Chapter 3

Generalized Hukuhara Clarke Derivative of Interval-valued Functions and its Application in Interval Optimization

3.1 Introduction

Clarke derivative [22] is often applied in the nonsmooth analysis where the functions do not have a unique linear approximation. Advances of nonsmooth analysis [17, 66] shows the essential need of this derivative to handle nondifferentiable functions, especially in the absence of convexity. The topics of optimization [42], control theory [42], variational method [3], etc. are wide application areas of Clarke derivative.

Convexity plays an important role in optimization theory because a local optimum becomes a global one and a necessary optimality condition becomes a sufficient condition. However, practically, not all optimization problems can be formulated as convex problems. Thus, various generalizations of convexity [3, 10] have been developed that enjoy the local-global property of optima and necessary-sufficient property of optimality conditions.

3.2 Motivation

Despite of many attempts to develop a calculus of IVFs, the existing ideas are not adequate to find solutions to nonsmooth IOPs as well as nonconvex IOPs. Although Bhurjee and Panda [8] proposed some optimality conditions and duality results for nonsmooth convex IOPs by converting them to a real-valued problem through a parametric representation of IVFs, one needs the explicit expression of the IVF for the parametric representation, which is often practically difficult. In the second chapter, the concepts of directional derivative and Gâteaux derivative for IVFs are derived to solve nonsmooth IOPs. However, these derivatives are not applicable for some nonsmooth IOPs (see Remark 3.4.4 of this chapter).

3.3 Contributions

In this chapter, the notions of upper and lower gH-Clarke derivative, gH-pseudoconvex, and quasiconvex for IVFs are proposed. To define the concept of gH-Clarke derivatives, the concepts of limit superior, limit inferior, and sublinear interval-valued functions are studied in the sequel. The upper gH-Clarke derivative of a gH-Lipschitz IVF is observed to be a sublinear IVF. It is found that every gH-Lipschitz continuous IVF is upper gH-Clarke differentiable. Further, for a convex and gH-Lipschitz IVF, it is shown that the upper gH-Clarke derivative coincides with the gH-directional derivative. With the help of the studied gH-pseudoconvex, quasiconvex and gH-Lipschitz IVFs, we present a few results on characterizing efficient solutions of an IOP with upper gH-Clarke and gH-Fréchet differentiable IVFs. Importantly, we report that at an efficient point of an IVF on a star-shaped set, the upper gH-Clarke derivative does not dominate zero. The entire study is supported by suitable illustrative examples.

Neat contributions of this chapter are as follows:

(i) For a convex IVF and gH-Lipschitz continuous IVF, it is proved that the gH-directional derivative coincides with the upper gH-Clarke derivative.

(ii) For a gH-pseudoconvex and gH-Lipschitz continuous IVF, it is shown that a point is an efficient solution of IOP if and only if zero is not strictly dominated by the upper gH-Clarke derivative.

3.4 Clarke Derivative of Interval-valued Functions

In this section, extended concepts of the gH-directional derivative, namely upper and lower gH-Clarke derivatives, for IVFs are given. A short discussion of the required notions of limit superior and sublinearity for IVFs is provided.

Definition 3.4.1 (Supremum and limit superior of an IVF)**.** *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$ and $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF. Then, the* supremum *of $\boldsymbol{F}$ over $\mathcal{S}$ is defined by*

$$\sup_{\mathcal{S}} \boldsymbol{F} = \left[\sup_{\mathcal{S}} \underline{f},\ \sup_{\mathcal{S}} \overline{f}\right],$$

where $\sup\limits_{\mathcal{S}} \underline{f} = \sup\left\{\underline{f}(x) : x \in \mathcal{S}\right\}$ *and* $\sup\limits_{\mathcal{S}} \overline{f} = \sup\left\{\overline{f}(x) : x \in \mathcal{S}\right\}$.

The limit superior *of* $\boldsymbol{F}$ *at a limit point* $\bar{x}$ *in* $\mathcal{S}$ *is defined by*

$$\limsup_{x\to\bar{x}} \boldsymbol{F}(x) = \left[\limsup_{x\to\bar{x}} \underline{f}(x),\ \limsup_{x\to\bar{x}} \overline{f}(x)\right],$$

where $\limsup\limits_{x\to\bar{x}} \underline{f}(x) = \lim\limits_{\delta\to 0}\left(\sup\limits_{x\in\overline{\mathcal{B}}(\bar{x},\delta)\cap\mathcal{S}} \underline{f}(x)\right)$ *and* $\limsup\limits_{x\to\bar{x}} \overline{f}(x) = \lim\limits_{\delta\to 0}\left(\sup\limits_{x\in\overline{\mathcal{B}}(\bar{x},\delta)\cap\mathcal{S}} \overline{f}(x)\right)$.

Definition 3.4.2 (Infimum and limit inferior of an IVF)**.** *Let* $\mathcal{S}$ *be a nonempty subset of* $\mathcal{X}$ *and* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *be an IVF. Then, the* infimum *of* $\boldsymbol{F}$ *is defined by*

$$\inf_{\mathcal{S}} \boldsymbol{F} = \left[\inf_{x\in\mathcal{S}} \underline{f},\ \inf_{x\in\mathcal{S}} \overline{f}\right],$$

where $\inf\limits_{\mathcal{S}} \underline{f} = \inf\left\{\underline{f}(x) : x \in \mathcal{S}\right\}$ *and* $\inf\limits_{\mathcal{S}} \overline{f} = \inf\left\{\overline{f}(x) : x \in \mathcal{S}\right\}$.

The limit inferior *of* $\boldsymbol{F}$ *at a limit point* $\bar{x}$ *in* $\mathcal{S}$ *is defined by*

$$\liminf_{x\to\bar{x}} \boldsymbol{F}(x) = \left[\liminf_{x\to\bar{x}} \underline{f}(x),\ \liminf_{x\to\bar{x}} \overline{f}(x)\right],$$

where $\liminf\limits_{x\to\bar{x}} \underline{f}(x) = \lim\limits_{\delta\to 0}\left(\inf\limits_{x\in\overline{\mathcal{B}}(\bar{x},\delta)\cap\mathcal{S}} \underline{f}(x)\right)$ *and* $\liminf\limits_{x\to\bar{x}} \overline{f}(x) = \lim\limits_{\delta\to 0}\left(\inf\limits_{x\in\overline{\mathcal{B}}(\bar{x},\delta)\cap\mathcal{S}} \overline{f}(x)\right)$.

Lemma 3.1. *Let* $\mathcal{S}$ *be a nonempty subset of* $\mathcal{X}$ *and* $\boldsymbol{F}$, $\boldsymbol{G} : \mathcal{S} \to I(\mathbb{R})$ *be two IVFs. Then, at any* $\bar{x} \in \mathcal{S}$*, the following properties are true:*

(i) $\limsup\limits_{x\to\bar{x}} (\boldsymbol{F}(x) \oplus \boldsymbol{G}(x)) \preceq \limsup\limits_{x\to\bar{x}} \boldsymbol{F}(x) \oplus \limsup\limits_{x\to\bar{x}} \boldsymbol{G}(x)$,

(ii) $\limsup\limits_{x\to\bar{x}} (\lambda \odot \boldsymbol{F}(x)) = \lambda \odot \limsup\limits_{x\to\bar{x}} \boldsymbol{F}(x)$ *for all* $\lambda \geq 0$*, and*

(iii) $\left\|\limsup\limits_{x\to\bar{x}} \boldsymbol{F}(x)\right\|_{I(\mathbb{R})} \leq \limsup\limits_{x\to\bar{x}} \|\boldsymbol{F}(x)\|_{I(\mathbb{R})}$.

Proof. See Appendix C.1. $\square$

Definition 3.4.3 (Upper gH-Clarke derivative)**.** *Let $\boldsymbol{F}$ be an IVF on a nonempty subset $\mathcal{S}$ of $\mathcal{X}$. For $\bar{x} \in \mathcal{S}$ and $h \in \mathcal{X}$, if the limit superior*

$$\limsup_{\substack{x \to \bar{x} \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \big(\boldsymbol{F}(x+\lambda h) \ominus_{gH} \boldsymbol{F}(x) \big) = \lim_{\delta \to 0} \left(\sup_{x \in \overline{\mathcal{B}}(\bar{x},\delta) \cap \mathcal{S},\ \lambda \in (0,\delta)} \frac{1}{\lambda} \odot \big(\boldsymbol{F}(x+\lambda h) \ominus_{gH} \boldsymbol{F}(x) \big) \right)$$

exists, then the limit superior value is called upper gH-Clarke derivative *of $\boldsymbol{F}$ at $\bar{x}$ in the direction h, and it is denoted by $\boldsymbol{F}_{\mathscr{C}}(\bar{x})(h)$. If this limit superior exists for all $h \in \mathcal{X}$, then $\boldsymbol{F}$ is said to be* upper gH-Clarke differentiable *at $\bar{x}$.*

Definition 3.4.4 (Lower gH-Clarke derivative)**.** *Let $\boldsymbol{F}$ be an IVF on a nonempty subset $\mathcal{S}$ of $\mathcal{X}$. For $\bar{x} \in \mathcal{S}$ and $h \in \mathcal{X}$, if the limit inferior*

$$\liminf_{\substack{x \to \bar{x} \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \big(\boldsymbol{F}(x+\lambda h) \ominus_{gH} \boldsymbol{F}(x) \big) = \lim_{\delta \to 0} \left(\inf_{x \in \overline{\mathcal{B}}(\bar{x},\delta) \cap \mathcal{S},\ \lambda \in (0,\delta)} \frac{1}{\lambda} \odot \big(\boldsymbol{F}(x+\lambda h) \ominus_{gH} \boldsymbol{F}(x) \big) \right)$$

exists, then the limit inferior value is called lower gH-Clarke derivative *of $\boldsymbol{F}$ at $\bar{x}$ in the direction h. If this limit inferior exists for all $h \in \mathcal{X}$, then $\boldsymbol{F}$ is said to be* lower gH-Clarke differentiable *at $\bar{x}$.*

Remark 3.4.1. *Conventionally, for real valued-functions, the terminologies Clarke derivative [17, 42] and upper Clarke derivative [20] are interchangeably used. In fact, the upper Clarke derivative is usually referred to as Clarke derivative. However, in order to avoid any confusion, we prefix upper and lower with the Clarke derivative corresponding to the values given by limit superior and limit inferior, respectively. In addition, throughout the article, we use the notation $\boldsymbol{F}_{\mathscr{C}}$ to refer the upper gH-Clarke derivative of an IVF $\boldsymbol{F}$.*

