{x})$*. If* $\bar{x}$ *is a local efficient point of (5.19), then there exist constants* u_0 *and* u_i *for* $i \in I(\bar{x})$ *such that*

$$\begin{cases} 0 \in \left(u_0 \odot \boldsymbol{F}_{\mathscr{H}}(\bar{x})(d) \oplus \displaystyle\sum_{i \in I(\bar{x})} u_i \odot \boldsymbol{G}_{i\mathscr{H}}(\bar{x})(d)\right), \\ u_0 \geq 0, u_i \geq 0 \text{ for } i \in I(\bar{x}), \\ (u_0, u_I) \neq \left(0, 0_v^{|I(\bar{x})|}\right), \end{cases}$$

where u_I *is the vector whose components are* u_i *for* $i \in I(\bar{x})$*.*

Further, if $\boldsymbol{G}_i$*, for all* $i \notin I(\bar{x})$*, are also gH-Hadamard differentiable at* $\bar{x}$*, then there* *exist constants* $u_0, u_1, u_2, \ldots, u_m$ *such that*

$$\begin{cases} 0 \in \left(u_0 \odot \boldsymbol{F}_{\mathscr{H}}(\bar{x})(d) \oplus \sum_{i=1}^{m} u_i \odot \boldsymbol{G}_{i\mathscr{H}}(\bar{x})(d) \right), \\ u_i \odot \boldsymbol{G}_i(\bar{x}) = \boldsymbol{0}, \ \ i = 1, 2, \ldots, m, \\ u_0 \geq 0, u_i \geq 0, \ \ i = 1, 2, \ldots, m, \\ (u_0, u) \neq (0, 0_v^m), \end{cases}$$

where u *is the vector* $(u_1, u_2, \ldots, u_m)$.

Proof. Since $\bar{x}$ is a local efficient point of (5.19), by Theorem 5.20, we get

$$\begin{aligned} &\hat{F}(\bar{x}) \cap \hat{G}(\bar{x}) = \emptyset, \\ \text{or, } &\nexists\, d \in \mathbb{R}^n \text{ s.t. } \mathbf{F}_{\mathscr{H}}(\bar{x})(d) \prec \mathbf{0} \text{ and } \mathbf{G}_{i\mathscr{H}}(\bar{x})(d) \prec \mathbf{0} \ \forall\, i \ \in I(\bar{x}), \\ \text{or, } &\mathbf{F}_{\mathscr{H}}(\bar{x})(d) \nprec \mathbf{0} \text{ and } \mathbf{G}_{i\mathscr{H}}(\bar{x})(d) \nprec \mathbf{0} \ \forall\, d \in \mathbb{R}^n \text{ and } i \in I(\bar{x}), \\ \text{or, } &0 \in \mathbf{F}_{\mathscr{H}}(\bar{x})(d) \text{ and } 0 \in \mathbf{G}_{i\mathscr{H}}(\bar{x})(d) \ \forall\, d \in \mathbb{R}^n \text{ and } i \ \in I(\bar{x}) \ \text{ by Lemma 1.10.} \end{aligned} \tag{5.20}$$

We can chose nonzero vector p with $p = [u_0, u_i]_{i \in I(\bar{x})}^{\top}$ such that

$$\begin{cases} 0 \in \left(u_0 \odot \mathbf{F}_{\mathscr{H}}(\bar{x})(d) \oplus \sum_{i \in I(\bar{x})} u_i \odot \mathbf{G}_{i\mathscr{H}}(\bar{x})(d) \right), \\ u_0, u_i \geq 0 \text{ for } i \in I(x_0), \\ (u_0, u_I) \neq (0, 0, \cdots, 0). \end{cases}$$

This proves the first part of the theorem.

