Chapter 2

Generalized Hukuhara Gâteaux and Fréchet Derivatives of Interval-valued Functions and their Application in Optimization with Interval-valued Functions

2.1 Introduction

To analyze the various types of functions and optimization problems, the concept of derivative is one of the main tool. For a vector-valued functions, there are two important concepts of derivative, i.e., Gâteaux (or weak) derivative and Fréchet (or strong) derivative, which are used to characterize the optimality condition in optimization theory. The Fréchet derivative is a derivative define on Banach spaces, and the Gâteaux is a generalization of the concept of directional derivative in differential calculus. Both derivatives are often used to formalize the functional derivative commonly used in Physics, particularly Quantum field theory.

2.2 Motivation

It is well known that what are the importance of directional derivative, Gâteaux derivative, and Fréchet derivative in real-world or optimization problems and how to characterize the optimal solutions by using these derivatives of the optimization problems. But, Now-a-days, due to inherent uncertainty in many real-world problems, the study of these derivatives for IVFs demands a significant study. In order to develop the calculus of IVFs as well as the optimality condition of optimization problems with interval-valued objective functions, the ideas of these derivatives are studied.

2.3 Contributions

In this chapter, the notions of gH-directional, gH-Gâteaux and gH-Fréchet derivative for IVFs are proposed. The existence of gH-Fréchet derivative is shown to imply the existence of gH-Gâteaux derivative and the existence of gH-Gâteaux derivative is observed to indicate the presence of gH-directional derivative. For a gH-Lipschitz continuous IVF, the existence of gH-Gâteaux derivative implies the existence of gH-Fréchet derivative is shown. It is observed that for a convex IVF on a linear space, the gH-directional derivative exists at any point for every direction. Concepts of linear and monotonic IVFs are studied in the sequel. Further, it is shown that the proposed derivatives are useful to check the convexity of an IVF and to characterize efficient points of an IOP. It is observed that at an efficient point of an IVF, none of its gH-directional derivatives dominates zero, and the gH-Gâteaux derivative must contain zero. The entire study is supported by suitable illustrative examples.

The main contributions of this chapter are as follows:

(i) For a convex IVF, it is proved that the gH-directional derivative exists at any point of the domain.

(ii) For a gH-Lipschitz continuous IVF, it is shown that gH-Gâteaux differentiable IVF is gH-Fréchet differentiable.

(iii) For a convex gH-Gâteaux differentiable IVF, it is proved that a point is an efficient point of IOP if and only if zero belongs to the gH-Gâteaux derivative.

2.4 Directional Derivative of Interval-valued Functions

In this section, we define the gH-directional derivative of IVFs and prove its existence for a convex IVF. Further, we present an optimality condition for efficient point of an IOP with the help this derivative.

Definition 2.4.1 (gH-directional derivative)**.** *Let* $\boldsymbol{F}$ *be an IVF on a nonempty subset* $\mathcal{S}$ *of* $\mathbb{R}^n$*. Let* $\bar{x} \in \mathcal{S}$ *and* $h \in \mathbb{R}^n$*. If the limit*

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x}))$$

*exists, then the limit is said to be gH-directional derivative of $\boldsymbol{F}$ at $\bar{x}$ in the direction h, and it is denoted by $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h)$. If this limit exists for all $h \in \mathbb{R}^n$, then $\boldsymbol{F}$ is said to be gH-*directional differentiable *at $\bar{x}$.*

The following example shows that given IVF is gH-directional differentiable at zero vector.

Example 2.1. *We consider the function $\boldsymbol{F}(x_1, x_2) = \boldsymbol{C}_1 \odot x_1 \oplus \boldsymbol{C}_2 \odot (x_2 e^{x_1})$, where $\boldsymbol{C}_1,\ \boldsymbol{C}_2 \in I(\mathbb{R})$. We note that*

$$\begin{aligned}
&\lim_{\lambda\to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\lambda h_1, \lambda h_2) \ominus_{gH} \boldsymbol{F}(0,0)) \\
&= \lim_{\lambda\to 0+} \frac{1}{\lambda} \odot \left(\lambda \odot \boldsymbol{C}_1 \odot h_1 \oplus \lambda \odot \boldsymbol{C}_2 \odot (h_2 e^{\lambda h_1})\right) \\
&= \boldsymbol{C}_1 \odot h_1 \oplus \boldsymbol{C}_2 \odot h_2.
\end{aligned}$$