Remark 3.4.2. *It is clear that $\boldsymbol{F}$ is lower gH-Clark differentiable at $\bar{x}$ if and only if $(\ominus \boldsymbol{F})$ is upper gH-Clark differentiable at $\bar{x}$. That is why we shall deal only with the upper gH-Clark differentiability.*

Example 3.1. *In this example, we calculate the upper gH-Clarke derivative at $\bar{x} = 0$ for the following IVF:*

$$\boldsymbol{F}(x) = \left[\sin(2|x|), e^{\frac{|x|}{4}}\right], x \in \mathbb{R}.$$

Here $\mathcal{X}$ is the Euclidean space $\mathbb{R}$, and $\mathcal{S} = \mathcal{X}$. At $\bar{x} = 0$ in $\mathcal{S}$, for any $h \in \mathbb{R}$, we have

$$\begin{aligned}
& \limsup_{\substack{x \to 0 \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\boldsymbol{F}(x + \lambda h) \ominus_{gH} \boldsymbol{F}(x)) \\
= \ & \limsup_{\substack{x \to 0 \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \left[\min\left\{ \sin(2|x + \lambda h|) - \sin(2|x|), e^{\frac{|x+\lambda h|}{4}} - e^{\frac{|x|}{4}} \right\}, \right. \\
& \qquad \left. \max\left\{ \sin(2|x + \lambda h|) - \sin(2|x|), e^{\frac{|x+\lambda h|}{4}} - e^{\frac{|x|}{4}} \right\} \right] \\
\preceq \ & \limsup_{\substack{x \to 0 \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \left[\min\left\{ 2|\lambda h|, e^{\frac{|x+\lambda h|}{4}} - e^{\frac{|x|}{4}} \right\}, \max\left\{ 2|\lambda h|, e^{\frac{|x+\lambda h|}{4}} - e^{\frac{|x|}{4}} \right\} \right] \\
& \qquad (\textit{since } |\sin x - \sin y| \leq |x - y|) \\
= \ & \left[\min\left\{ 2|h|, \tfrac{|h|}{4} \right\}, \ \max\left\{ 2|h|, \tfrac{|h|}{4} \right\} \right], \ \textit{by Theorem 3.42 of [42]} \\
= \ & |h| \odot \left[\tfrac{1}{4}, 2 \right]. \qquad (3.1)
\end{aligned}$$

Further, taking $x = \lambda h$, we obtain

$$
\begin{aligned}
&\limsup_{\substack{x \to 0 \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \left(\boldsymbol{F}(x + \lambda h) \ominus_{gH} \boldsymbol{F}(x) \right) \\
= \; & \limsup_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\left[\sin(2|x + \lambda h|), e^{\frac{|x+\lambda h|}{4}} \right] \ominus_{gH} \left[\sin(2|h|), e^{\frac{|h|}{4}} \right] \right) \\
\succeq \; & \limsup_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\left[\sin(4\lambda|h|), e^{\frac{2\lambda|h|}{4}} \right] \ominus_{gH} \left[\sin(2\lambda|h|), e^{\frac{\lambda|h|}{4}} \right] \right) \\
= \; & \limsup_{\lambda \to 0+} \frac{1}{\lambda} \odot \left[\min\left\{ \sin(4\lambda|h|) - \sin(2\lambda|h|), e^{\frac{2\lambda|h|}{4}} - e^{\frac{\lambda|h|}{4}} \right\}, \right. \\
& \qquad\qquad\qquad \left. \max\left\{ \sin(4\lambda|h|) - \sin(2\lambda|h|), e^{\frac{2\lambda|h|}{4}} - e^{\frac{\lambda|h|}{4}} \right\} \right] \\
= \; & \left[\min\left\{ 2|h|, \tfrac{|h|}{4} \right\}, \ \max\left\{ 2|h|, \tfrac{|h|}{4} \right\} \right] \\
= \; & |h| \odot \left[\tfrac{1}{4}, 2 \right].
\end{aligned}
\tag{3.2}
$$

From the inequalities (3.1) and (3.2), we have $\boldsymbol{F}_{\mathscr{C}}(\bar{x})(h) = |h| \odot \left[\frac{1}{4}, 2\right]$.

Lemma 3.2. *If $\underline{f}$ and $\overline{f}$ are upper Clarke differentiable at $\bar{x} \in \mathcal{S} \subseteq \mathcal{X}$, then the IVF $\boldsymbol{F}$ is upper gH-Clarke differentiable at $\bar{x} \in \mathcal{S}$.*

Proof. See Appendix C.2. □

Lemma 3.3. *Let $\boldsymbol{F}$ be an IVF on a nonempty subset $\mathcal{S}$ of $\mathcal{X}$.*

(i) *$\boldsymbol{F}$ is gH-continuous on $\mathcal{S}$ if and only if $\underline{f}$ and $\overline{f}$ are continuous on $\mathcal{S}$.*

(ii) *$\boldsymbol{F}$ is gH-Lipschitz continuous on $\mathcal{S}$ if and only if $\underline{f}$ and $\overline{f}$ are Lipschitz continuous on $\mathcal{S}$.*

(iii) *If $\boldsymbol{F}$ is a gH-Lipschitz continuous on $\mathcal{S}$, then $\boldsymbol{F}$ is gH-continuous on $\mathcal{S}$.*

Proof. See Appendix C.3. □

A consequence of Lemma 3.3 is that gH-continuity and gH-Lipschitz continuity of IVFs can be defined classically, i.e., without the concept of gH-difference. Then, the prefix gH- in continuity and Lipschitz continuity could be omitted.

Remark 3.4.3. *Converse of (iii) of Lemma 3.3 is not true. For example, consider $\mathcal{X}$ as the Euclidean space $\mathbb{R}$, $\mathcal{S} = [0, 10]$, and the IVF $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$, which is defined by*

$$\boldsymbol{F}(x) = \sqrt{x} \odot [2, 5].$$

Since $\underline{f}(x) = 2\sqrt{x}$ and $\overline{f}(x) = 5\sqrt{x}$ are continuous on $\mathcal{S}$, $\boldsymbol{F}$ is gH-continuous on $\mathcal{S}$ by (i) of Lemma 3.3. If $\boldsymbol{F}$ is gH-Lipschitz continuous on $\mathcal{S}$, then by (ii) of Lemma 3.3, $\underline{f}$ and $\overline{f}$ are Lipschitz continuous on $\mathcal{S}$, which is not true. Consequently, $\boldsymbol{F}$ is not gH-Lipschitz continuous on $\mathcal{S}$.

The following theorem extends the well-known result from [42] for Lipschitz continuous functions to gH-Lipschitz continuous IVFs with the help of Lemma 3.2.

Theorem 3.4. *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$ with $\bar{x} \in int(\mathcal{S})$ and $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be a gH-Lipschitz continuous IVF at $\bar{x}$ with a Lipschitz constant K'. Then, $\boldsymbol{F}$ is upper gH-Clarke differentiable at $\bar{x}$ and*

$$\|\boldsymbol{F}_{\mathscr{C}}(\bar{x})(h)\|_{I(\mathbb{R})} \leq K'\|h\| \text{ for all } h \in \mathcal{X}.$$

Proof. Since $\mathbf{F}$ is gH-Lipschitz continuous on $\mathcal{S}$, for any $h \in \mathcal{X}$, we get for $\lambda > 0$ that

$$\left\| \frac{1}{\lambda} \odot (\mathbf{F}(x + \lambda h) \ominus_{gH} \mathbf{F}(x)) \right\|_{I(\mathbb{R})} \leq \frac{1}{\lambda} K' \|x + \lambda h - x\| = K'\|h\|, \tag{3.3}$$

if x and λ are sufficiently close to $\bar{x}$ and 0, respectively. From inequality (3.3) we have

$$\left|\frac{1}{\lambda}\left(\underline{f}(x+\lambda h)-\underline{f}(x)\right)\right| \leq K'\|h\| \text{ and } \left|\frac{1}{\lambda}\left(\overline{f}(x+\lambda h)-\overline{f}(x)\right)\right| \leq K'\|h\|.$$

Hence, by (ii) of Lemma 3.3, the limit superior $\underline{f}_{\mathscr{C}}(\bar{x})(h)$ and $\overline{f}_{\mathscr{C}}(\bar{x})(h)$ exist at $\bar{x}$ (cf. p. 69 of [42]). By Lemma 3.2, the limit superior $\mathbf{F}_{\mathscr{C}}(\bar{x})(h)$ exists.

Furthermore, by gH-Lipschitz continuity of $\mathbf{F}$ on $\mathcal{S}$, we have the following for all $h \in \mathcal{X}$:

$$\begin{aligned}
\|\mathbf{F}_{\mathscr{C}}(\bar{x})(h)\|_{I(\mathbb{R})} &= \left\|\limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \frac{1}{\lambda}\odot\left(\mathbf{F}(x+\lambda h)\ominus_{gH}\mathbf{F}(x)\right)\right\|_{I(\mathbb{R})}\\
&\leq \limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \left\|\frac{1}{\lambda}\odot\left(\mathbf{F}(x+\lambda h)\ominus_{gH}\mathbf{F}(x)\right)\right\|_{I(\mathbb{R})} \text{ by Lemma 3.1}\\
&\leq K'\|h\| \text{ by (3.3).}
\end{aligned}$$

□

For convex and gH-Lipschitz continuous IVFs, upper gH-Clarke derivative and gH-directional derivative coincide as the next theorem states.

