Chapter 4

Generalized Hukuhara Hadamard Semidervative of Interval-valued Functions and its Application in Interval Optimization

4.1 Introduction

For the first initiation to nondifferentiable optimization, semidifferentials have been preferred over subdifferentials that necessitate a good command of set-valued analysis. The emphasis will be on Hadamard semidifferentiable functions for which the resulting semidifferential calculus retains all the nice features of the classical differential calculus, including the chain rule. Convex continuous and semiconvex functions are Hadamard semidifferentiable, and an explicit expression of the semidifferential of an extremum with respect to parameters can be obtained. So, it works well for most nondifferentiable optimization problems including semiconvex or semiconcave problems. The Hadamard semidifferential calculus readily extends to functions defined on differential manifolds and on groups that naturally occur in optimization problems with respect to the shape or the geometry.

4.2 Motivation

From the literature on the analysis of IVFs, one can notice that the existing derivatives are neither sufficient to retain the two most important features of classical differential calculus—Continuity of functions and the chain rule, and nor sufficient to characterize the optimal solutions of IOPs. Although some optimality conditions are proposed for IOPs by using gH-directional and gH-Gâteaux derivatives, these derivatives are not sufficient to preserve the continuity of IVFs and chain rule for the composition of IVFs. Even though gH-Fréchet derivative preserves linearity and continuity but it does not hold the chain rule for the composition of IVFs whose lower and upper functions are equal at each points (see the example for Proposition 3.5 [75]). However, Hadamard semiderivative preserves linearity and continuity as well as the chain rule.

4.3 Contributions

In this chapter, we define gH-Hadamard semiderivative of IVFs and prove that if an IVF is gH-Hadamard semidifferentiable, then IVF is gH-continuous. For a convex gH-Lipschitz continuous IVF, we show that the gH-Hadamard derivative exists and is equals to gH-directional derivative. Further, we prove that the composition of

Hadamard semidifferentiable real-valued function and gH-Hadamard semidifferentiable IVF is again a gH-Hadamard semidifferentiable IVF. Besides, with the help of gH-hadamard semiderivative, we provide a necessary and sufficient condition for characterizing the efficient solutions to IOPs by using better dominance relation.

Novel derivations of this chapter are as follows:

(i) For a gH-Lipschitz continuous and gH-differentiable IVF, it is proved that gH-Hadamard semiderivative exists at every point of the domain.

(ii) It is shown that the composition of Hadamard semidifferentiable real-valued function and gH-Hadamard semidifferentiable IVF is a gH-Hadamard semidifferentiable IVF.

(iii) Extended Karush-Kuhn-Tucker condition for constraint IOP is derived.

4.4 Hadamard Semiderivative of Interval-valued Function

In this section, extended concepts of the gH-directional derivative, namely gH-Hadamard semiderivatives and some results to this derivative, for IVFs is given.

Definition 4.4.1 (gH-Hadamard semiderivative of IVF)**.** *Let $\boldsymbol{F}$ be an IVF on a nonempty subset $\mathcal{S}$ of $\mathcal{X}$. For $\bar{x} \in \mathcal{S}$ and $v \in \mathcal{X}$, if the limit*

$$\lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})\right)$$

exists finitely, then the limit value, denoted by $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v)$*, is called* gH-Hadamard semiderivative *of* $\boldsymbol{F}$ *at* $\bar{x}$ *in the direction* v*. If this limit exists for all* $v \in \mathcal{X}$*, then* $\boldsymbol{F}$ *is said to be* gH-Hadamard semidifferentiable *at* $\bar{x}$*.*

Remark 4.4.1. *The existence of limit as* $h \to v$ *and* $\lambda \to 0+$ *in Definition 4.4.1 is equivalent to using two sequences* $\{h_n\}$ *and* $\{\lambda_n\}$*, with* $\lambda_n > 0\ \forall\ n$*, converging to* v *and* 0*, respectively. That is,* $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v)$ *exists if for all sequences* $\{\lambda_n\}$ *and* $\{h_n\}$ *with* $\lambda_n > 0$ *for all* n*,* $\lim_{n\to\infty} \lambda_n = 0$ *and* $\lim_{n\to\infty} h_n = v$*, we have*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \odot (\boldsymbol{F}(\bar{x} + \lambda_n h_n) \ominus_{gH} \boldsymbol{F}(\bar{x})) = \boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v).$$

Example 4.1. *In this example, we calculate the* gH*-Hadamard semiderivative of the IVF* $\boldsymbol{F}(x) = \|x\| \odot \boldsymbol{C}$*,* $x \in \mathbb{R}^n$*. For any* $v \in \mathbb{R}^n$ *and* $\bar{x} = 0$*, we see that*

$$\lim_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) = \lim_{\substack{\lambda\to 0+\\ h\to v}} \left(\left(\frac{1}{\lambda} \odot \lambda\right) \odot (\|h\| \odot \boldsymbol{C})\right) = \|v\| \odot \boldsymbol{C}.$$

Lemma 4.1. *If* $\underline{f}$ *and* $\overline{f}$ *are Hadamard semidifferentiable at* $\bar{x} \in \mathcal{S} \subseteq \mathcal{X}$*, then* $\boldsymbol{F}$ *is* gH*-Hadamard semidifferentiable IVF at* $\bar{x} \in \mathcal{S}$ *and*

$$\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) = \left[\min\left\{\underline{f}_{\mathscr{H}'}(\bar{x})(v), \overline{f}_{\mathscr{H}'}(\bar{x})(v)\right\}, \max\left\{\underline{f}_{\mathscr{H}'}(\bar{x})(v), \overline{f}_{\mathscr{H}'}(\bar{x})(v)\right\}\right]$$

Proof. See Appendix D.1 □

Note 8. *By definitions of* gH*-directional derivative (Definition 2.4.1) and* gH*-Hadamard semiderivative (Definition 4.4.1), it is clear that if* $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(h)$ *exists, then* $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h)$ *exists and it is equal to* $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(h)$*. However, the converse is not* *true. For instance, consider the IVF $\boldsymbol{F} : \mathbb{R}^2 \to I(\mathbb{R})$, which is defined by*

$$\boldsymbol{F}(x, y) = \begin{cases} \left(\frac{x^6}{(y-x^2)^2+x^8}\right) \odot [5,\ 8] & \text{if } (x, y) \neq (0, 0) \\ \boldsymbol{0} & \text{otherwise.} \end{cases}$$

For $\bar{x} = (0, 0)$ and arbitrary $h = (h_1, h_2) \in \mathbb{R}^2$,

$$\begin{aligned} \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\left(\frac{\lambda^6 h_1^6}{(\lambda h_2 - \lambda^2 h_1^2)^2 + \lambda^8 h_1^8} \right) \odot [5,\ 8] \right) \\ &= \boldsymbol{0}. \end{aligned}$$

Hence, $\boldsymbol{F}$ is gH-directional differentiable at $\bar{x}$ with $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h) = \boldsymbol{0}$.

Let $\lambda_n = \frac{1}{n}$ and $h_n = (\frac{1}{n}, \frac{1}{n^3}),\ n \in \mathbb{N}$. Then, for $\bar{x} = (0, 0)$ we have

$$\lim_{n \to \infty} \frac{1}{\lambda_n} \odot (\boldsymbol{F}(\bar{x} + \lambda_n h_n) \ominus_{gH} \boldsymbol{F}(\bar{x})) = \lim_{n \to \infty} n^5 \odot [5, 8]. \tag{4.1}$$

Hence, $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(0)$ does not exist.