For $i \in I(\bar{x})$, $\mathbf{G}_i(\bar{x}) = \mathbf{0}$. Therefore, $u_i \odot \mathbf{G}_i(\bar{x}) = \mathbf{0}$. If $\mathbf{G}_i$ for all $i \notin I(\bar{x})$ are also gH-differentiable at $\bar{x}$, by setting $u_i = 0$ for $i \notin I(\bar{x})$ the second part of the theorem is followed. □

Definition 5.22. [26] The set of m intervals $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m\}$ is said to be linearly independent if for m real numbers $c_1, c_2, \ldots, c_m$:

$$0 \in c_1 \odot \mathbf{X}_1 \oplus c_2 \odot \mathbf{X}_2 \oplus \ldots \oplus c_m \odot \mathbf{X}_m \quad \text{if and only if} \quad c_1 = 0, c_2 = 0, \ldots, c_m = 0.$$

Theorem 5.23. (Extended Karush-Kuhn-Tucker necessary optimality condition). *Let $\mathcal{S}$ be a non-empty open set in $\mathbb{R}^n$ and $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ and $\boldsymbol{G}_i : \mathbb{R}^n \to I(\mathbb{R})$, $i = 1, 2, \ldots, m$, be IVFs. Suppose that $\bar{x}$ is a feasible point of the following IOP:*

$$\left.\begin{array}{l} \min \ \ \boldsymbol{F}(x) \\ \text{such that} \quad \boldsymbol{G}_i(x) \preceq \mathbf{0}, \quad i = 1, 2, \ldots, m \\ x \in \mathcal{S}. \end{array}\right\}$$

Define $I(\bar{x}) = \{i : \boldsymbol{G}_i(\bar{x}) = 0\}$. Let

(i) $\boldsymbol{F}$ and $\boldsymbol{G}_i$ be gH-Hadamard differentiable at $\bar{x}$ for all $i \in I(\bar{x})$,

(ii) $\boldsymbol{G}_i$ be gH-continuous for all $i \notin I(\bar{x})$, and

(iii) the collection of intervals $\{\boldsymbol{G}_{i\mathscr{H}}(\bar{x})(d) \ : \ i \in I(\bar{x})\}$ be linearly independent.

If $\bar{x}$ is a local efficient solution, then there exist constants $u_i \geq 0$ for all $i \in I(\bar{x})$ such that

$$0 \in \left(u_0 \odot \boldsymbol{F}_{\mathscr{H}}(\bar{x})(d) \oplus \sum_{i \in I(\bar{x})} u_i \odot \boldsymbol{G}_{i\mathscr{H}}(\bar{x})(d) \right)$$

If $\boldsymbol{G}_i$'s, for $i \notin I(\bar{x})$, are also gH-differentiable at $\bar{x}$, then there exist constants u_1, u_2, …, u_m such that

$$\begin{cases} 0 \in \left(u_0 \odot \boldsymbol{F}_{\mathscr{H}}(\bar{x})(d) \oplus \sum_{i=1}^{m} u_i \odot \boldsymbol{G}_{i\mathscr{H}}(\bar{x})(d) \right), \\ u_i \odot \boldsymbol{G}_i(\bar{x}) = \boldsymbol{0},\ i = 1, 2, \ldots, m, \\ u_i \geq 0,\ i = 1, 2, \ldots, m. \end{cases}$$

Proof. By Theorem 5.21, there exist real constants u_0 and u_i' for all $i \in I(\bar{x})$, not all zeros, such that

$$\begin{cases} 0 \in \left(u_0 \odot \mathbf{F}_{\mathscr{H}}(\bar{x})(d) \oplus \sum_{i \in I(\bar{x})} u_i' \odot \mathbf{G}_{i\mathscr{H}}(\bar{x})(d) \right), \\ u_0 \geq 0, u_i' \geq 0 \text{ for all } i \in I(\bar{x}). \end{cases} \tag{5.21}$$

Then, we must have $u_0 > 0$. Since otherwise, the set $\{\mathbf{G}_{i\mathscr{H}}(\bar{x})(d) \ : \ i \in I(\bar{x})\}$ will become linearly dependent.

Define $u_i = u_i'/u_0$. Then, $u_i \geq 0$ for all $i \in I(\bar{x})$ and

$$0 \in \left(u_0 \odot \mathbf{F}_{\mathscr{H}}(\bar{x})(d) \oplus \sum_{i \in I(\bar{x})} u_i \odot \mathbf{G}_{i\mathscr{H}}(\bar{x})(d) \right).$$

For $i \in I(\bar{x})$, $\mathbf{G}_i(\bar{x}) = \mathbf{0}$. Therefore, $0 \in u_i \odot \mathbf{G}_i(\bar{x})$. If the functions $\mathbf{G}_i$ for $i \notin I(\bar{x})$ are also gH-Hadamard differentiable at x_0, then by setting $u_i = 0$ for $i \notin I(\bar{x})$, the latter part of the theorem is followed. □