Theorem 3.5. *Let $\mathcal{X}$ be convex, and the IVF $\boldsymbol{F}: \mathcal{X} \to I(\mathbb{R})$ be convex on $\mathcal{X}$ and gH-Lipschitz continuous at some $\bar{x} \in \mathcal{X}$. Then, the upper gH-Clarke derivative of $\boldsymbol{F}$ at $\bar{x}$ coincides with the gH-directional derivative of $\boldsymbol{F}$ at $\bar{x}$ in the direction $h \in \mathcal{X}$.*

Proof. Since $\mathbf{F}$ is a convex IVF on $\mathcal{X}$, we get by Theorem 2.3 that the gH-directional derivative of $\mathbf{F}$ exists at $\bar{x} \in \mathcal{X}$ in every direction h. Also, as $\mathbf{F}$ is gH-Lipschitz continuous at $\bar{x}$, from Theorem 3.4, we get that the upper gH-Clarke derivative of $\mathbf{F}$ exists at any $\bar{x} \in \mathcal{X}$ in every direction h. Thus, by Definitions 2.4.1 and 3.4.3, we observe that

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(h) \preceq \mathbf{F}_{\mathscr{C}}(\bar{x})(h) \text{ for all } h. \tag{3.4}$$

For the proof of the reverse inequality, we write

$$\begin{aligned} \mathbf{F}_{\mathscr{C}}(\bar{x})(h) &= \limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \frac{1}{\lambda} \odot (\mathbf{F}(x+\lambda h) \ominus_{gH} \mathbf{F}(x)) \\ &= \lim_{\substack{\delta\to 0+\\ \epsilon\to 0+}} \sup_{\|x-\bar{x}\|<\delta} \sup_{0<\lambda<\epsilon} \frac{1}{\lambda} \odot (\mathbf{F}(x+\lambda h) \ominus_{gH} \mathbf{F}(x)). \end{aligned}$$

Since $\mathbf{F}$ is convex on $\mathcal{X}$, Lemma 2.1 leads to the equality

$$\mathbf{F}_{\mathscr{C}}(\bar{x})(h) = \lim_{\substack{\delta\to 0+\\ \epsilon\to 0+}} \sup_{\|x-\bar{x}\|<\delta} \frac{1}{\epsilon} \odot (\mathbf{F}(x+\epsilon h) \ominus_{gH} \mathbf{F}(x)),$$

and for an arbitrary $\alpha > 0$,

$$\mathbf{F}_{\mathscr{C}}(\bar{x})(h) = \lim_{\epsilon\to 0+} \sup_{\|x-\bar{x}\|<\epsilon\alpha} \frac{1}{\epsilon} \odot (\mathbf{F}(x+\epsilon h) \ominus_{gH} \mathbf{F}(x)).$$

Because of the gH-Lipschitz continuity of $\mathbf{F}$ at $\bar{x}$, we have for sufficiently small $\epsilon > 0$ and $\|x-\bar{x}\| < \epsilon\alpha$ that

$$\begin{aligned} & \left\|\tfrac{1}{\epsilon} \odot (\mathbf{F}(x+\epsilon h) \ominus_{gH} \mathbf{F}(x)) \ominus_{gH} \tfrac{1}{\epsilon} \odot (\mathbf{F}(\bar{x}+\epsilon h) \ominus_{gH} \mathbf{F}(\bar{x}))\right\|_{I(\mathbb{R})} \\ \leq\ & \left\|\tfrac{1}{\epsilon} \odot (\mathbf{F}(x+\epsilon h) \ominus_{gH} \mathbf{F}(\bar{x}+\epsilon h))\right\|_{I(\mathbb{R})} + \left\|\tfrac{1}{\epsilon} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}))\right\|_{I(\mathbb{R})} \\ & \text{by (iv) of Lemma 1.6} \\ \leq\ & \tfrac{1}{\epsilon} K'\|x-\bar{x}\| + \tfrac{1}{\epsilon} K'\|x-\bar{x}\|, \text{ where } K' \text{ is the Lipschitz constant of } \mathbf{F} \text{ at } \bar{x} \in \mathcal{X} \\ \leq\ & 2K'\alpha. \end{aligned}$$

Then, by (iii) of Lemma 1.6, we have

$$\begin{aligned} \mathbf{F}_{\mathscr{C}}(\bar{x})(h) &\preceq \lim_{\epsilon\to 0+} \frac{1}{\epsilon} \odot (\mathbf{F}(x+\epsilon h) \ominus_{gH} \mathbf{F}(x)) \oplus [2K\alpha, 2K\alpha] \\ &= \mathbf{F}_{\mathscr{D}}(\bar{x})(h) \oplus [2K'\alpha, 2K'\alpha]. \end{aligned}$$

Since $\alpha > 0$ is chosen arbitrarily, we obtain

$$\mathbf{F}_{\mathscr{C}}(\bar{x})(h) \preceq \mathbf{F}_{\mathscr{D}}(\bar{x})(h) \text{ for all } h. \tag{3.5}$$

From (3.4) and (3.5), we get

$$\mathbf{F}_{\mathscr{C}}(\bar{x})(h) = \mathbf{F}_{\mathscr{D}}(\bar{x})(h).$$

□

Remark 3.4.4. *An upper gH-Clarke diffrentiable IVF* $\boldsymbol{F}$ *may not be gH-directional differentiable. For example, take* $\mathcal{X}$ *as the Euclidean space* $\mathbb{R}$, $\mathcal{S} = \mathbb{R}$ *and the IVF* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$, *which is defined by*

$$\boldsymbol{F}(x) = \begin{cases} \left[\frac{2\sin^2 x}{x}, 2|x| + \frac{\sin^2 2x}{3x}\right], & \text{if } x \neq 0 \\ [2, 7], & \text{if } x = 0. \end{cases}$$

For all nonzero h *in* $\mathcal{X}$, *we have*

$$\begin{aligned} & \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(x + \lambda h) \ominus_{gH} \boldsymbol{F}(x)) \\ = & \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\left[\tfrac{2\sin^2(x+\lambda h)}{x+\lambda h}, 2|x + \lambda h| + \tfrac{\sin^2 2(x+\lambda h)}{3(x+\lambda h)}\right] \ominus_{gH} \left[\tfrac{2\sin^2 x}{x}, 2|x| + \tfrac{\sin^2 2x}{3x}\right] \right) \\ = & \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \Bigg[\min \left\{ \tfrac{2\sin^2(x+\lambda h)}{x+\lambda h} - \tfrac{2\sin^2 x}{x}, (2|x + \lambda h| - 2|x|) + \left(\tfrac{\sin^2 2(x+\lambda h)}{3(x+\lambda h)} - \tfrac{\sin^2 2x}{3x} \right) \right\}, \\ & \qquad \max \left\{ \tfrac{2\sin^2(x+\lambda h)}{x+\lambda h} - \tfrac{2\sin^2 x}{x}, (2|x + \lambda h| - 2|h|) + \left(\tfrac{\sin^2 2(x+\lambda h)}{3(x+\lambda h)} - \tfrac{\sin^2 2x}{3x} \right) \right\} \Bigg] \\ = & \Bigg[\min \left\{ \tfrac{2xh\sin 2x - 2h\sin^2 x}{x^2}, 2|h| + \tfrac{1}{3} \left(\tfrac{2xh\sin 4x - h\sin^2 x}{x^2} \right) \right\}, \\ & \max \left\{ \tfrac{2xh\sin 2x - 2h\sin^2 x}{x^2}, 2|h| + \tfrac{1}{3} \left(\tfrac{2xh\sin 4x - h\sin^2 x}{x^2} \right) \right\} \Bigg] \end{aligned}$$

Thus,

$$\boldsymbol{F}_{\mathscr{C}}(0)(h) = \left[2h,\ 2|h| + \tfrac{4}{3}h\right].$$

However, the limit

$$\begin{aligned} &\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\lambda h) \ominus_{gH} \boldsymbol{F}(0)) \\ = &\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\left[\tfrac{2\sin^2 \lambda h}{\lambda h},\ 2|\lambda h| + \tfrac{\sin^2 2\lambda h}{6\lambda h}\right] \ominus_{gH} [2, 7]\right) \end{aligned}$$

does not exist. Consequently, $\boldsymbol{G}$ *is not gH-directional differentiable at* $\bar{x} = 0$.

Remark 3.4.5. *Let* $\mathcal{S}$ *be a nonempty subset of* $\mathcal{X}$ *and* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *has gH-directional derivative at* $\bar{x} \in \mathcal{S}$*. Then,* $\boldsymbol{F}$ *is not necessarily upper gH-Clarke differentiable at* $\bar{x} \in \mathcal{X}$*. For instance, take* $\mathcal{X}$ *as the Euclidean space* $\mathbb{R}^2$*,* $\mathcal{S} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0, x_2 \geq 0\}$ *and the IVF* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$*, which is defined by*

$$\boldsymbol{F}(x_1, x_2) = \begin{cases} x_1^2 \left(1 + \frac{1}{x_2}\right) \odot [3,\ 8] & \textit{if } x = (x_1, x_2) \neq (0, 0) \\ \boldsymbol{0} & \textit{otherwise.} \end{cases}$$

Then, at $\bar{x} = (0, 0)$ *and* $h = (h_1, h_2) \in \mathcal{X}$ *such that for sufficiently small* $\lambda > 0$ *so that* $\bar{x} + \lambda h \in \mathcal{S}$*, we have*

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) \quad = \quad \begin{cases} \frac{h_1^2}{h_2} \odot [3,\ 8] & \textit{if } h_2 \neq 0 \\ \boldsymbol{0} & \textit{otherwise.} \end{cases}$$

Hence, $\boldsymbol{F}$ *has a gH-directional derivative at* $\bar{x}$ *in every direction* $h \in \mathcal{X}$*.*
Again, for $x = (x_1, x_2) \in \mathcal{S}$ *and* $h = (h_1, h_2) \in \mathcal{X}$*, we have*

$$\begin{aligned} &\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(x + \lambda h) \ominus_{gH} \boldsymbol{F}(x)) \\ = &\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left((x_1 + \lambda h_1)^2 \left(1 + \frac{1}{x_2 + \lambda h_2}\right) \odot [3, 8] \ominus_{gH} x_1^2 \left(1 + \frac{1}{x_2}\right) \odot [3,\ 8]\right) \end{aligned}$$

$$= \left[\min\left\{3\left(2x_1h_1 + \frac{2x_1h_1}{x_2} - \frac{x_1^2h_2}{x_2^2}\right),\ 8\left(2x_1h_1 + \frac{2x_1h_1}{x_2} - \frac{x_1^2h_2}{x_2^2}\right)\right\},\right.$$
$$\left.\max\left\{3\left(2x_1h_1 + \frac{2x_1h_1}{x_2} - \frac{x_1^2h_2}{x_2^2}\right),\ 8\left(2x_1h_1 + \frac{2x_1h_1}{x_2} - \frac{x_1^2h_2}{x_2^2}\right)\right\}\right].$$

Along $x_2 = mx_1$, *where* m *is any real number,*

$$\lim_{\substack{x\to 0\\ \lambda\to 0+}} \frac{1}{\lambda} \odot (\boldsymbol{F}(x+\lambda h) \ominus_{gH} \boldsymbol{F}(x))$$
$$=\left[\min\left\{3\left(\frac{2h_1}{m} - \frac{h_2}{m^2}\right),\ 8\left(\frac{2h_1}{m} - \frac{h_2}{m^2}\right)\right\}, \max\left\{3\left(\frac{2h_1}{m} - \frac{h_2}{m^2}\right),\ 8\left(\frac{2h_1}{m} - \frac{h_2}{m^2}\right)\right\}\right].$$

Hence, for $h_2 > 0$, $\left(\frac{2h_1}{m} - \frac{h_2}{m^2}\right) \to -\infty$ *as* $m \to 0$. *Consequently,*

$$\limsup_{\substack{x\to 0\\ \lambda\to 0+}} \frac{1}{\lambda} \odot (\boldsymbol{F}(x+\lambda h) \ominus_{gH} \boldsymbol{F}(x)) \quad \textit{does not exist.}$$

This implies that $\boldsymbol{F}$ *has no upper* gH*-Clarke derivative at* $\bar{x} \in \mathcal{S}$.