Remark 4.4.2. *For an IVF $\boldsymbol{F}$ if $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(0)$ exists, $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(0)$ exists and $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(0) = \boldsymbol{F}_{\mathscr{H}'}(\bar{x})(0) = \boldsymbol{0}$.*

Theorem 4.2. *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$ and $\boldsymbol{F}$ is an IVF on $\mathcal{S}$. If $\boldsymbol{F}$ is gH-Hadamard semidifferentiable at $\bar{x} \in \mathcal{S}$ in every direction $v \in \mathcal{X}$, then the IVF $\boldsymbol{F}_{\mathscr{H}'}(\bar{x}) : \mathcal{X} \to I(\mathbb{R})$ is gH-continuous and for all $\delta \geq 0$,*

$$\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(\delta v) = \delta \odot \boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) \text{ for all } v \in \mathcal{X}.$$

Proof. For an arbitrary $v \in \mathcal{X}$ and $\delta \geq 0$, we have

$$\begin{aligned} \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda \delta h) \ominus_{gH} \mathbf{F}(\bar{x})) &= \delta \odot (\lim_{\substack{\lambda \to 0+ \\ h \to v}} \tfrac{1}{\lambda\delta} \odot (\mathbf{F}(\bar{x} + \lambda \delta h) \ominus_{gH} \mathbf{F}(\bar{x}))) \\ &= \delta \odot \mathbf{F}_{\mathscr{H}'}(\bar{x})(v). \end{aligned}$$

Since $\mathbf{F}_{\mathscr{H}'}(\bar{x})(v)$, for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $\lambda \in (0, \delta)$ and $h \in \mathcal{X}$,

$$\left\| \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) \ominus_{gH} \mathbf{F}_{\mathscr{H}'}(\bar{x})(v) \right\|_{I(\mathbb{R})} < \epsilon \text{ whenever } \|h - v\| < \delta.$$

As $\lambda \to 0+$, $\mathbf{F}_{\mathscr{D}}(\bar{x})(h)$ exists. Therefore, for all $h \in \mathcal{X}$ with $\|h - v\| < \delta$ we have

$$\begin{aligned} & \left\| \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) \ominus_{gH} \mathbf{F}_{\mathscr{H}'}(\bar{x})(v) \right\|_{I(\mathbb{R})} < \epsilon \\ \implies \quad & \|\mathbf{F}_{\mathscr{D}}(\bar{x})(h) \ominus_{gH} \mathbf{F}_{\mathscr{H}'}(\bar{x})(v)\|_{I(\mathbb{R})} < \epsilon. \end{aligned} \tag{4.2}$$

Since $\mathbf{F}_{\mathscr{D}}(\bar{x})(h) = \mathbf{F}_{\mathscr{H}'}(\bar{x})(h)$, from (4.2), $\mathbf{F}_{\mathscr{H}'}(\bar{x})$ is gH-continuous IVF at every $v \in \mathcal{X}$. □

The next theorem gives a necessary condition for the existence of gH-Hadamard semiderivative of IVFs.

Theorem 4.3. *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$ and $\boldsymbol{F}$ is an IVF on $\mathcal{S}$. If $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(0)$ exists at $\bar{x} \in \mathcal{S}$, then $\boldsymbol{F}$ is gH-continuous at $\bar{x}$. In addition, for any $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\frac{\|\boldsymbol{F}(y) \ominus_{gH} \boldsymbol{F}(\bar{x})\|_{I(\mathbb{R})}}{\|y - x\|^{\alpha}} < \epsilon \text{ for all } y \in \mathcal{B}_{\delta}(\bar{x}).$$

Proof. As $\mathbf{F}_{\mathscr{H}'}(\bar{x})(0)$ exists, $\mathbf{F}_{\mathscr{D}}(\bar{x})(0)$ exists and $\mathbf{F}_{\mathscr{D}}(\bar{x})(0) = \mathbf{F}_{\mathscr{H}'}(\bar{x})(0) = \mathbf{0}$. For $y \neq \bar{x}$, denoting

$$\lambda = \|y - \bar{x}\|^{\alpha} \text{ and } h = \frac{y - \bar{x}}{\|y - \bar{x}\|^{\alpha}} = \|y - \bar{x}\|^{1-\alpha} \frac{y - \bar{x}}{\|y - \bar{x}\|^{\alpha}},$$

we obtain

$$\frac{\|\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x})\|_{I(\mathbb{R})}}{\|y-\bar{x}\|^{\alpha}} = \left\|\frac{1}{\lambda} \odot (\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x}))\right\|_{I(\mathbb{R})}.$$

Also, $y \to \bar{x}$ implies $\lambda \to 0+$ and $h \to 0$. Therefore,

$$\lim_{y\to\bar{x}} \frac{\|\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x})\|_{I(\mathbb{R})}}{\|y-\bar{x}\|^{\alpha}} = \lim_{\lambda\to 0+} \left\|\frac{1}{\lambda} \odot (\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x}))\right\|_{I(\mathbb{R})} = \|\mathbf{F}_{\mathscr{D}}(\bar{x})(0)\|_{I(\mathbb{R})} = 0.$$

Therefore, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $y \in \mathcal{B}_{\delta}(\bar{x})$,

$$\frac{\|\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x})\|_{I(\mathbb{R})}}{\|y-\bar{x}\|^{\alpha}} = \left\|\frac{1}{\lambda} \odot (\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) \ominus_{gH} \mathbf{0}\right\|_{I(\mathbb{R})} < \epsilon.$$

This also yields gH-continuity of $\mathbf{F}$ at $\bar{x}$. □

The next theorem gives sufficient condition for the existence of gH-Hadamard semiderivative of IVFs.

Theorem 4.4. *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$ and $\boldsymbol{F}$ is a gH-Lipschitz continuous IVF at $\bar{x} \in \mathcal{S}$. If $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(v)$ exists, then $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v)$ exists and equals to $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(v)$, and*

$$\|\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) \ominus_{gH} \boldsymbol{F}_{\mathscr{H}'}(\bar{x})(w)\|_{I(\mathbb{R})} = \|v-w\| \text{ for all } v, w \in \mathcal{X}.$$

Proof. Since $\mathbf{F}$ is gH-Lipschitz continuous at $\bar{x}$, $\underline{f}$ and $\overline{f}$ are Lipschitz continuous at $\bar{x}$ by Lemma 1.8. As $\mathbf{F}_{\mathscr{D}}(\bar{x})(v)$ exists, $\underline{f}_{\mathscr{D}}(\bar{x})(v)$ and $\overline{f}_{\mathscr{D}}(\bar{x})(v)$ exist. Also, from Theorem 3.5 of [23], $\underline{f}$ and $\overline{f}$ are Hadamard semidifferentiable at $\bar{x}$ and

$$\underline{f}_{\mathscr{H}'}(\bar{x})(v) = \underline{f}_{\mathscr{D}}(\bar{x})(v) \text{ and } \overline{f}_{\mathscr{H}'}(\bar{x})(v) = \overline{f}_{\mathscr{D}}(\bar{x})(v) \text{ for all } v \in \mathcal{X}. \tag{4.3}$$

From (4.3) and Lemma 4.1, $\mathbf{F}_{\mathscr{H}'}(\bar{x})(v)$ exists and

$$\begin{aligned}\mathbf{F}_{\mathscr{H}'}(\bar{x})(v) &= \left[\min\left\{\underline{f}_{\mathscr{H}'}(\bar{x})(v), \overline{f}_{\mathscr{H}'}(\bar{x})(v)\right\}, \max\left\{\underline{f}_{\mathscr{H}'}(\bar{x})(v), \overline{f}_{\mathscr{H}'}(\bar{x})(v)\right\}\right] \\ &= \left[\min\left\{\underline{f}_{\mathscr{D}}(\bar{x})(v), \overline{f}_{\mathscr{D}}(\bar{x})(v)\right\}, \max\left\{\underline{f}_{\mathscr{D}}(\bar{x})(v), \overline{f}_{\mathscr{D}}(\bar{x})(v)\right\}\right] \\ &= \mathbf{F}_{\mathscr{D}}(\bar{x})(v).\end{aligned}$$

Similarly, $\mathbf{F}_{\mathscr{H}'}(\bar{x})(w)$ exists and equals to $\mathbf{F}_{\mathscr{D}}(\bar{x})(w)$.

Let $y = \bar{x} + \lambda v$ and $z = \bar{x} + \lambda w$ with $\lambda > 0$ such that $y, z \in \mathcal{X}$. Then, from (iv) of Lemma 1.6 and gH-Lipschitz continuity of $\mathbf{F}$ at $\bar{x}$, we have

$$\begin{aligned}&\|(\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x})) \ominus_{gH} (\mathbf{F}(z) \ominus_{gH} \mathbf{F}(\bar{x}))\|_{I(\mathbb{R})} \leq \|\mathbf{F}(y) \ominus_{gH} \mathbf{F}(z)\|_{I(\mathbb{R})} \\ \text{or, } &\frac{1}{\lambda} \odot \left(\|(\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x})) \ominus_{gH} (\mathbf{F}(z) \ominus_{gH} \mathbf{F}(\bar{x}))\|_{I(\mathbb{R})}\right) \leq \|v - w\| \text{ for all } v, w \in \mathcal{X} \\ \text{or, } &\|\mathbf{F}_{\mathscr{D}}(\bar{x})(v) \ominus_{gH} \mathbf{F}_{\mathscr{D}}(\bar{x})(w)\|_{I(\mathbb{R})} \leq \|v - w\| \text{ for all } v, w \in \mathcal{X} \\ \text{or, } &\|\mathbf{F}_{\mathscr{H}'}(\bar{x})(v) \ominus_{gH} \mathbf{F}_{\mathscr{H}'}(\bar{x})(w)\|_{I(\mathbb{R})} \leq \|v - w\| \text{ for all } v, w \in \mathcal{X}.\end{aligned}$$