Theorem 5.24. (Extended Karush-Kuhn-Tucker sufficient condition for efficient points). *Let $\mathcal{S}$ be a nonempty convex subset of $\mathcal{X}$; $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ and $\boldsymbol{G}_i : \mathcal{S} \to$*

$I(\mathbb{R})$, $i = 1, 2, \cdots, m$ be interval-valued gH-Hadamard differentiable convex functions. Suppose that $\bar{x} \in \mathcal{S}$ is a feasible point of the following IOP:

$$\left.\begin{aligned} & \textit{min} \quad \boldsymbol{F}(x) \\ & \textit{such that} \quad \boldsymbol{G}_i(\bar{x}) \preceq \boldsymbol{0}, \quad i = 1, 2, \cdots, m \\ & x \in \mathcal{S}. \end{aligned}\right\} \tag{5.22}$$

If there exist real constants $u_1, u_2, \ldots, u_m$ for which

$$\begin{cases} \boldsymbol{F}_{\mathscr{H}}(\bar{x})(v) \oplus \sum_{i=1}^{m} u_i \odot \boldsymbol{G}_{i\mathscr{H}}(\bar{x})(v) \nprec \boldsymbol{0}, & \textit{for all } v \in \mathcal{S}, \\ u_i \odot \boldsymbol{G}_i(\bar{x}) = \boldsymbol{0}, \ i = 1, 2, \cdots, m \\ u_i \geq 0, \ i = 1, 2, \cdots, m, \end{cases}$$

then $\bar{x}$ is an efficient point of the IOP.

Proof. By the hypothesis, for every $v \in \mathcal{S}$ satisfying $\mathbf{G}_i(v) \preceq \mathbf{0}$ for all $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} & \mathbf{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}) \oplus \sum_{i=1}^{m} u_i \mathbf{G}_{i\mathscr{H}}(\bar{x})(v - \bar{x}) \nprec \mathbf{0}, \\ \Longrightarrow \quad & (\mathbf{F}(v) \ominus_{gH} \mathbf{F}(\bar{x})) \oplus \left(\sum_{i=1}^{m} u_i \left(\mathbf{G}_i(v) \ominus_{gH} \mathbf{G}_i(\bar{x}) \right) \right) \nprec \mathbf{0}, \quad \text{by (5.3) of Theorem 5.5} \\ \Longrightarrow \quad & (\mathbf{F}(v) \ominus_{gH} \mathbf{F}(\bar{x})) \oplus \left(\sum_{i=1}^{m} u_i \left(\mathbf{G}_i(v) \right) \right) \nprec \mathbf{0}, \\ \Longrightarrow \quad & \mathbf{F}(v) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0} \ \text{ since } \mathbf{G}_i(v) \preceq \mathbf{0}, \\ \Longrightarrow \quad & \mathbf{F}(v) \nprec \mathbf{F}(\bar{x}). \end{aligned}$$

Hence, $\bar{x}$ is an efficient point of the IOP (5.22). □

5.7 Concluding Remarks

In this chapter, the concept gH-Hadamard derivative of IVF has been studied. It has been noticed that the gH-Hadamard derivative at any point is the gH-Fréchet derivative at that point and vise-versa. Also, a gH-Hadamard differentiable IVF is found to be gH-continuous. It has been shown the gH-Hadamard derivative is helpful to characterize the convexity of IVF. It has been observed that the composition of a Hadamard differentiable real-valued function and a gH-Hadamard differentiable IVF is gH-Hadamard differentiable IVF and the chain rule is applicable. Further, for finite comparable IVF, it has been proven that the gH-Hadamard derivative of the maximum of all finite comparable IVFs is the maximum of their gH-Hadamard derivative.

In addition, it has been shown that for a convex IVF if gH-Hadamard derivative at any point does not dominate to zero, then that point is an efficient point of that IOP. Also, it has been proven that if the set of feasible points is convex and the gH-Hadamard differentiable at an efficient point of IOP, then gH-Hadamard derivative does not dominate to zero and also contains zero. Further, for constraint IOPs, we have proved extended KKT necessary and sufficient condition to characterize the efficient solutions by using gH-Hadamard derivative.

****