Definition 3.4.5 (Sublinear IVF)**.** *Let* $\mathcal{S}$ *be a linear subspace of* $\mathcal{X}$. *An IVF* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *is said to be* sublinear *on* $\mathcal{S}$ *if*

(i) $\boldsymbol{F}(\lambda x) = \lambda \odot \boldsymbol{F}(x)$ *for all* $x \in \mathcal{S}$ *and for all* $\lambda \geq 0$, *and*

(ii) $\boldsymbol{F}(x+y) \nsucceq \boldsymbol{F}(x) \oplus \boldsymbol{F}(y)$ *for all* $x,\ y \in \mathcal{S}$.

Example 3.2. *Let* $\mathcal{X}$ *be the Euclidean space* $\mathbb{R}^2$ *and* $\mathcal{S} = \mathcal{X}$. *Then, the IVF* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *that is defined by*

$$\boldsymbol{G}(x_1, x_2) = |x_1| \odot [3, 8] \oplus |x_2| \odot [7, 11]$$

is sublinear on $\mathcal{S}$. *The reason is as follows.*

Here, $\boldsymbol{F}(x_1, x_2) = \left[\underline{f}(x_1, x_2), \overline{f}(x_1, x_2)\right] = \left[3|x_1| + 7|x_2|, 8|x_1| + 11|x_2|\right]$.

For all $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{S}$ *and* $\lambda \geq 0$, *we have*

(i) $\boldsymbol{F}(\lambda x) = \left[3|\lambda x_1| + 7|\lambda x_2|, 8|\lambda x_1| + 11|\lambda x_2|\right] = \lambda \odot \left[\underline{f}(x), \overline{f}(x)\right] = \lambda \odot \boldsymbol{F}(x)$.

(ii) *for all* $x_1, x_2, y_1, y_2 \in \mathbb{R}$, *we have*

$$\begin{aligned} & 3|x_1 + y_1| + 7|x_2 + y_2| \leq 3|x_1| + 3|y_1| + 7|x_2| + 7|y_2| \\ \Longrightarrow \quad & \underline{f}(x + y) \leq \underline{f}(x) + \underline{f}(y), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} & 8|x_1 + y_1| + 11|x_2 + y_2| \leq 8|x_1| + 8|y_1| + 11|x_2| + 11|y_2| \\ \Longrightarrow \quad & \overline{f}(x + y) \leq \overline{f}(x) + \overline{f}(y), \end{aligned} \tag{3.7}$$

From the inequalities (3.6) and (3.7), we obtain

$$\begin{aligned} & \left[\underline{f}(x+y), \overline{f}(x+y)\right] \preceq \left[\underline{f}(x) + \underline{f}(y), \overline{f}(x) + \overline{f}(y)\right] \\ \Longrightarrow \quad & \boldsymbol{F}(x+y) \preceq \boldsymbol{F}(x) \oplus \boldsymbol{F}(y) \\ \Longrightarrow \quad & \boldsymbol{F}(x+y) \not\succ \boldsymbol{F}(x) \oplus \boldsymbol{F}(y) \end{aligned}$$

Hence, $\boldsymbol{F}$ *is a sublinear IVF on* $\mathcal{S}$.

Note 6. *Let* Q *be a real positive definite matrix of order* $n \times n$ *and* $\mathcal{S}$ *be a linear subspace of* $\mathcal{X}$. *Consider the IVF* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$, *which is defined by*

$$\boldsymbol{F}(x) = \left(\sqrt{x^T Q x}\right) \odot \boldsymbol{C}, \text{ where } \boldsymbol{C} \not\prec \boldsymbol{0}.$$

Then, $\boldsymbol{F}$ *is a sublinear IVF on* $\mathcal{S}$. *The reason is as follows.*

The function $\boldsymbol{F}(x)$ *can be written as* $g(x) \odot \boldsymbol{C}$, *where* $g(x) = \sqrt{x^T Q x}$. *By Example 1.2.3 of [36],* g *satisfies the following conditions:*

(a) *for* $\lambda \geq 0$ *and* $x \in \mathcal{S}$,

$$g(\lambda x) = \lambda g(x), \tag{3.8}$$

and

(b) *for all* $x, y \in \mathcal{S}$

$$g(x + y) \leq g(x) + g(y). \tag{3.9}$$

From (3.8), we have

$$g(\lambda x) \odot \boldsymbol{C} = \lambda g(x) \odot \boldsymbol{C}, \ \ \textit{or}, \ \ \boldsymbol{F}(\lambda x) = \lambda \odot \boldsymbol{F}(x).$$

Since $\boldsymbol{C} \nprec \boldsymbol{0}$, *from (3.9) and Lemma 1.7, we obtain*

$$\begin{aligned}
& g(x + y) \odot \boldsymbol{C} \nsucc (g(x) + g(y)) \odot \boldsymbol{C} \\
\implies \ & (g(x + y)) \odot \boldsymbol{C} \nsucc g(x) \odot \boldsymbol{C} \oplus g(y) \odot \boldsymbol{C} \ \textit{since } g(x) \textit{ and } g(y) \textit{ are nonnegative} \\
\implies \ & \boldsymbol{F}(x + y) \nsucc \boldsymbol{F}(x) \oplus \boldsymbol{F}(y).
\end{aligned}$$

Hence, $\boldsymbol{F}$ *is a sublinear IVF on* $\mathcal{S}$.

Note 7. *Let* $\mathcal{S}$ *be a linear subspace of* $\mathcal{X}$ *and* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *be a convex IVF on* $\mathcal{S}$ *such that for all* $x \in \mathcal{S}$,

$$\boldsymbol{F}(\alpha x) = \alpha \odot \boldsymbol{F}(x) \ \ \textit{for every } \alpha \geq 0. \tag{3.10}$$

Then, $\boldsymbol{F}$ is a sublinear IVF on $\mathcal{S}$. The reason is as follows.

For $x, y \in \mathcal{S}$ and $\lambda_1, \lambda_2 > 0$, we have

$$\begin{aligned}\boldsymbol{F}(\lambda_1 x + \lambda_2 y) &= \boldsymbol{F}\Big(\lambda\Big(\tfrac{\lambda_1}{\lambda}x + \tfrac{\lambda_2}{\lambda}y\Big)\Big), \text{ where } \lambda = \lambda_1 + \lambda_2\\ &= \lambda \odot \boldsymbol{F}\Big(\tfrac{\lambda_1}{\lambda}x + \tfrac{\lambda_2}{\lambda}y\Big) \text{ by (3.10)}\\ &\preceq \lambda_1 \odot \boldsymbol{F}(x) \oplus \lambda_2 \odot \boldsymbol{F}(y) \text{ by the convexity of } \boldsymbol{F}.\end{aligned}$$

Taking $\lambda_1 = \lambda_2 = 1$, we obtain

$$\boldsymbol{F}(x+y) \preceq \boldsymbol{F}(x) \oplus \boldsymbol{F}(y) \text{ for all } x, y \in \mathcal{S}.$$

Hence, $\boldsymbol{F}$ is a sublinear IVF on $\mathcal{S}$.

Remark 3.4.6. *A sublinear IVF may not be convex. For instance, take $\mathcal{X}$ as the Euclidean space $\mathbb{R}$, $\mathcal{S} = \mathcal{X}$ and the IVF $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ that is given by*

$$\boldsymbol{F}(x) = |x| \odot [-3, 2].$$

Clearly, by Example 3.2, $\boldsymbol{F}$ is a sublinear IVF on $\mathcal{S}$. However, $\underline{f}(x) = -3|x|$ is not convex on $\mathcal{S}$. Therefore, by Lemma 1.8, $\boldsymbol{F}$ is not a convex IVF on $\mathcal{S}$.

Theorem 3.6. *Let $\mathcal{S}$ be a subset of $\mathcal{X}$ with nonempty interior, and let $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF that is upper gH-Clarke differentiable at $\bar{x} \in int(\mathcal{S})$. Then, the upper gH-Clarke derivative $\boldsymbol{F}_{\mathscr{C}}(\bar{x})$ of $\boldsymbol{F}$ is a sublinear IVF on $\mathcal{S}$.*

Proof. For an arbitrary $h \in \mathcal{S}$ and $\alpha \geq 0$, we have

$$\begin{aligned}\limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \frac{1}{\lambda} \odot (\mathbf{F}(x+\lambda\alpha h) \ominus_{gH} \mathbf{F}(x)) &= \alpha \odot (\limsup_{\substack{x\to\bar{x}\\ \lambda\to 0+}} \tfrac{1}{\lambda\alpha} \odot (\mathbf{F}(x+\lambda\alpha h) \ominus_{gH} \mathbf{F}(x)))\\ &= \alpha \odot \mathbf{F}_{\mathscr{C}}(\bar{x})(h).\end{aligned}$$

Thus, $\mathbf{F}_{\mathscr{C}}(\bar{x})(\alpha h) = \alpha \odot \mathbf{F}_{\mathscr{C}}(\bar{x})(h)$.

Next, for all $h_1, h_2 \in \mathcal{S}$, we get

$$\begin{aligned}
& \mathbf{F}_{\mathscr{C}}(\bar{x})(h_1 + h_2) \\
= \; & \limsup_{\substack{x \to \bar{x} \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \left(\mathbf{F}(x + \lambda(h_1 + h_2)) \ominus_{gH} \mathbf{F}(x)\right) \\
\not\succ \; & \limsup_{\substack{x \to \bar{x} \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \Big[\mathbf{F}(x + \lambda h_1 + \lambda h_2) \ominus_{gH} \mathbf{F}(x + \lambda h_2) \oplus \mathbf{F}(x + \lambda h_2) \ominus_{gH} \mathbf{F}(x)\Big], \\
& \text{by (iii) of Lemma 1.5} \\
= \; & \mathbf{F}_{\mathscr{C}}(\bar{x})(h_1) \oplus \mathbf{F}_{\mathscr{C}}(\bar{x})(h_2).
\end{aligned}$$

Hence, $\mathbf{F}_{\mathscr{C}}(\bar{x})$ is a sublinear IVF on $\mathcal{S}$. □

3.5 Pseudoconvex and Quasiconvex Interval-valued Functions

Here we study the concepts of gH-pseudoconvex and quasiconvex IVFs. In the case of gH-pseudoconvex IVF, a necessary condition for the existence of an efficient solution becomes a sufficient condition. Further, we show that the class of gH-pseudoconvex IVFs includes the class of all differentiable convex IVFs and is included in the class of all differentiable quasiconvex IVFs.