□

Theorem 4.5. *Let $\mathcal{S}$ be a nonempty convex subset of $\mathcal{X}$ and the function $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ has gH-Hadamard semiderivative at every $\bar{x} \in \mathcal{S}$. If the function $\boldsymbol{F}$ is convex on $\mathcal{S}$, then*

$$\boldsymbol{F}(v) \ominus_{gH} \boldsymbol{F}(\bar{x}) \nprec \boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v - \bar{x}) \text{ for all } v \in \mathcal{S}.$$

Proof. Since $\mathbf{F}$ is convex on $\mathcal{S}$, for any $\bar{x},\ h \in \mathcal{S}$ and $\lambda,\ \lambda' \in (0, 1]$ with $\lambda + \lambda' = 1$, we have

$$\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) = \mathbf{F}(\lambda h + \lambda'\bar{x}) \preceq \lambda \odot \mathbf{F}(h) \oplus \lambda' \odot \mathbf{F}(\bar{x}).$$

Consequently,

$$\begin{aligned}\mathbf{F}(\bar{x}+\lambda(h-\bar{x}))\ominus_{gH}\mathbf{F}(\bar{x}) &\preceq (\lambda\odot\mathbf{F}(h)\oplus\lambda'\odot\mathbf{F}(\bar{x}))\ominus_{gH}\mathbf{F}(\bar{x})\\ &= \big[\min\{\lambda\underline{f}(h)-\lambda\underline{f}(\bar{x}),\lambda\overline{f}(h)-\lambda\overline{f}(\bar{x})\},\\ &\quad\ \max\{\lambda\underline{f}(h)-\lambda\underline{f}(\bar{x}),\lambda\overline{f}(h)-\lambda\overline{f}(\bar{x})\}\big]\\ &= \lambda\odot(\mathbf{F}(h)\ominus_{gH}\mathbf{F}(\bar{x})),\end{aligned}$$

which implies

$$\frac{1}{\lambda}\odot(\mathbf{F}(\bar{x}+\lambda(h-\bar{x}))\ominus_{gH}\mathbf{F}(\bar{x}))\preceq\mathbf{F}(h)\ominus_{gH}\mathbf{F}(\bar{x}).$$

From Theorem 4.3, $\mathbf{F}$ is gH-continuous. Thus, as $\lambda\to 0+$ and $h\to v$, we obtain

$$\mathbf{F}_{\mathscr{H}'}(\bar{x})(v-\bar{x})\preceq\mathbf{F}(v)\ominus_{gH}\mathbf{F}(\bar{x})\quad\text{for all } v\in\mathcal{S}. \tag{4.4}$$

If possible, let

$$\mathbf{F}(v')\ominus_{gH}\mathbf{F}(\bar{x}')\prec\mathbf{F}_{\mathscr{H}'}(\bar{x}')(v'-\bar{x}')\text{ for some } v'\in\mathcal{X}.$$

Then,

$$\mathbf{F}(v')\ominus_{gH}\mathbf{F}(\bar{x}')\prec\mathbf{F}_{\mathscr{H}'}(\bar{x}')(v'-\bar{x}'),$$

which contradicts (5.3). Hence,

$$\mathbf{F}(v)\ominus_{gH}\mathbf{F}(\bar{x})\nprec\mathbf{F}_{\mathscr{H}'}(\bar{x})(v-\bar{x})\text{ for all } v\in\mathcal{S}.$$

□

Note 9. *Converse of Theorem 4.5 is not true. For example, let us consider the IVF* $\boldsymbol{F} : \mathbb{R} \to I(\mathbb{R})$ *that is defined by*

$$\boldsymbol{F}(x) = x^2 \odot [-4, 6] = [-4x^2, 6x^2].$$

At $\bar{x} = 0 \in \mathbb{R}$*, for arbitrary* $h \in \mathbb{R}$*, we have*

$$\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) = \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) = \boldsymbol{0}.$$

Hence, $\boldsymbol{F}(v) \ominus_{gH} \boldsymbol{F}(\bar{x}) \not\preceq \boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v - \bar{x})$ *for all* $v \in \mathbb{R}$*.*

However, $\underline{f}$ *is not convex on* $\mathbb{R}$*. Thus, from Lemma 1.8,* $\boldsymbol{F}$ *is not convex on* $\mathbb{R}$*.*

Theorem 4.6. *Let* $\mathcal{S}$ *be a nonempty convex subset of* $\mathcal{X}$ *and the function* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *be a convex gH-continuous IVF on* $\mathcal{S}$*. Then,* $\boldsymbol{F}$ *is a gH-Hadamard semidifferentiable IVF.*

Proof. Since $\mathbf{F}$ is convex on $\mathcal{S}$, for any $\bar{x},\ h \in \mathcal{S}$ and $\lambda,\ \lambda' \in (0, 1]$ with $\lambda + \lambda' = 1$, we have

$$\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) = \mathbf{F}(\lambda h + \lambda' \bar{x}) \preceq \lambda \odot \mathbf{F}(h) \oplus \lambda' \odot \mathbf{F}(\bar{x}).$$

Consequently,

$$\begin{aligned}
\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) &\preceq (\lambda \odot \mathbf{F}(h) \oplus \lambda' \odot \mathbf{F}(\bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \\
&= \big[\min\{\lambda \underline{f}(h) - \lambda \underline{f}(\bar{x}), \lambda \overline{f}(h) - \lambda \overline{f}(\bar{x})\}, \\
&\qquad \max\{\lambda \underline{f}(h) - \lambda \underline{f}(\bar{x}), \lambda \overline{f}(h) - \lambda \overline{f}(\bar{x})\}\big] \\
&= \lambda \odot (\mathbf{F}(h) \ominus_{gH} \mathbf{F}(\bar{x})),
\end{aligned}$$

which implies

$$\frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})) \preceq \mathbf{F}(h) \ominus_{gH} \mathbf{F}(\bar{x}).$$

By Lemma 2.2,

$$\mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(h) \preceq \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})) \preceq \mathbf{F}(h) \ominus_{gH} \mathbf{F}(\bar{x}).$$

Due to gH-continuity of $\mathbf{F}$, we obtain

$$\mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(v) \preceq \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) \preceq \mathbf{F}(v) \ominus_{gH} \mathbf{F}(\bar{x}).$$

This implies that $\mathbf{F}$ is gH-Hadamard semidifferentiable at $\bar{x}$. □

Note 10. *The result of Theorem 4.6 is not true for convex IVFs which are not gH-continuous. For instance, let us consider the IVF $\boldsymbol{F} : [0, 2] \to I(\mathbb{R})$ that is defined by*

$$\boldsymbol{F}(x) = \begin{cases} [2,\ 8] & \text{if } x = 0, \\ \left(1 - \frac{x}{2}\right) \odot [1, 2] & \text{if } 0 < x < 2, \\ [1, 4] & \text{if } x = 2. \end{cases}$$

Then,

$$\underline{f}(x) = \begin{cases} 2 & \text{if } x = 0, \\ \left(1 - \frac{x}{2}\right) & \text{if } 0 < x < 2, \\ 1 & \text{if } x = 2. \end{cases} \text{ and } \overline{f}(x) = \begin{cases} 8 & \text{if } x = 0, \\ 2\left(1 - \frac{x}{2}\right) & \text{if } 0 < x < 2, \\ 4 & \text{if } x = 2. \end{cases}$$

Hence, $\boldsymbol{F}$ is convex as $\underline{f}$ and $\overline{f}$ are convex on $[0, 2]$ by Lemma 1.8. Also, $\boldsymbol{F}$ is not gH-continuous as $\underline{f}$ and $\overline{f}$ are not continuous on $[0, 2]$.

At $\bar{x} = 0, v = 1$ and $0 < \lambda \leq 2$, we have

$$\frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda v) \ominus_{gH} \boldsymbol{F}(\bar{x})) \quad = \quad \frac{1}{\lambda} \odot \left(\left(1 - \frac{\lambda}{2}\right) \odot [1, 2] \ominus_{gH} [2, 8] \right).$$

This implies that $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(v)$ does not exist. Hence, $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v)$ does not exists by Note 8.

Definition 4.4.2 (Semiconvex IVF)**.** *Let $\mathcal{S}$ be a convex subset of $\mathbb{R}^n$, then the IVF $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ is called* semiconvex *on $\mathcal{S}$ if there exists a monotonic increasing IVF $\boldsymbol{E} : \mathbb{R}_+ \to I(\mathbb{R}_+)$ such that $\boldsymbol{E}(\delta) \to \boldsymbol{0}$ as $\delta \to 0+$ and*

$$\boldsymbol{F}(\lambda_1 x_1 + \lambda_2 x_2) \preceq \lambda_1 \odot \boldsymbol{F}(x_1) \oplus \lambda_2 \odot \boldsymbol{F}(x_2) \oplus \lambda_1 \lambda_2 \|x - y\| \odot \boldsymbol{E}(\|x - y\|)$$

for all $x, y \in \mathcal{S}$ and $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_1 + \lambda_2 = 1$.

Lemma 4.7. *If $\boldsymbol{F} : \mathbb{R}^n \in I(\mathbb{R})$ is semiconvex on a convex subset $\mathcal{S}$ of $\mathbb{R}^n$, then there exists compact interval $\boldsymbol{A} \succeq \boldsymbol{0}$ such that $\boldsymbol{G}(x) = \boldsymbol{F}(x) \oplus \|x\|^2 \odot \boldsymbol{A}$ is convex on $\mathcal{S}$.*

Proof. Let $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$. Then, for arbitrary $\mathbf{A} \succeq \mathbf{0}$, we have

$$\begin{aligned} & \mathbf{G}(\lambda_1 x + \lambda_2 y) \\ = \; & \mathbf{F}(\lambda_1 x + \lambda_2 y) \oplus \|\lambda_1 x + \lambda_2 y)\|^2 \odot \mathbf{A} \\ \preceq \; & \lambda_1 \odot \mathbf{F}(x) \oplus \lambda_2 \odot \mathbf{F}(y) \oplus \lambda_1 \lambda_2 \|x - y\| \odot \mathbf{E}(\|x - y\|) \oplus \|\lambda_1 x + \lambda_2 y)\|^2 \odot \mathbf{A}. \end{aligned} \tag{4.5}$$

Let $\|x - y\| = \delta$ and $\mathbf{E}(\delta) = \delta \odot \mathbf{A}$. Then, $\mathbf{E}$ is a monotonic increasing IVF and $\mathbf{E}(\delta) \to \mathbf{0}$ as $\delta \to 0+$. Also, due arbitrariness of $\mathbf{A}$, (4.5) implies

$$\begin{aligned}
&\lambda_1 \odot \mathbf{F}(x) \oplus \lambda_2 \odot \mathbf{F}(y) \oplus \lambda_1\lambda_2\|x-y\| \odot \mathbf{E}\left(\|x-y\|\right) \oplus \|\lambda_1 x + \lambda_2 y)\|^2 \odot \mathbf{A} \\
&= \lambda_1 \odot \mathbf{F}(x) \oplus \lambda_2 \odot \mathbf{F}(y) \oplus \lambda_1\lambda_2\|x-y\|^2 \odot \mathbf{A} \oplus \|\lambda_1 x + \lambda_2 y)\|^2 \odot \mathbf{A} \\
&= \lambda_1 \odot \mathbf{F}(x) \oplus \lambda_2 \odot \mathbf{F}(y) \oplus \lambda_1\|x\|^2 \odot \mathbf{A} \oplus \lambda_2\|y\|^2 \odot \mathbf{A} \\
&= \lambda_1 \odot \left(\mathbf{F}(x) \oplus \|x\|^2\right) \oplus \lambda_2 \odot \left(\mathbf{F}(x) \oplus \|x\|^2\right) \\
&= \lambda_1 \odot \mathbf{G}(x) \oplus \lambda_2 \oplus \mathbf{G}(y). \qquad (4.6)
\end{aligned}$$

From (4.5) and (4.6), we have

$$\mathbf{G}(\lambda_1 x + \lambda_2 y) \preceq \lambda_1 \odot \mathbf{G}(x) \oplus \lambda_2 \oplus \mathbf{G}(y).$$

Hence, $\mathbf{G}$ is a convex IVF on $\mathcal{S}$. □

Lemma 4.8. *Let $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF and $\boldsymbol{F}$ is semiconvex on $\mathcal{S}$. Then, $\underline{f}$ and $\overline{f}$ are also semiconvex on $\mathcal{S}$.*

Proof. See Appendix D.2 □

Theorem 4.9. *Let $\mathcal{S}$ be a nonempty convex subset of $\mathcal{X}$ and the function $\boldsymbol{F}$: $\mathcal{S} \to I(\mathbb{R})$ be a semiconvex gH-continuous IVF on $\mathcal{S}$. Then, $\boldsymbol{F}$ is a gH-Hadamard semidifferentiable IVF on $\mathcal{S}$.*

Proof. Since $\mathbf{F}$ is semiconvex, then there exists compact interval $\mathbf{A} = [\underline{a}, \overline{a}] \succeq \mathbf{0}$ such that the IVF $\mathbf{G}(x) = \mathbf{F}(x) \oplus \|x\|^2 \odot \mathbf{A}$ is convex on $\mathcal{S}$, by Lemma 4.7.

Also,

$$\underline{g}(x) = \underline{f}(x) + \underline{a}\|x\|^2 \text{ and } \overline{g}(x) = \overline{f}(x) + \overline{a}\|x\|^2$$
$$\implies \underline{g}(x) - \underline{a}\|x\|^2 = \underline{f}(x) \text{ and } \overline{g}(x) - \overline{a}\|x\|^2 = \overline{f}(x).$$

Since $\underline{g}$, $\overline{g}$, and $\|x\|^2$ are convex and continuous on $\mathcal{S}$, $\underline{f}$ and $\overline{f}$ are Hadamard semidifferentiable on $\mathcal{S}$ by Theorem 4.6. Consequently, $\mathbf{F}$ is a gH-Hadamard semidifferentiable IVF on $\mathcal{S}$ by Lemma 4.1. □

Theorem 4.10. *Let $\mathcal{S}$ be a nonempty convex subset of $\mathbb{R}^n$ with int $(\mathcal{S}) \neq \emptyset$ and the function $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be a semiconvex on $\mathcal{S}$. Then,*

(i) $\boldsymbol{F}$ has gH-directional derivative at each $\bar{x} \in$ int $(\mathcal{S})$ and for all direction $v \in \mathbb{R}^n$.

(ii) For each $\bar{x} \in$ int $(\mathcal{S})$, there exists $\delta > 0$ such that $\boldsymbol{F}$ is gH-Lipschitz continuous in $\mathcal{B}(\bar{x}, \delta)$.

(iii) For all $v \in \mathbb{R}^n$, $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v)$ exists, and

$$\|\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) \ominus_{gH} \boldsymbol{F}_{\mathscr{H}'}(\bar{x})(w)\|_{I(\mathbb{R})} = \|v - w\| \text{ for all } v, w \in \mathcal{X}.$$

Proof. (i) Since $\mathbf{F}$ is a semiconvex IVF on $\mathcal{S}$, $\underline{f}$ and $\overline{f}$ are semiconvex on $\mathcal{S}$ by Lemma 4.8. Also, from Theorem 5.11 of [23], $\underline{f}$ and $\overline{f}$ has directional derivative at $\bar{x} \in$ int $(\mathcal{S})$ for all $v \in \mathbb{R}^n$. Consequently, $\mathbf{F}$ has gH-directional derivative at $\bar{x} \in$ int $(\mathcal{S})$ for all direction $v \in \mathbb{R}^n$.

(ii) Since $\underline{f}$ and $\overline{f}$ are semiconvex on $\mathcal{S}$, then for each $\bar{x} \in$ int $(\mathcal{S})$, there exists $\delta > 0$ such that $\underline{f}$ and $\overline{f}$ are Lipschitz continuous in $\mathcal{B}(\bar{x}, \delta)$ by Theorem 5.11 of [23], and from Lemma 1.8, $\mathbf{F}$ is also gH-Lipschitz continuous in $\mathcal{B}(\bar{x}, \delta)$.