Definition 3.5.1. *(gH-pseudoconvex IVF). Let $\mathcal{S}$ be a nonempty convex subset of $\mathcal{X}$ and $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF which has gH-directional derivative $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(y - \bar{x})$ at some $\bar{x} \in \mathcal{S}$ in every direction $y - \bar{x}$, $y \in \mathcal{S}$. Then, $\boldsymbol{F}$ is called gH-pseudoconvex at $\bar{x}$ if for all $y \in \mathcal{S}$,*

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(y - \bar{x}) \not\prec \boldsymbol{0} \implies \boldsymbol{F}(y) \ominus_{gH} \boldsymbol{F}(\bar{x}) \not\prec \boldsymbol{0}.$$

Remark 3.5.1. *Definition 3.5.1 does not assume that the intervals in the range set* $\boldsymbol{F}(\mathcal{S})$ *of* $\boldsymbol{F}$ *are comparable. This makes the definition nonrestrictive. Furthermore, in the degenerate case, i.e., in the case of* $\underline{f} = \overline{f}$ *on* $\mathcal{S}$*, the definition reduces to*

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(y - \bar{x}) \geq 0 \implies \boldsymbol{F}(y) \geq \boldsymbol{F}(\bar{x}),$$

which is the conventional pseudoconvexity. Thus, Definition 3.5.1 is a true and nonrestrictive generalization of the conventional pseudoconvexity.

Remark 3.5.2. *Notice that one may define* gH*-pseudoconvexity of an IVF* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *at* $\bar{x} \in \mathcal{S}$ *by*

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(y - \bar{x}) \succeq \boldsymbol{0} \implies \boldsymbol{F}(y) \ominus_{gH} \boldsymbol{F}(\bar{x}) \succeq \boldsymbol{0} \text{ for all } y \in \mathcal{S}. \tag{3.11}$$

$$\text{i.e.,} \quad \boldsymbol{F}_{\mathscr{D}}(\bar{x})(y - \bar{x}) \succeq \boldsymbol{0} \implies \boldsymbol{F}(y) \succeq \boldsymbol{F}(\bar{x}) \text{ for all } y \in \mathcal{S}.$$

However, this definition is very restrictive as it is applicable only for an IVF $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *for which the intervals in the range set* $\boldsymbol{F}(\mathcal{S})$ *are comparable. Thus, Definition 3.5.1 of* gH*-pseudoconvex IVF is more general than that in the sense of* (3.11)*, and the results that are derived using Definition 3.5.1 are true for* gH*-pseudoconvex IVFs in the sense of* (3.11)*.*

The next theorem analyzes the relation between convex and gH-pseudoconvex interval-valued functions.

Theorem 3.7. *Let* $\mathcal{S}$ *be a nonempty convex subset of* $\mathcal{X}$*, and* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *be a convex IVF on* $\mathcal{S}$ *which has* gH*-directional derivative at some* $\bar{x} \in \mathcal{S}$ *in every direction* $y - \bar{x}, y \in \mathcal{S}$*. Then,* $\boldsymbol{F}$ *is* gH*-pseudoconvex at* $\bar{x}$*.*

Proof. Since $\mathbf{F}$ is a convex IVF on $\mathcal{S}$, for every $\bar{x}, y \in \mathcal{S}$ and $\lambda, \lambda' \in (0,1)$ with $\lambda + \lambda' = 1$, we have

$$\mathbf{F}(\bar{x} + \lambda(y - \bar{x})) = \mathbf{F}(\lambda y + \lambda'\bar{x}) \preceq \lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(\bar{x}).$$

Consequently,

$$\begin{aligned}
\mathbf{F}(\bar{x} + \lambda(y - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) &\preceq (\lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(\bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \\
&= \big[\min\{\lambda \underline{f}(y) + \lambda' \underline{f}(\bar{x}) - \underline{f}(\bar{x}), \lambda \overline{f}(y) + \lambda' \overline{f}(\bar{x}) - \overline{f}(\bar{x})\}, \\
&\qquad \max\{\lambda \underline{f}(y) + \lambda' \underline{f}(\bar{x}) - \underline{f}(\bar{x}), \lambda \overline{f}(y) + \lambda' \overline{f}(\bar{x}) - \overline{f}(\bar{x})\}\big] \\
&= \lambda \odot (\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x})),
\end{aligned}$$

which implies

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(y - \bar{x}) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } y \in \mathcal{S}. \tag{3.12}$$

From the inequality (iv) of Lemma 1.5 and (3.12), we obtain

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(y - \bar{x}) \nprec \mathbf{0} \implies \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0} \text{ for all } y \in \mathcal{S}.$$

Hence, $\mathbf{F}$ is gH-pseudoconvex at $\bar{x} \in \mathcal{S}$. □

In the following example, we show an IVF which is gH-pseudoconvex but not convex.

Example 3.3. *Consider $\mathcal{X}$ as the Euclidean space $\mathbb{R}$, $\mathcal{S} = \mathcal{X}$ and the IVF $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$, which is defined by*

$$\begin{aligned}
\boldsymbol{F}(x) &= x \odot [1, 2] \oplus x^3 \odot [5, 8] \\
&= \begin{cases} [x + 5x^3, 2x + 8x^3], & \textit{for } x \geq 0 \\ [2x + 8x^3, x + 5x^3], & \textit{for } x < 0. \end{cases}
\end{aligned}$$

The IVF $\boldsymbol{F}$ *is depicted in Figure 3.1 by the shaded region.*

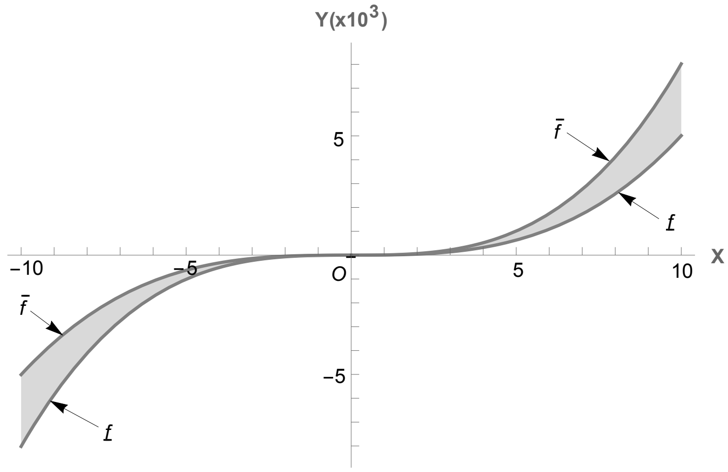

FIGURE 3.1: The IVF **F** of Example 3.3

Note that for any $y \in \mathcal{S}$ *and* $\bar{x} = 0$, *we have*

$$\begin{aligned} \boldsymbol{F}_{\mathscr{D}}(\bar{x})(y - \bar{x}) &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\boldsymbol{F}(\bar{x} + \lambda(y - \bar{x})) \ominus_{gH} \boldsymbol{F}(\bar{x})\right) \\ &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\lambda y \odot [1, 2] \oplus \lambda^3 y^3 \odot [5, 8]\right) \\ &= y \odot [1, 2]. \end{aligned}$$

Again

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(y - \bar{x}) \nprec 0 \implies y \odot [1, 2] \nprec 0 \implies y \geq 0.$$

Further, for $y \geq 0$,

$$\boldsymbol{F}(y) \ominus_{gH} \boldsymbol{F}(0) = y \odot [1, 2] \oplus y^3 \odot [5, 8] \nprec \boldsymbol{0}.$$

Hence, $\boldsymbol{F}$ *is gH-pseudoconvex at* $\bar{x} = 0$. *However, the functions* $\underline{f}$ *and* $\overline{f}$ *are not convex on* $\mathcal{S}$ *(see Figure 3.1). Therefore, the IVF* $\boldsymbol{F}$ *is not convex on* $\mathcal{S}$.

Definition 3.5.2 (Quasiconvex IVF)**.** *Let $\mathcal{S}$ be a nonempty convex subset of $\mathcal{X}$. An IVF $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ is said to be* quasiconvex *on $\mathcal{S}$ if for all $x, y \in \mathcal{S}$*

either $\boldsymbol{F}(x) \not\prec \boldsymbol{F}(\lambda x + \lambda' y)$ or $\boldsymbol{F}(y) \not\prec \boldsymbol{F}(\lambda x + \lambda' y)$, where $\lambda, \lambda' \in [0, 1]$ with $\lambda + \lambda' = 1$.

Remark 3.5.3. *It is a convention to prefix gH in a nomenclature if we use gH-difference in its definition. As there is no role of gH-difference in Definition 3.5.2, we have not prefixed gH with* quasiconvex.

Remark 3.5.4. *Definition 3.5.2 does not assume that $\boldsymbol{F}(x)$ and $\boldsymbol{F}(y)$ are comparable for all $x, y \in \mathcal{S}$. This makes the definition nonrestrictive. In addition, in the degenerate case, i.e., in the case of $\underline{f} = \overline{f}$ on $\mathcal{S}$, the definition reduces to the conventional quasiconvexity. Thus, Definition 3.5.2 is a true and nonrestrictive generalization of the conventional quasiconvexity. In the example of Remark 3.5.5 the IVF $\boldsymbol{F}$ is quasiconvex but there are a few points in the domain of $\boldsymbol{F}$ for which $\boldsymbol{F}(x)$ and $\boldsymbol{F}(y)$ are not comparable.*

Using the definition of quasiconvex IVFs, we obtain the following relation between gH-pseudoconvex and quasiconvex IVFs.