(iii) Due to gH-directional differentiability of $\mathbf{F}$, $\mathbf{F}_{\mathscr{H}'}(\bar{x})(v)$ exists by Theorem 4.4 and

$$\|\mathbf{F}_{\mathscr{H}'}(\bar{x})(v) \ominus_{gH} \mathbf{F}_{\mathscr{H}'}(\bar{x})(w)\|_{I(\mathbb{R})} = \|v - w\| \text{ for all } v, w \in \mathbb{R}^n.$$

□

Theorem 4.11. *Let $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ be an IVF and $\bar{x} \in \mathbb{R}^n$. Then, for a given direction $v \in \mathbb{R}^n$, the following conditions are equivalent:*

(i) *$\boldsymbol{F}$ is gH-Hadamard semidifferentiable at $\bar{x}$;*

(ii) *there exists an IVF $\boldsymbol{G}(\bar{x})(v)$ such that for any path $f : \mathbb{R}_+ \to \mathbb{R}^n$ with $f(0) = \bar{x}$ for which $f_{\mathscr{D}}(0)(1)$ exists, we have*

$$(\boldsymbol{F} \circ f)_{\mathscr{D}}(0)(1) = \boldsymbol{G}(\bar{x})(v), \text{ where } v = f_{\mathscr{D}}(0)(1).$$

Proof. (i) $\implies$ (ii). Let $\{\lambda_n\}$ be a sequence of positive real numbers with $\lambda_n \to 0+$ and $h_n = \frac{1}{\lambda_n}(f(\lambda_n) - f(0))$ for all $n \in \mathbb{N}$. Since $f_{\mathscr{D}}(0)(1)$ exists,

$$\lim_{n\to\infty} h_n = \lim_{n\to\infty} \frac{1}{\lambda_n} \odot (f(\lambda_n) - f(0)) = f_{\mathscr{D}}(0)(1) = v. \tag{4.7}$$

If $\mathbf{F}$ is gH-Hadamard semidifferentiable at $\bar{x}$, then

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\lambda_n} \odot (\mathbf{F}(\bar{x} + \lambda_n h_n) \ominus_{gH} \mathbf{F}(\bar{x})) = \mathbf{F}_{\mathscr{H}'}(\bar{x})(v) \\
\text{or, } &\lim_{n\to\infty} \frac{1}{\lambda_n} \odot (\mathbf{F}(f(\lambda_n)) \ominus_{gH} \mathbf{F}(f(0))) = \mathbf{F}_{\mathscr{H}'}(\bar{x})(v) \\
&\left(\text{since } f(0) = \bar{x} \text{ and } h_n = \frac{1}{\lambda_n}(f(\lambda_n) - f(0))\right) \\
\text{or, } &\lim_{n\to\infty} \frac{1}{\lambda_n} \odot ((\mathbf{F} \circ f)(\lambda_n) \ominus_{gH} (\mathbf{F} \circ f)(0)) = \mathbf{F}_{\mathscr{H}'}(\bar{x})(v).
\end{aligned}$$

Hence, $\mathbf{F}$ has gH-directional derivative at $\bar{x}$ and $(\mathbf{F}\circ f)_{\mathscr{D}}(0)(1) = \mathbf{F}_{\mathscr{H}'}(\bar{x})(v)$. Taking $\mathbf{G}(\bar{x})(v) = \mathbf{F}_{\mathscr{H}'}(\bar{x})(v)$, we get the desired result.

(ii) $\implies$ (i). If possible, let $\mathbf{F}$ be not gH-Hadamard semidifferentiable at $\bar{x}$. Then, there exist two sequences $h_n \to v$ and $\lambda_n \to 0+$ such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \odot \left(\mathbf{F}(\bar{x} + \lambda_n h_n) \ominus_{gH} \mathbf{F}(\bar{x})\right) \text{ does not exist.} \tag{4.8}$$

Since $h_n \to v$ and $\lambda_n \to 0+$, for every $\epsilon > 0$ there exists a natural number N and real number a such that

$$\|h_n\| \leq a, \|h_n - v\| < \epsilon, \text{ and } \lambda_n < \epsilon/a \text{ for all } n > N. \tag{4.9}$$

By using the sequences $\{h_n\}$ and $\{\lambda_n\}$, we construct a function $f : \mathbb{R}_+ \to \mathbb{R}^n$ as follows:

$$f(\lambda) = \begin{cases} \bar{x} & \text{if } \lambda = 0 \\ \bar{x} + \lambda h_n & \text{if } \lambda_n \leq \lambda < \lambda_{n-1}, n \geq 2 \\ \bar{x} + \lambda h_1 & \text{if } \lambda \geq \lambda_1. \end{cases}$$

The function f yields $h(0) = \bar{x}$ and $f_{\mathscr{D}}(0)(1) = v$ (for details, see p. 92 of [23]). By hypothesis, $(\mathbf{F} \circ f)_{\mathscr{D}}(0)(1)$ exists and equals to $\mathbf{G}(\bar{x})(v)$, where $v = f_{\mathscr{D}}(0)(1)$. From the construction of f,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{\lambda_n} \odot \left((\mathbf{F}\circ f)(\lambda_n) \ominus_{gH} (\mathbf{F}\circ f)(0)\right) = \mathbf{G}(\bar{x})(v) \\ \text{or,}\ &\lim_{n\to\infty} \frac{1}{\lambda_n} \odot \left(\mathbf{F}(f(\lambda_n)) \ominus_{gH} \mathbf{F}(f(0))\right) = \mathbf{G}(\bar{x})(v) \\ \text{or,}\ &\lim_{n\to\infty} \frac{1}{\lambda_n} \odot \left(\mathbf{F}(\bar{x} + \lambda_n h_n) \ominus_{gH} \mathbf{F}(\bar{x})\right) = \mathbf{G}(\bar{x})(v), \end{aligned}$$

which is a contradiction to (4.8). Therefore, $\mathbf{F}$ is gH-Hadamard semidifferentiable at $\bar{x}$.

□

Theorem 4.12 (Chain rule). *Let $G : \mathbb{R}^m \to \mathbb{R}^n$ and $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ be two functions. Assume that for a point $\bar{x} \in \mathbb{R}^m$ and a direction $v \in \mathbb{R}^m$,*

(a) *$G_{\mathscr{D}}(\bar{x})(v)$ exists for any v in $\mathbb{R}^m$,*

(b) *$\boldsymbol{F}_{\mathscr{H}'}(\bar{y})(k)$ exists, where $\bar{y} = G(\bar{x})$ and $k = G_{\mathscr{D}}(\bar{x})(v)$.*

Then,

(i) *$(\boldsymbol{F} \circ G)_{\mathscr{D}}(\bar{x})(v)$ exists and*

$$(\boldsymbol{F} \circ G)_{\mathscr{D}}(\bar{x})(v) = \boldsymbol{F}_{\mathscr{H}'}(\bar{y})(k), \text{ where } \bar{y} = G(\bar{x}), k = G_{\mathscr{D}}(\bar{x})(v);$$

(ii) *if $G_{\mathscr{H}'}(\bar{x})(v)$ exists, then $(\boldsymbol{F} \circ G)_{\mathscr{H}'}(\bar{x})(v)$ exists and*

$$(\boldsymbol{F} \circ G)_{\mathscr{H}'}(\bar{x})(v) = \boldsymbol{F}_{\mathscr{H}'}(\bar{y})(\bar{k}), \text{ where } \bar{y} = G(\bar{x}), \bar{k} = G_{\mathscr{H}'}(\bar{x})(v).$$

Proof. (i) For $\lambda > 0$, define

$$Q(\lambda) = \frac{1}{\lambda} \odot (\mathbf{F}(G(\bar{x} + \lambda v)) \ominus_{gH} \mathbf{F}(G(\bar{x}))) \text{ and } \theta(\lambda) = \frac{1}{\lambda} (G(\bar{x} + \lambda v) - G(\bar{x})) . \tag{4.10}$$

Then,

$$Q(\lambda) = \frac{1}{\lambda} \odot (\mathbf{F}(G(\bar{x}) + \lambda\theta(\lambda)) \ominus_{gH} \mathbf{F}(G(\bar{x}))) . \tag{4.11}$$

Since $\theta(\lambda) \to G_{\mathscr{D}}(\bar{x})(v)$ as $\lambda \to 0+$, from (4.10), (4.11) and the hypothesis (b), we have

$$\begin{aligned}
&\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(G(\bar{x} + \lambda v)) \ominus_{gH} \mathbf{F}(G(\bar{x}))) = \mathbf{F}_{\mathscr{H}'}(\bar{y})(k) \\
\text{or, } &\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot ((\mathbf{F} \circ G)(\bar{x} + \lambda v) \ominus_{gH} (\mathbf{F} \circ G)(\bar{x}))) = \mathbf{F}_{\mathscr{H}'}(\bar{y})(k) \\
\text{or, } &(\mathbf{F} \circ G)_{\mathscr{D}}(\bar{x})(v) = \mathbf{F}_{\mathscr{H}'}(\bar{y})(k).
\end{aligned}$$

(ii) For $\lambda > 0$ and $h \in \mathbb{R}^m$, define

$$Q(\lambda, h) = \frac{1}{\lambda} \odot (\mathbf{F}(G(\bar{x} + \lambda h)) \ominus_{gH} \mathbf{F}(G(\bar{x}))) \text{ and } \Phi(\lambda, h) = \frac{1}{\lambda} (G(\bar{x} + \lambda h) - G(\bar{x})). \tag{4.12}$$

Then

$$Q(\lambda, h) = \frac{1}{\lambda} \odot (\mathbf{F}(G(\bar{x}) + \lambda \Phi(\lambda, h)) \ominus_{gH} \mathbf{F}(G(\bar{x}))). \tag{4.13}$$

Since $\Phi(\lambda, h) \to G_{\mathscr{H}'}(\bar{x})(v)$ as $\lambda \to +0$ and $h \to v$, from (4.12), (4.13), and the hypothesis (b), we have

$$\begin{aligned}
&\lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\mathbf{F}(G(\bar{x} + \lambda h)) \ominus_{gH} \mathbf{F}(G(\bar{x}))) = \mathbf{F}_{\mathscr{H}'}(\bar{y})(\bar{k}) \\
\text{or, } &\lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot ((\mathbf{F} \circ G)(\bar{x} + \lambda h) \ominus_{gH} (\mathbf{F} \circ G)(\bar{x}))) = \mathbf{F}_{\mathscr{H}'}(\bar{y})(\bar{k}) \\
\text{or, } &(\mathbf{F} \circ G)_{\mathscr{H}'}(\bar{x})(v) = \mathbf{F}_{\mathscr{H}'}(\bar{y})(\bar{k}).
\end{aligned}$$

□

Remark 4.4.3. *The weaker assumption—the existence of $G_{\mathscr{D}}(\bar{x})(v)$ and $\boldsymbol{F}_{\mathscr{D}}(\bar{y})(k)$ with $\bar{y} = G(\bar{x}), k = G_{\mathscr{D}}(\bar{x})(v)$—is not sufficient to prove Theorem 4.12. For the proof of this theorem we require a stronger assumption (b) of Theorem 4.12. This is illustrated by the following example that the composition $\boldsymbol{F} \circ G$, of a gH-Gâteaux* *differentiable IVF* $\boldsymbol{F}$ *and a Gâteaux differentiable vector-valued function* G*, is not necessarily* gH*-Gâteaux differentiable and even not* gH*-directionally differentiable in any direction* $v \neq 0$*.*

Example 4.2. *Consider the function* $\boldsymbol{F} : \mathbb{R}^2 \to I(\mathbb{R})$*, which is defined by*

$$\boldsymbol{F}(x, y) = \begin{cases} \left(\frac{x^6}{(y-x^2)^2+x^8}\right) \odot [2,\ 6] & \textit{if } (x, y) \neq (0,0) \\ \boldsymbol{0} & \textit{otherwise,} \end{cases}$$

and $G : \mathbb{R} \to \mathbb{R}^2$ *is defined by*

$$G(x) = (x, x^2).$$

It is clear that G *is Gâteaux differentiable function at* $\bar{x} = 0$ *in every direction. Note that* $\bar{y} = G(\bar{x}) = (0, 0)$ *and for any* $h \in \mathbb{R}$*, we have*

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{y} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{y})) = \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\left(\frac{\lambda^6 h_1^6}{(\lambda h_2 - \lambda^2 h_1^2)^2 + \lambda^8 h_1^8} \right) \odot [2,\ 6] \right) = \boldsymbol{0}.$$

Then, due to linearity and gH*-continuity of the limit value,* $\boldsymbol{F}$ *is also* gH*-Gâteaux differentiable IVF at* $\bar{y} = G(\bar{x})$*.*

The composition of $\boldsymbol{F}$ *and* G *is*

$$\boldsymbol{H}(x) = (\boldsymbol{F} \circ G)(x) = \begin{cases} \left(\frac{1}{x^2}\right) \odot [2,\ 6] & \textit{if } (x, y) \neq (0,0) \\ \boldsymbol{0} & \textit{otherwise.} \end{cases}$$

Since for $h \neq 0$*, the limit*

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{H}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{H}(\bar{x})) = \lim_{\lambda \to 0+} \frac{1}{\lambda^3 h} \odot [2, 6]$$

does not exist, $\boldsymbol{H} = \boldsymbol{F} \circ G$ *is not* gH*-directionally differentiable IVF at* $G(\bar{x}) = 0$ *in any direction* $h \neq 0$*.*

Theorem 4.13. *Let I be a finite set of indices and $\boldsymbol{F}_i : \mathcal{X} \to I(\mathbb{R})$ be family of IVFs such that $\boldsymbol{F}_{i_{\mathscr{H}'}}(\bar{x})(h)$ exists and for all $x \in \mathcal{X}$, $\boldsymbol{F}(x) = \max_{i \in I} \boldsymbol{F}_i(x)$. Then,*

$$\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(h) = \max_{i \in \mathcal{A}(\bar{x})} \boldsymbol{F}_{i_{\mathscr{H}'}}(\bar{x})(h), \text{ where } \mathcal{A}(\bar{x}) = \{i : \boldsymbol{F}_i(\bar{x}) = \boldsymbol{F}(\bar{x})\}.$$

Proof. Let $\bar{x} \in \mathcal{X}$, and $d \in \mathcal{X}$ be such that $\bar{x} + \lambda d \in \mathcal{X}$ for all $\lambda > 0$. Then,

$$\begin{aligned}
&\mathbf{F}_i(\bar{x} + \lambda d) \preceq \mathbf{F}(\bar{x} + \lambda d) \text{ for all } i \in I \\
\text{or, } &\mathbf{F}_i(\bar{x} + \lambda d) \ominus_{gH} \mathbf{F}(\bar{x}) \preceq \mathbf{F}(\bar{x} + \lambda d) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } i \in I \\
\text{or, } &\mathbf{F}_i(\bar{x} + \lambda d) \ominus_{gH} \mathbf{F}_i(\bar{x}) \preceq \mathbf{F}(\bar{x} + \lambda d) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for each } i \in \mathcal{A}(\bar{x}) \\
\text{or, } &\lim_{\substack{\lambda \to 0+ \\ d \to h}} \frac{1}{\lambda} \odot (\mathbf{F}_i(\bar{x} + \lambda d) \ominus_{gH} \mathbf{F}_i(\bar{x})) \preceq \lim_{\substack{\lambda \to 0+ \\ d \to h}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda d) \ominus_{gH} \mathbf{F}(\bar{x})) \\
\text{or, } &\max_{i \in \mathcal{A}(\bar{x})} \mathbf{F}_{i_{\mathscr{H}'}}(\bar{x})(h) \preceq \mathbf{F}_{\mathscr{H}'}(\bar{x})(h). \qquad (4.14)
\end{aligned}$$

To prove the reverse inequality, we claim that there exists a neighbourhood $\mathcal{N}(\bar{x})$ such that $\mathcal{A}(x) \subset \mathcal{A}(\bar{x})$ for all $x \in \mathcal{N}(\bar{x})$. Assume contrarily that there exists a sequence $\{x_k\}$ in $\mathcal{X}$ with $x_k \to \bar{x}$ such that $\mathcal{A}(x_k) \not\subset \mathcal{A}(\bar{x})$. We can choose $i_k \in \mathcal{A}(x_k)$ but $i_k \notin \mathcal{A}(\bar{x})$. Since $\mathcal{A}(x_k)$ is closed, $i_k \to \bar{i} \in \mathcal{A}(x_k)$. By gH-continuity of $\mathbf{F}$ we have

$$\mathbf{F}_{\bar{i}}(x_k) = \mathbf{F}(x_k) \implies \mathbf{F}_{\bar{i}}(\bar{x}) = \mathbf{F}(\bar{x}),$$

which is a contradiction to $i_k \notin \mathcal{A}(\bar{x})$. Thus, $\mathcal{A}(x) \subset \mathcal{A}(\bar{x})$ for all $x \in \mathcal{N}(\bar{x})$.