Theorem 3.8. *Let $\mathcal{S}$ be a nonempty convex subset of $\mathcal{X}$, and $\boldsymbol{F}$ be an IVF defined on an open superset of $\mathcal{S}$. If $\boldsymbol{F}$ is gH-Fréchet differentiable and gH-pseudoconvex at every $\bar{x} \in \mathcal{S}$, then $\boldsymbol{F}$ is quasiconvex on $\mathcal{S}$.*

Proof. Suppose $\mathbf{F}$ is not quasiconvex on $\mathcal{S}$. Then, there exists a $\hat{\lambda} \in [0, 1]$ such that for all $x, y \in \mathcal{S}$, we have

$$\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\hat{\lambda}x + (1 - \hat{\lambda})y) \prec \mathbf{0} \text{ and } \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\hat{\lambda}x + (1 - \hat{\lambda})y) \prec \mathbf{0}. \tag{3.13}$$

Since $\mathbf{F}$ is gH-Fréchet differentiable on $\mathcal{S}$, $\mathbf{F}$ is gH-continuous on $\mathcal{S}$. Consequently, there is a $\bar{\lambda} \in (0, 1)$ with

$$\mathbf{F}(\lambda x + (1-\lambda)y) \preceq \mathbf{F}(\bar{\lambda}x + (1-\bar{\lambda})y) \text{ for all } \lambda \in (0,1). \tag{3.14}$$

As $\mathbf{F}$ is gH-Fréchet differentiable, $\mathbf{F}$ is also gH-directionally differentiable. So, with the help of Theorem 2.4, we have

$$\mathbf{0} \not\prec \mathbf{F}_{\mathscr{D}}(\bar{x})(x-\bar{x}) \text{ and } \mathbf{0} \not\prec \mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}), \text{ where } \bar{x} = \bar{\lambda}x + (1-\bar{\lambda})y. \tag{3.15}$$

Since

$$y - \bar{x} = y - \bar{\lambda}x - (1-\bar{\lambda})y = -\bar{\lambda}(x-y) \tag{3.16}$$

and

$$x - \bar{x} = x - \bar{\lambda}x - (1-\bar{\lambda})y = (1-\bar{\lambda})(x-y), \tag{3.17}$$

due to inequality (3.15) and equation (3.17), we obtain

$$\begin{aligned} & \mathbf{0} \not\prec \mathbf{F}_{\mathscr{D}}(\bar{x})\,(x-y) \\ \implies\quad & \mathbf{0} \not\succ \mathbf{F}_{\mathscr{D}}(\bar{x})\left(-\bar{\lambda}(x-y)\right) \\ \implies\quad & \mathbf{0} \not\succ \mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}) \text{ by equation (3.16).} \end{aligned} \tag{3.18}$$

From (3.15) and (3.18), we have

$$\text{either } \mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}) = \mathbf{0} \text{ or `}\mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}) \text{ and } \mathbf{0} \text{ are not comparable'.} \tag{3.19}$$

Since $\mathbf{F}$ is gH-pseudoconvex at $\bar{x} \in \mathcal{S}$ we have from (3.19) that

$$\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x}) \not\prec \mathbf{0} \text{ for all } y \in \mathcal{S}. \tag{3.20}$$

However, for any $y \in \mathcal{S}$,

$$\begin{aligned}\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x}) &= \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{\lambda}x + (1-\bar{\lambda})y) \\ &\preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\hat{\lambda}x + (1-\hat{\lambda})y), \text{by (3.14)} \\ &\prec \mathbf{0}, \text{ by (3.13)}. \end{aligned} \tag{3.21}$$

Hence, (3.20) contradicts (3.21).

Therefore, $\mathbf{F}$ is a quasiconvex IVF on $\mathcal{S}$. □

In the following example, we show that there are some gH-Fréchet differentiable IVFs which are quasiconvex but not gH-pseudoconvex.

Example 3.4. *Consider $\mathcal{X}$ as the Euclidean space $\mathbb{R}$, $\mathcal{S} = \mathbb{R}_+$, and the IVF $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$, which is defined by*

$$\boldsymbol{F}(x) = x^3 \odot [3,\ 9].$$

For any $\bar{x}$ in $\mathcal{S}$ and h in $\mathcal{X}$ such that $\bar{x} + \lambda h \in \mathcal{S}$ with $\lambda \geq 0$, we have

$$\begin{aligned}\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h) &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})\right), \textit{ provided limit exists} \\ &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot [3(\bar{x} + \lambda h)^3 - 3\bar{x}^3,\ 9(\bar{x} + \lambda h)^3 - 9\bar{x}^3] \\ &= \bar{x}^2 h \odot [9,\ 27].\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{\|h\| \to 0} \frac{1}{\|h\|} \odot \left(\|\boldsymbol{F}(\bar{x} + h) \ominus_{gH} \boldsymbol{F}(\bar{x}) \ominus_{gH} \boldsymbol{F}_{\mathscr{D}}(\bar{x})(h)\|_{I(\mathbb{R})}\right) \\ = &\lim_{\|h\| \to 0} \frac{1}{\|h\|} \odot \left(\|[3(\bar{x} + h)^3 - 3\bar{x}^3,\ 9(\bar{x} + h)^3 - 9\bar{x}^3] \ominus_{gH} \bar{x}^2 h \odot [9,\ 27]\|_{I(\mathbb{R})}\right) \\ = &\ 0.\end{aligned}$$

Hence, $\boldsymbol{F}$ is gH-Fréchet differentiable at every $\bar{x} \in \mathcal{S}$ and every direction $h \in \mathcal{X}$. Also, for any $x, y \in \mathcal{S}$ and $\lambda, \lambda' \in [0,1]$ with $\lambda + \lambda' = 1$, we have

$$\begin{aligned} &\textit{either } \boldsymbol{F}(x) \succeq \boldsymbol{F}(\lambda x + \lambda' y) \textit{ or } \boldsymbol{F}(y) \succeq \boldsymbol{F}(\lambda x + \lambda' y) \\ \Longrightarrow &\textit{either } \boldsymbol{F}(x) \nprec \boldsymbol{F}(\lambda x + \lambda' y) \textit{ or } \boldsymbol{F}(y) \nprec \boldsymbol{F}(\lambda x + \lambda' y). \end{aligned}$$

Therefore, $\boldsymbol{F}$ is a quasiconvex IVF on $\mathcal{S}$. Now,

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(y - \bar{x}) \nprec \boldsymbol{0} \implies \bar{x}^2 y \odot [9, 27] \nprec \boldsymbol{0} \implies y \geq 0.$$

Further, for $\bar{x} = 2$ with $y = 1$,

$$\boldsymbol{F}(y) \ominus_{gH} \boldsymbol{F}(\bar{x}) = 1^3 \odot [3, 9] \ominus_{gH} 2^3 \odot [3, 9] = [-63, -21] \prec \boldsymbol{0}.$$

Since $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(y - \bar{x}) \nprec 0 \not\Longrightarrow \boldsymbol{F}(y) \ominus_{gH} \boldsymbol{F}(\bar{x}) \nprec \boldsymbol{0}$ for $\bar{x} = 2$ with $y = 1$, $\boldsymbol{F}$ is not gH-pseudoconvex at $\bar{x} \in \mathcal{S}$.

Theorem 3.9. *Let $\mathcal{S}$ be a nonempty convex subset of $\mathcal{X}$ and $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be a convex IVF on $\mathcal{S}$ which is gH-Fréchet differentiable on $\mathcal{S}$. Then, $\boldsymbol{F}$ is a quasiconvex IVF on $\mathcal{S}$.*

Proof. Since **F** is convex and gH-Freéchet differentiable, by Theorem 3.7, **F** is gH-pseudoconvex. Also, by Theorem 3.8, **F** is a quasiconvex IVF on $\mathcal{S}$. □

Remark 3.5.5. *Converse of Theorem 3.9 is not true. For example, consider $\mathcal{X}$ as the Euclidean space $\mathbb{R}$, $\mathcal{S} = [0, 2]$, and the IVF $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ that is defined by*

$$\boldsymbol{F}(x) = \begin{cases} x \odot [1, 2] \oplus [-1, 0], & \textit{for } 0 \leq x \leq 1 \\ [1, 0] \oplus x \odot [-1, 2], & \textit{for } 1 \leq x \leq 2 \end{cases}$$

$$= [-|x-1|, 2x].$$

The IVF is depicted in Figure 5.1 by gray shaded region.

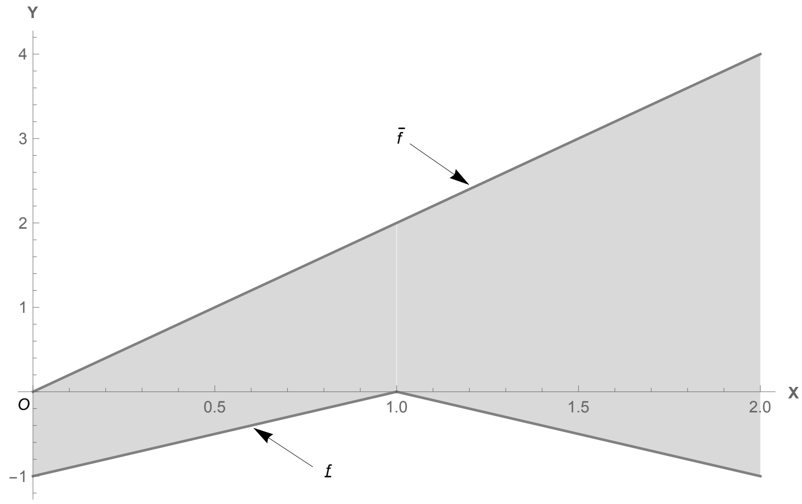

FIGURE 3.2: The IVF **F** of Remark 3.5.5

From Figure 5.1, we obtain that for any $\lambda \in (0, 1)$,

$$\boldsymbol{F}(2) \not\preceq \boldsymbol{F}(2\lambda + 0(1-\lambda)) = \boldsymbol{F}(2\lambda).$$

and thus, $\boldsymbol{F}$ *is quasiconvex on* $\mathcal{S}$. *However, the function* $\underline{f}(x) = -|x-1|$ *is not convex on* $\mathcal{S}$ *(see Figure 5.1). Therefore, the IVF* $\boldsymbol{F}$ *is not convex on* $\mathcal{S}$.

The results of Theorem 3.7, Theorem 3.8, and Theorem 3.9 can be summarized as follows. If $\mathbf{F}$ is an IVF on a nonempty convex subset $\mathcal{S}$ of $\mathcal{X}$, which is gH-Fréchet differentiable at every $\bar{x} \in \mathcal{X}$, then the following implications are satisfied

$$\begin{aligned}\mathbf{F} \text{ is convex IVF on } \mathcal{S} \quad &\Longrightarrow \quad \mathbf{F} \text{ is } gH\text{-pseudoconvex IVF at every } \bar{x} \in \mathcal{S} \\ &\Longrightarrow \quad \mathbf{F} \text{ is quasiconvex IVF on } \mathcal{S}.\end{aligned}$$

3.6 Characterization of Efficient Solutions

In this section, we present several characterizations of efficient solutions for IOPs based on the properties of quasiconvex and gH-pseudoconvex IVFs, contingent cone, upper gH-Clarke and gH-Fréchet derivatives.