Let us choose a sequence $\{\lambda_k\}, \lambda_k \to 0+$ such $\bar{x} + \lambda_k d \in \mathcal{N}(\bar{x})$ for all $d \in \mathcal{X}$. Then,

$$\begin{aligned}
&\mathbf{F}_i(\bar{x}) \preceq \mathbf{F}(\bar{x}) \text{ for all } i \in I \\
&\text{or, } \mathbf{F}(\bar{x} + \lambda_k d) \ominus_{gH} \mathbf{F}(\bar{x}) \preceq \mathbf{F}(\bar{x} + \lambda_k d) \ominus_{gH} \mathbf{F}_i(\bar{x}) \text{ for all } i \in \mathcal{A}(\bar{x}) \\
&\text{or, } \mathbf{F}(\bar{x} + \lambda_k d) \ominus_{gH} \mathbf{F}(\bar{x}) \preceq \mathbf{F}_i(\bar{x} + \lambda_k d) \ominus_{gH} \mathbf{F}_i(\bar{x}) \text{ for all } i \in \mathcal{A}(\bar{x} + \lambda_k d) \\
&\text{or, } \lim_{\substack{k\to\infty \\ d\to h}} \frac{1}{\lambda_k} \odot (\mathbf{F}(\bar{x} + \lambda_k d) \ominus_{gH} \mathbf{F}(\bar{x})) \preceq \lim_{\substack{k\to\infty \\ d\to h}} \frac{1}{\lambda_k} \odot (\mathbf{F}_i(\bar{x} + \lambda_k d) \ominus_{gH} \mathbf{F}_i(\bar{x})) \\
&\text{or, } \mathbf{F}_{\mathscr{H}'}(\bar{x})(h) \preceq \max_{i\in\mathcal{A}(\bar{x})} \mathbf{F}_{i_{\mathscr{H}'}}(\bar{x})(h). \qquad (4.15)
\end{aligned}$$

From (4.14) and (4.15), we obtain

$$\mathbf{F}_{\mathscr{H}'}(\bar{x})(h) = \max_{i\in\mathcal{A}(\bar{x})} \mathbf{F}_{i_{\mathscr{H}'}}(\bar{x})(h).$$

□

4.5 Characterization of Efficient Solutions

In this section, we present the characterization of efficient solutions for IOPs based on the properties of gH-Hadamard semidifferentiable IVFs.

Theorem 4.14 (Necessary condition for efficient points). *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$, $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF, and $\bar{x} \in \mathcal{S}$ be an efficient point of the IOP (1.5). If the function $\boldsymbol{F}$ has a gH-Hadamard semiderivative at $\bar{x}$ in the direction $(v - \bar{x})$ for any $x \in \mathcal{S}$, then*

$$\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v - \bar{x}) \nprec \boldsymbol{0} \text{ for all } v \in \mathcal{S}. \qquad (4.16)$$

Proof. Let $\bar{x} \in \mathcal{S}$ be an efficient point of the IVF $\mathbf{F}$. For any point $x \in \mathcal{X}$, the gH-Hadamard semiderivative of $\mathbf{F}$ at $\bar{x}$ in the direction $(v - \bar{x})$ is given by

$$\mathbf{F}_{\mathscr{H}'}(\bar{x})(v-\bar{x}) = \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})\big). \tag{4.17}$$

Since the point $\bar{x}$ is an efficient point of the function $\mathbf{F}$, for any $h \in \mathcal{X}$ and $\lambda > 0$ with $\bar{x} + \lambda(h - \bar{x}) \in \mathcal{S}$, we get

$$\mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \nprec \mathbf{F}(\bar{x})$$
$$\text{or, } \mathbf{F}(\bar{x} + \lambda(h - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0}.$$

This implies that

$$\max\left\{\underline{f}(\bar{x} + \lambda(h - \bar{x}) - \underline{f}(\bar{x}), \overline{f}(\bar{x} + \lambda(h - \bar{x}) - \overline{f}(\bar{x})\right\} \geq 0.$$

Since $\lambda > 0$, we obtain

$$\lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \max\left\{\underline{f}(\bar{x} + \lambda(h - \bar{x}) - \underline{f}(\bar{x}), \overline{f}(\bar{x} + \lambda(h - \bar{x}) - \overline{f}(\bar{x})\right\} \geq 0$$
$$\text{or,} \quad \max\left\{\underline{f}_{\mathscr{D}}(\bar{x})(v), \overline{f}_{\mathscr{D}}(\bar{x})(v)\right\} \geq 0. \tag{4.18}$$

By Lemma 4.1,

$$\mathbf{F}_{\mathscr{H}'}(\bar{x})(v) = \left[\min\left\{\underline{f}_{\mathscr{H}'}(\bar{x})(v), \overline{f}_{\mathscr{H}'}(\bar{x})(v)\right\}, \max\left\{\underline{f}_{\mathscr{H}'}(\bar{x})(v), \overline{f}_{\mathscr{H}'}(\bar{x})(v)\right\}\right] \tag{4.19}$$

From (4.17), (4.18) and (4.19), we have

$$\mathbf{F}_{\mathscr{H}'}(\bar{x})(v - \bar{x}) \nprec \mathbf{0} \text{ for all } v \in \mathcal{S}.$$

□

Note 11. *One may think that in Theorem 4.14, instead of using the "not better strict dominance" relation of compact intervals (Definition 1.4.3), we may use "not strict dominance" relation of compact intervals (Definition 1.4.2). However, this assumption is not sufficient. For instance, consider* $\mathcal{X} = \mathbb{R}$, $\mathcal{S} = [-1, 7]$, *and the IVF* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *that is defined by*

$$\boldsymbol{F}(x) = [x^2 - 4x + 4, x^2 + 5].$$

For any $x, h \in \mathcal{S}$ *such that* $x + \lambda h \in \mathcal{S}$, *we have the following for any* $v \in \mathcal{X}$:

$$\lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\boldsymbol{F}(x + \lambda h) \ominus_{gH} \boldsymbol{F}(x)) = 2v \odot [x - 2, x].$$

Hence, $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) = 2v \odot [\bar{x} - 2, \bar{x}]$. *Note that* $\bar{x} = 0$ *is an efficient point of the IOP (1.5) because*

$$\boldsymbol{F}(y) \nprec \boldsymbol{F}(\bar{x}) \text{ for all } y \in \mathcal{S}.$$

However, $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) \prec \boldsymbol{0}$ *for all* $v > 0$.

Theorem 4.15 (Sufficient condition for efficient points)**.** *Let* $\mathcal{S}$ *be a nonempty convex subset of* $\mathcal{X}$ *and* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *be a convex IVF. If the function* $\boldsymbol{F}$ *has a gH-Hadamard semderivative at* $\bar{x} \in \mathcal{S}$ *in the direction* $(v - \bar{x})$ *with*

$$\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v - \bar{x}) \nprec \boldsymbol{0} \text{ for all } v \in \mathcal{X}, \tag{4.20}$$

then $\bar{x}$ *is an efficient point of the IOP (1.5).*

Proof. Suppose at $\bar{x} \in \mathcal{S}$, for each direction $(v - \bar{x})$, we have

$$\mathbf{F}_{\mathscr{H}'}(\bar{x})(v - \bar{x}) \nprec \mathbf{0} \text{ for all } v \in \mathcal{X}.$$

If possible, let $\bar{x}$ be not an efficient point of $\mathbf{F}$. Then, there exists at least one $y \in \mathcal{S}$ such that $\mathbf{F}(y) \prec \mathbf{F}(\bar{x})$. Therefore, for any $\lambda \in (0, 1]$, we have

$$\begin{aligned}
&\lambda \odot \mathbf{F}(y) \prec \lambda \odot \mathbf{F}(\bar{x}) \\
\text{or, } &\lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(\bar{x}) \prec \lambda \odot \mathbf{F}(\bar{x}) \oplus \lambda' \odot \mathbf{F}(\bar{x}), \text{ where } \lambda' = 1 - \lambda \\
\text{or, } &\lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(\bar{x}) \prec (\lambda + \lambda') \odot \mathbf{F}(\bar{x}) = \mathbf{F}(\bar{x}).
\end{aligned}$$

Due to the convexity of $\mathbf{F}$ on $\mathcal{S}$, we have

$$\begin{aligned}
&\mathbf{F}(\bar{x} + \lambda(y - \bar{x})) = \mathbf{F}(\lambda y + \lambda'\bar{x}) \preceq \lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(\bar{x}) \prec \mathbf{F}(\bar{x}) \\
\text{or, } &\mathbf{F}(\bar{x} + \lambda(y - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \prec \mathbf{0} \\
\text{or, } &\lim_{\substack{\lambda \to 0+ \\ y \to v}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda(y - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})) \preceq \mathbf{0} \\
\text{or, } &\mathbf{F}_{\mathscr{H}'}(\bar{x})(v - \bar{x}) \preceq \mathbf{0}.
\end{aligned} \tag{4.21}$$

From (4.21), we have the following two possibilities.