Theorem 3.10. *Let $\mathcal{S}$ be a nonempty convex subset of $\mathcal{X}$ and $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF. Consider the IOP:*

$$\min_{x \in \mathcal{S}} \boldsymbol{F}(x). \tag{3.22}$$

(i) If $\boldsymbol{F}$ is continuous and convex on $\mathcal{S}$, then for every efficient point $\bar{x} \in \mathcal{S}$ of the IOP (3.22),

$$\boldsymbol{F}(\bar{x} + h) \nprec \boldsymbol{F}(\bar{x}) \text{ for all } h \in \mathcal{T}(\mathcal{S}, \bar{x}). \tag{3.23}$$

(ii) If the set $\mathcal{S}$ is star-shaped with respect to some $\bar{x} \in \mathcal{S}$ and the inequality (3.23) is true for $\bar{x}$, then $\bar{x}$ is an efficient point of the IOP (3.22).

Proof. (i) For $h = 0_{\mathcal{X}}$, result is trivial. We assume that the inequality (3.23) does not hold at an efficient point $\bar{x} \in \mathcal{S}$ of the IOP (3.22). Then, there is a vector $h \in \mathcal{T}(\mathcal{S}, \bar{x}) \setminus \{0_{\mathcal{X}}\}$ such that

$$\begin{aligned} & \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x} + h) \succ \mathbf{0} \\ \implies \quad & \overline{f}(\bar{x}) - \overline{f}(x \bar{+} h) > 0 \text{ and } \underline{f}(\bar{x}) - \underline{f}(\bar{x} + h) \geq 0. \end{aligned}$$

Since $\mathbf{F}$ is gH-continuous, $\overline{f}$ and $\underline{f}$ are continuous by (i) of Lemma 3.3. Therefore, there exist ϵ_1 and ϵ_2 with $\epsilon_1 > \epsilon_2 \geq 0$ such that

$$\overline{f}(\bar{x}) - \overline{f}(\bar{x} + h) > \epsilon_1 > 0 \text{ and } \underline{f}(\bar{x}) - \underline{f}(\bar{x} + h) \geq \epsilon_2 \geq 0.$$

$$\implies \quad \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}+h) \succ \mathbf{A} \succ \mathbf{0}, \text{ where } \mathbf{A} = [\epsilon_2, \epsilon_1]. \tag{3.24}$$

As $h \in \mathcal{T}(\mathcal{S}, \bar{x}) \setminus \{0_{\mathcal{X}}\}$, there exists a sequence $\{x_n\}$ in $\mathcal{S}$ and a sequence $\{\lambda_n\}$ of positive real numbers such that

$$\bar{x} = \lim_{n\to+\infty} x_n \text{ and } h = \lim_{n\to+\infty} h_n,$$

where $h_n = \lambda_n(x_n - \bar{x})$ for all $n \in \mathbb{N}$. As $h \neq 0_{\mathcal{X}}$, we obtain $\lim_{n\to+\infty} \frac{1}{\lambda_n} = 0$. Since $\lim_{n\to+\infty} \frac{1}{\lambda_n} = 0$, without loss of generality, we assume $0 < \frac{1}{\lambda_n} < 1$ for all $n \in \mathbb{N}$. Since $\mathbf{F}$ is convex and gH-continuous on $\mathcal{S}$, for sufficiently large $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbf{F}(x_n) &= \mathbf{F}\left(\tfrac{1}{\lambda_n}\bar{x} + x_n - \bar{x} + \bar{x} - \tfrac{1}{\lambda_n}\bar{x}\right) \\
&= \mathbf{F}\left(\tfrac{1}{\lambda_n}(\bar{x} + h_n) + \left(1 - \tfrac{1}{\lambda_n}\right)\bar{x}\right) \\
&\preceq \tfrac{1}{\lambda_n} \odot \mathbf{F}(\bar{x} + h_n) \oplus \left(1 - \tfrac{1}{\lambda_n}\right) \odot \mathbf{F}(\bar{x}) \\
&\preceq \tfrac{1}{\lambda_n} \odot (\mathbf{F}(\bar{x} + h) \oplus \mathbf{A}) \oplus \left(1 - \tfrac{1}{\lambda_n}\right) \odot \mathbf{F}(\bar{x}) \\
&\prec \tfrac{1}{\lambda_n} \odot \mathbf{F}(\bar{x}) \oplus \left(1 - \tfrac{1}{\lambda_n}\right) \odot \mathbf{F}(\bar{x}), \text{ by (3.24)} \\
&\prec \mathbf{F}(\bar{x}),
\end{aligned}$$

which is a contradiction to $\bar{x}$ an efficient point of the IOP (3.22). Therefore, the relation (3.23) must be true.

(ii) Let the set $\mathcal{S}$ be star-shaped with respect to some $\bar{x} \in \mathcal{S}$. Then, it follows from Theorem 4.8 of [42] that

$$\mathcal{S} \setminus \{\bar{x}\} \subset \mathcal{T}(\mathcal{S}, \bar{x}).$$

Hence, by (3.23), we get

$$\mathbf{F}(\bar{x}+h) \nprec \mathbf{F}(\bar{x}) \text{ for all } h \in \mathcal{S} \setminus \{\bar{x}\},$$

i.e., $\bar{x}$ is an efficient point of the IOP (3.22).

$\square$

In the next theorem, using the notion of upper gH-Clarke derivative, we present a necessary condition for efficient solutions.

Theorem 3.11. *Let $\mathcal{W}$ be a subset of $\mathcal{X}$ with nonempty interior. Suppose $\mathcal{S}$ is a nonempty subset of $\mathcal{W}$, and $\bar{x} \in \mathcal{S} \cap int(\mathcal{W})$ is an efficient point of the following IOP:*

$$\min_{x \in \mathcal{S}} \boldsymbol{F}(x). \tag{3.25}$$

If the set $\mathcal{S}$ is star-shaped with respect to $\bar{x}$ and the IVF $\boldsymbol{F}$ of the IOP (3.25) is gH-Lipschitz continuous at $\bar{x}$, then the following inequality holds:

$$\boldsymbol{F}_{\mathscr{C}}(\bar{x})(x-\bar{x}) \nprec \boldsymbol{0} \text{ for all } x \in \mathcal{S}.$$

Proof. Since $\mathbf{F}$ is gH-Lipschitz continuous at $\bar{x}$, due to Theorem 3.4, $\mathbf{F}$ is upper gH-Clarke differentiable at $\bar{x}$ and

$$\limsup_{\substack{x \to 0 \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x}+\lambda(x-\bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})\right) = \mathbf{F}_{\mathscr{C}}(\bar{x})(x-\bar{x}). \tag{3.26}$$

As $\bar{x}$ is an efficient point of $\mathbf{F}$ and $\mathcal{S}$ is star-shaped with respect to $\bar{x}$, for any $x \in \mathcal{S}$ and $\lambda > 0$ with $\bar{x}+\lambda(x-\bar{x}) \in \mathcal{S}$, we have the following inequality:

$$\mathbf{F}(\bar{x}+\lambda(x-\bar{x}) \nprec \mathbf{F}(\bar{x})$$

$$\begin{aligned}
&\implies \mathbf{F}(\bar{x}+\lambda(x-\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0}, \text{ by Lemma 1.4}\\
&\implies \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x}+\lambda(x-\bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})) \nprec \mathbf{0}\\
&\implies \limsup_{\substack{x\to 0\\ \lambda\to 0+}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x}+\lambda(x-\bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})) \nprec \mathbf{0}\\
&\implies \mathbf{F}_{\mathscr{C}}(\bar{x})(x-\bar{x}) \nprec \mathbf{0} \text{ for all } x \in \mathcal{S} \text{ from (3.26).}
\end{aligned}$$

□

Remark 3.6.1. *Converse of Theorem 3.11 is not true. For example, consider $\mathcal{X}$ as the Euclidean space $\mathbb{R}$ and the IOP*

$$\min_{x\in\mathcal{S}=(-\infty,0]} \boldsymbol{F}(x), \ \textit{with } \boldsymbol{F}(x) = x^2 \odot [-5,-3] = \left[-5x^2, -3x^2\right]. \tag{3.27}$$

For any x in $\mathcal{S}$ and an arbitrary $h \in \mathcal{X}$ with $x+\lambda h$, we have

$$\begin{aligned}
&\lim_{\lambda\to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(x+\lambda h) \ominus_{gH} \boldsymbol{F}(x))\\
=\ &\lim_{\lambda\to 0+} \frac{1}{\lambda} \odot \Big[\min\{5x^2 - 5(x+\lambda h)^2,\ 3x^2 - 3(x+\lambda h)^2\},\\
&\qquad\qquad \max\{5x^2 - 5(x+\lambda h)^2,\ 3x^2 - 3(x+\lambda h)^2\}\Big]\\
=\ &[\min\{-10xh, -6xh\},\ \max\{-10xh, -6xh\}].
\end{aligned}$$

Then,

$$\lim_{\substack{x\to 0\\ \lambda\to 0+}} \frac{1}{\lambda} \odot (\boldsymbol{F}(x+\lambda h) \ominus_{gH} \boldsymbol{F}(x)) = [\min\{0,0\}, \max\{0,0\}] = \boldsymbol{0}.$$

Consequently, considering $\bar{x} = 0$, we have $\boldsymbol{F}_{\mathscr{C}}(\bar{x})(h) = \boldsymbol{0}$ for all $h \in \mathcal{X}$. Therefore, $\boldsymbol{F}_{\mathscr{C}}(\bar{x})(h) \nprec \boldsymbol{0}$ for all $h \in \mathcal{X}$.

Let $\mathcal{W} = (-\infty, 12]$, and thus $\bar{x} \in \mathcal{S} \cap int(\mathcal{W})$. Further, $\mathcal{S}$ is starshaped at $\bar{x}$. As $\underline{f}(x) = -5x^2$ and $\overline{f}(x) = -3x^2$ are Lipschitz continuous at $\bar{x}$, by (ii) of Lemma 3.3, the IVF $\boldsymbol{F}$ is Lipschitz continuous at $\bar{x}$. However, $\bar{x}$ is not an efficient point of the

IOP (3.27) as

$$\boldsymbol{F}(x) \prec \boldsymbol{0} = \boldsymbol{F}(\bar{x}) \text{ for all } x \in (-\infty, 0).$$

In the next result, with the help of tangent cone, we derive a necessary optimality condition for IOPs whose objective functions are gH-Fréchet differentiable.

Theorem 3.12. *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$ and $\boldsymbol{F}$ be an IVF defined on an open superset of $\mathcal{S}$. If $\bar{x}$ is an efficient point of the IOP (3.22) with $\boldsymbol{F}(\bar{x}) \preceq \boldsymbol{F}(x)$ for all $x \in \mathcal{S}$ and $\boldsymbol{F}$ is gH-Fréchet differentiable at $\bar{x}$, then*

$$\boldsymbol{F}_{\mathscr{F}}(\bar{x})(h) \nprec \boldsymbol{0} \text{ for all } h \in \mathcal{T}(\mathcal{S}, \bar{x}).$$

Proof. For $h = 0_{\mathcal{X}}$, we have $\mathbf{F}_{\mathscr{F}}(\bar{x})(h) = \mathbf{0}$, and hence the result is trivial.
Let $\bar{x} \in \mathcal{S}$ be an efficient point of $\mathbf{F}$ on $\mathcal{S}$, and let h be an arbitrary element of $\mathcal{T}(\mathcal{S}, \bar{x}) \setminus \{0_{\mathcal{X}}\}$.

By the definition of h, there exists a sequence $\{x_n\}$ of elements in $\mathcal{S}$ and a sequence $\{\lambda_n\}$ of positive real numbers such that

$$\bar{x} = \lim_{n \to +\infty} x_n \text{ and } h = \lim_{n \to +\infty} h_n,$$

where $h_n = \lambda_n(x_n - \bar{x})$ for all $n \in \mathbb{N}$. Since $\mathbf{F}$ is gH-Fréchet differentiable at $\bar{x}$,

$$\begin{aligned}
& \mathbf{F}_{\mathscr{F}}(\bar{x})(h) \\
= \;& \mathbf{F}_{\mathscr{F}}(\bar{x}) \left(\lim_{n \to +\infty} \lambda_n (x_n - \bar{x}) \right) \\
= \;& \lim_{n \to +\infty} \lambda_n \odot \mathbf{F}_{\mathscr{F}}(\bar{x})(x_n - \bar{x}) \\
\nprec \;& \lim_{n \to +\infty} \lambda_n \odot \left[\left(\mathbf{F}(x_n) \ominus_{gH} \mathbf{F}(\bar{x}) \right) \ominus_{gH} \left(\mathbf{F}(x_n) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{F}}(\bar{x})(x_n - \bar{x}) \right) \right], \\
& \text{by inequality (i) of Lemma 1.5} \\
\nprec \;& (-1) \odot \lim_{n \to +\infty} \lambda_n \odot \left(\mathbf{F}(x_n) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{F}}(\bar{x})(x_n - \bar{x}) \right), \text{ by (ii) of Lemma 1.5}
\end{aligned}$$

$$= (-1) \odot \lim_{n \to +\infty} \left(\|h_n\| \odot \frac{\mathbf{F}(x_n) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{F}}(\bar{x})(x_n - \bar{x})}{\|x_n - \bar{x}\|} \right)$$
$$= \mathbf{0}.$$

Hence, $\mathbf{F}_{\mathscr{F}}(\bar{x})(h) \nprec \mathbf{0}$ for all $h \in \mathcal{T}(\mathcal{S}, \bar{x})$.

□

Remark 3.6.2. *One may think that in Theorem 3.12, instead of considering the fact that the IVF $\boldsymbol{F}$ is defined on an open superset of $\mathcal{S}$, we may consider that*

(i) *the IVF $\boldsymbol{F}$ is simply defined on $\mathcal{S}$ or,*

(ii) *$\mathcal{S}$ is open and the IVF $\boldsymbol{F}$ is defined on $\mathcal{S}$.*

However, this assumptions are quite restrictive. For instance, let $\mathcal{S} = [0,1] \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational number, $\mathcal{W} = (-2, +\infty)$ and $\mathcal{X}$ as the Euclidean space $\mathbb{R}$. Then, by Theorem 4.8 and Theorem 4.9 of [42], $\mathcal{T}(\mathcal{S}, \bar{x}) = [0, +\infty)$, where $\bar{x} = 0$. Let $\boldsymbol{F}$ be an IVF defined on open superset $\mathcal{W}$ of $\mathcal{S}$ by

$$\boldsymbol{F}(x) = x^2 \odot [1, 2].$$

Then, $\bar{x}$ is an efficient point of the IOP (3.22) with $\boldsymbol{F}(\bar{x}) \preceq \boldsymbol{F}(x)$ for all $x \in \mathcal{S}$ and $\boldsymbol{F}$ is gH-Fréchet differentiable at $\bar{x} = 0$ with

$$\boldsymbol{F}_{\mathscr{F}}(\bar{x})(h) = \boldsymbol{0} \nprec \boldsymbol{0} \text{ for all } h \in \mathcal{T}(\mathcal{S}, \bar{x}).$$

Hence, the result of Theorem 3.12 is true.

If we restrict $\boldsymbol{F}$ on $\mathcal{S}$ only then, for any $h \in \mathcal{T}(\mathcal{S}, \bar{x}) \subseteq \mathbb{R}$, $\bar{x} + \lambda_n h \notin \mathcal{S}$ for some sequence $\{\lambda_n\}$ of positive real numbers such that $\lambda_n \to 0$. Therefore, we can not find gH-Fréchet derivative at $\bar{x} = 0 \in \mathcal{S}$ in any direction $h \in \mathcal{T}(\mathcal{S}, \bar{x})$. Hence, the result *of Theorem 3.12 is not applicable for this case.*

Further, $\mathcal{S} = [0,1] \cap \mathbb{Q}$ assures the restrictiveness of assumption (ii).

Remark 3.6.3. *The condition '$\boldsymbol{F}_{\mathscr{F}}(\bar{x})(h) \not\prec \boldsymbol{0}$ for all $h \in \mathcal{T}(\mathcal{S}, \bar{x})$' in Theorem 3.12 is necessary for an efficient point but not sufficient. For instance, consider $\mathcal{S} = (-\infty, 0]$ and $\mathcal{X}$ as the Euclidean space $\mathbb{R}$. Then, by Theorem 4.8 and Theorem 4.9 of [42], $\mathcal{T}(\mathcal{S}, \bar{x}) = (-\infty, 0]$, where $\bar{x} = 0$. Let $\boldsymbol{F}$ be an IVF with IOP:*

$$\min_{x \in \mathcal{S} = [0, +\infty)} \boldsymbol{F}(x) = x^2 \odot [-6, -2]. \tag{3.28}$$

Note that for $\bar{x}$ in $\mathcal{S}$ and h in $\mathcal{T}(\mathcal{S}, \bar{x})$ with $\bar{x} + \lambda h \in \mathcal{S}$, we have

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) = \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left((\lambda h)^2 \odot [-6, -2]\right) = \boldsymbol{0}$$

and

$$\lim_{\|h\| \to 0} \frac{\|\boldsymbol{F}(\bar{x} + h) \ominus_{gH} \boldsymbol{F}(\bar{x}) \ominus_{gH} \boldsymbol{0}\|_{I(\mathbb{R})}}{\|h\|} = \lim_{\|h\| \to 0} \frac{\|h^2 \odot [-6, -2]\|_{I(\mathbb{R})}}{\|h\|} = 0.$$

Hence, $\boldsymbol{F}$ is gH-Fréchet differentiable at $\bar{x} \in \mathcal{S}$ with $\boldsymbol{F}_{\mathscr{F}}(\bar{x})(h) = \boldsymbol{0} \not\prec \boldsymbol{0}$ for all $h \in \mathcal{T}(\mathcal{S}, \bar{x})$. However,

$$\boldsymbol{F}(x) \preceq \boldsymbol{0} = \boldsymbol{F}(0) \text{ for all } x \in \mathcal{S}.$$

Therefore, $\bar{x} = 0$ is not an efficient solution of IOP (3.28).

Next, we look for some conditions under which Theorem 3.12 becomes a sufficient optimality condition. In the next theorem, we show that the result in Theorem 3.12 becomes a sufficient optimality condition for gH-pseudoconvex IVFs.

Theorem 3.13. *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$ and $\boldsymbol{F}$ be a gH-directionally differentiable IVF defined on an open superset of $\mathcal{S}$. Let $\mathcal{S}$ be star-shaped with* *respect to some $\bar{x} \in \mathcal{S}$, $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h)$ be gH-directional derivative at $\bar{x}$ in the direction of h, and $\boldsymbol{F}$ be gH-pseudoconvex at $\bar{x}$. Then,*

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h) \not\prec \boldsymbol{0} \text{ for all } h \in \mathcal{T}(\mathcal{S}, \bar{x}), \tag{3.29}$$

if and only if $\bar{x}$ is an efficient point of the IOP (3.22) on $\mathcal{S}$.

Proof. Let the IVF $\mathbf{F}$ be satisfied the condition (3.29) at some $\bar{x} \in \mathcal{S}$. Since $\mathcal{S}$ is star-shaped with respect to $\bar{x} \in \mathcal{S}$. Then, by Theorem 4.8 of [42] we have

$$\mathcal{S} \setminus \{\bar{x}\} \subset \mathcal{T}(\mathcal{S}, \bar{x}),$$

and thus

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(x - \bar{x}) \not\prec \mathbf{0} \text{ for all } x \in \mathcal{S}.$$

Since $\mathbf{F}$ is gH-pseudoconvex at $\bar{x}$,

$$\begin{aligned} & \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \not\prec \mathbf{0} \text{ for all } x \in \mathcal{S} \\ \implies & \mathbf{F}(x) \not\prec \mathbf{F}(\bar{x}) \text{ for all } x \in \mathcal{S}, \text{ by Lemma 2.1 of [28]}). \end{aligned}$$

Hence, $\bar{x}$ is an efficient point of the IOP (3.22).

The converse part is followed by Theorem 3.2 of [28]. □

3.7 Concluding Remarks

In this chapter, the notions of upper and lower gH-Clarke derivative, gH-pseudoconvex, and quasiconvex for IVFs have been proposed. To describe the properties of Clarke derivative, the concepts of limit superior, limit inferior, and sublinear for IVFs have been studied. Further, by using the derived concepts, the existence of upper gH-Clarke derivative, the relation of upper gH-Clarke derivative with gH-directional derivative, the relation of convex with gH-pseudoconvex, and the relation of gH-peudoconvex with quasiconvex have been shown for IVFs. With the help of the studied gH-pseudoconvex, quasiconvex, and gH-Lipschitz IVFs, we have presented a few results on characterizing efficient solutions to an IOP with upper Clarke and Fréchet differentiable IVF.

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