(a) If $\mathbf{F}_{\mathscr{H}'}(\bar{x})(v - \bar{x}) = \mathbf{0}$, then $\mathbf{F}_{\mathscr{D}}(\bar{x})(v - \bar{x}) = \mathbf{0}$. By Lemma 4.1 and Theorem 4.4 we have

$$\underline{f}_{\mathscr{D}}(\bar{x})(v - \bar{x}) = 0 \text{ and } \overline{f}_{\mathscr{D}}(\bar{x})(v - \bar{x}) = 0. \tag{4.22}$$

Due to Lemma 1.8, $\underline{f}$ and $\overline{f}$ are convex on $\mathcal{S}$. From (4.22), we observe that $\bar{x}$ is a minimum point of $\underline{f}$ and $\overline{f}$. Consequently, $\bar{x}$ is an efficient point of $\mathbf{F}$. This is contradictory to the assumption that $\bar{x}$ is not efficient point of $\mathbf{F}$.

(b) If $\mathbf{F}_{\mathscr{H}'}(\bar{x})(v-\bar{x}) \prec \mathbf{0}$, then this contradicts the assumption that $\mathbf{F}_{\mathscr{H}'}(\bar{x})(v-\bar{x}) \nprec \mathbf{0}$ for all $v \in \mathcal{X}$.

Hence, $\bar{x}$ is the efficient point of the IOP (1.5). □

Note 12. *Converse of Theorem 4.15 is not true. For example, consider $\mathcal{X} = \mathbb{R}$, $\mathcal{S} = [-1, 2]$, and the IVF $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ that is defined by*

$$\boldsymbol{F}(x) = [4x^2 - 4x + 1, 2x^2 + 75].$$

Since $\underline{f}$ and $\overline{f}$ are convex and Lipschitz continuous on $\mathcal{S}$, $\boldsymbol{F}$ is convex and gH-Lipschitz continuous IVF on $\mathcal{S}$ by Lemma 1.8 and Lemma 3.3. Also, from Theorem 4.4, $\boldsymbol{F}$ has gH-Hadamard semiderivative at $\bar{x} = 0 \in \mathcal{S}$ in every direction $v \in \mathcal{X}$. Since

$$\begin{aligned} & \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) \\ = & \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\left[4(\lambda h)^2 - 4(\lambda h) + 1, 2(\lambda h)^2 + 75\right] \ominus_{gH} [1, 75] \right) \\ = & \; v \odot [-4, 0], \end{aligned}$$

$\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) = v \odot [-4, 0]$ for all $v \in \mathcal{X}$. From Figure 5.1, it is clear that $\bar{x} = 0$ is an efficient solution of the IOP (1.5).

However, for all $v > 0$ we have $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) \prec \boldsymbol{0}$.

Example 4.3. *In this example, we show that the condition '$\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v - \bar{x}) \nprec \boldsymbol{0}$' for a gH-Hadamard differentiable IVF in Theorem 4.15 is sufficient for convex IOPs but not sufficient for nonconvex IOPs. Let us consider $\mathcal{X} = \mathbb{R}$, $\mathcal{X} = \mathcal{S}$ and the IVF $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ that is defined by*

$$\boldsymbol{F}(x) = x^2 \odot [-5, -2].$$

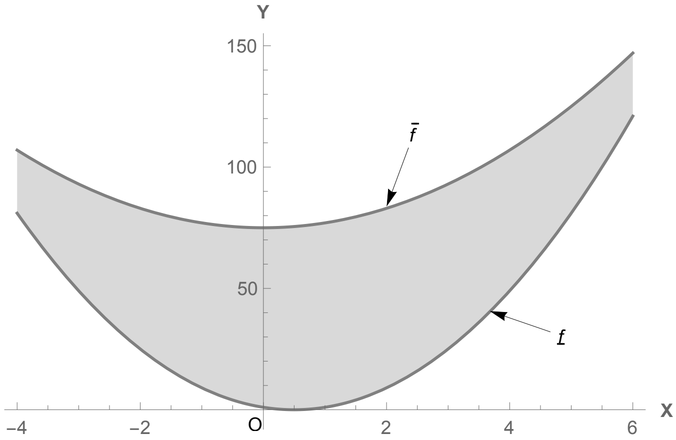

FIGURE 4.1: The IVF **F** of Note 16

Since $\underline{f}$ is not convex on $\mathcal{S}$, $\boldsymbol{F}$ is not a convex IVF on $\mathcal{S}$ by Lemma 1.8. At $\bar{x} = 0$, for arbitrary $h \in \mathcal{S}$, we have

$$\lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) = \boldsymbol{0}.$$

Hence, $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) = \boldsymbol{0}$. Note that $\boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) \nprec \boldsymbol{0}$, but $\bar{x}$ is not an efficient point of the IOP (1.5) because

$$\boldsymbol{F}(y) \prec \boldsymbol{F}(\bar{x}) \text{ for all } y \in \mathcal{S} \text{ and } y \neq \bar{x}.$$

Theorem 4.16 (Extended Karush-Kuhn-Tucker sufficient condition for efficient points)**.** *Let $\mathcal{S}$ be a nonempty convex subset of $\mathcal{X}$; $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ and $\boldsymbol{G}_i : \mathcal{S} \to I(\mathbb{R})$ on $\mathcal{S}$, $i = 1, 2, \ldots, m$ be interval-valued gH-Hadamard differentiable convex function on $\mathcal{S}$. Suppose $\bar{x} \in \mathcal{S}$ be a feasible point of the IOP:*

$$\left.\begin{array}{rl} \min & \boldsymbol{F}(x) \\ \text{subject to} & \boldsymbol{G}_i(\bar{x}) \preceq \boldsymbol{0}, \;\; i = 1, 2, \cdots, m \\ & x \in \mathcal{S}. \end{array}\right\} \quad (4.23)$$

If there exist real constants $u_1, u_2, \ldots, u_m$ for which

$$\begin{cases} \boldsymbol{F}_{\mathscr{H}'}(\bar{x})(v) \oplus \sum_{i=1}^{m} u_i \boldsymbol{G}_{i\mathscr{H}'}(\bar{x})(v) \nprec \boldsymbol{0} \text{ for all } v \in \mathcal{S}, \\ u_i \odot \boldsymbol{G}_i(\bar{x}) = \boldsymbol{0}, \ i = 1, 2, \cdots, m, \ \text{and} \\ u_i \geq 0, \ i = 1, 2, \cdots, m. \end{cases}$$

then $\bar{x}$ is an efficient point of the IOP (4.23) under consideration.

Proof. By the hypothesis, for every $v \in \mathcal{S}$ satisfying $\mathbf{G}_i(v) \preceq \mathbf{0}$ for all $i = 1, 2, \cdots, m$. we have

$$\begin{aligned} & \mathbf{F}_{\mathscr{H}}(\bar{x})(v - \bar{x}) \oplus \sum_{i=1}^{m} u_i \mathbf{G}_{i\mathscr{H}}(\bar{x})(v - \bar{x}) \nprec \mathbf{0} \\ \Longrightarrow \quad & (\mathbf{F}(v) \ominus_{gH} \mathbf{F}(\bar{x})) \oplus \left(\sum_{i=1}^{m} u_i \left(\mathbf{G}_i(v) \ominus_{gH} \mathbf{G}_i(\bar{x}) \right) \right) \nprec \mathbf{0} \text{ from (4.4) of Theorem 4.5} \\ \Longrightarrow \quad & (\mathbf{F}(v) \ominus_{gH} \mathbf{F}(\bar{x})) \oplus \left(\sum_{i=1}^{m} u_i \left(\mathbf{G}_i(v) \right) \right) \nprec \mathbf{0} \\ \Longrightarrow \quad & \mathbf{F}(v) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0} \text{ since } \mathbf{G}_i(v) \preceq \mathbf{0} \\ \Longrightarrow \quad & \mathbf{F}(v) \nprec \mathbf{F}(\bar{x}). \end{aligned}$$

Hence, $\bar{x}$ is an efficient point of IOP (4.23). □

4.6 Concluding Remarks

In this chapter, a concept of gH-Hadamard semiderivative for IVFs has been studied. It has been observed that gH-continuous is necessary condition and gH-Lipschitz continuity is sufficient condition for existence of gH-Hadamard semiderivative. It has been proved that a gH-Hadamard differentiable IVF follows the chain rule and max rule. In addition, by using this derivative, the optimality condition to find the efficient solutions of IOPs has been derived. Moreover, for constraint IOPs, it has been proved extended KKT sufficient condition to characterize the efficient solutions by using gH-Hadamard semiderivative.

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