Thus, the gH-directional derivative of $\boldsymbol{F}$ at $(0,0)$ in direction (h_1, h_2) exists and $\boldsymbol{F}_{\mathscr{D}}(0,0)(h) = \boldsymbol{C}_1 \odot h_1 \oplus \boldsymbol{C}_2 \odot h_2$.

Note 3. *According to Bao [5], the gH-directional derivative of $\boldsymbol{F}$ at $\bar{x}$ in the direction h exists if and only if both the limits*

$$\lim_{\lambda\to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) \ \text{ and } \lim_{\lambda\to 0-} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x}) \ominus_{gH} \boldsymbol{F}(\bar{x} - \lambda h))$$

exist and they are equal. However, we note that

$$\begin{aligned}
&\lim_{\lambda\to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) \\
= \ &\lim_{\delta\to 0-} \frac{1}{-\delta} \odot (\boldsymbol{F}(\bar{x} + (-\delta) h) \ominus_{gH} \boldsymbol{F}(\bar{x}))\,, \ \text{where } \lambda = -\delta \\
= \ &\lim_{\delta\to 0-} \frac{1}{\delta} \odot (\boldsymbol{F}(\bar{x}) \ominus_{gH} \boldsymbol{F}(\bar{x} - \delta h))\,, \ \text{by Note 1.}
\end{aligned}$$

Hence, the existence of the limit in Definition 2.4.1 suffices to check the existence of the gH-directional derivative.

Definition 2.4.2 (Monotonic IVF). *An IVF $\boldsymbol{F}(x) = \left[\underline{f}(x), \overline{f}(x)\right]$ from a nonempty subset $\mathcal{S}$ of $\mathbb{R}^n$ to $I(\mathbb{R})$ is said to be* monotonic increasing *if for all $x_1,\ x_2 \in \mathcal{S}$,*

$$x_1 \leq x_2 \text{ implies } \boldsymbol{F}(x_1) \preceq \boldsymbol{F}(x_2).$$

The function $\boldsymbol{F}$ is said to be monotonic decreasing *if for all $x_1,\ x_2 \in \mathcal{S}$,*

$$x_1 \leq x_2 \text{ implies } \boldsymbol{F}(x_2) \preceq \boldsymbol{F}(x_1).$$

Remark 2.4.1. *It is easy to verify that if an IVF $\boldsymbol{F}$ is monotonic increasing (or monotonic decreasing) on $\mathcal{S} \subseteq \mathbb{R}^n$ then both the real-valued functions $\underline{f}$ and $\overline{f}$ are monotonic increasing (or monotonic decreasing) on $\mathcal{S} \subseteq \mathbb{R}^n$ and vice-versa.*

Lemma 2.1. *Let $\mathcal{S}$ be a linear subspace of $\mathbb{R}^n$ and $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be a convex function on $\mathcal{S}$. Then for any $\bar{x} \in \mathcal{S}$ and $h \in \mathbb{R}^n$, the IVF $\boldsymbol{\Phi} : \mathbb{R}^+ \backslash \{0\} \to I(\mathbb{R})$, defined by*

$$\boldsymbol{\Phi}(\lambda) = \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) \quad \text{for all} \quad \lambda > 0,$$

is monotonically increasing.

Proof. As $\mathbf{F}$ is a convex function, for any $0 < s \leq t$, we have

$$\begin{aligned}
& \mathbf{F}(\bar{x} + sh) \ominus_{gH} \mathbf{F}(\bar{x}) \\
= \ & \mathbf{F}(\lambda_1(\bar{x} + th) + \lambda_2 \bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}), \text{ where } \lambda_1 = \frac{s}{t} \text{ and } \lambda_2 = \frac{t-s}{t} \\
\preceq \ & (\lambda_1 \odot \mathbf{F}(z) \oplus \lambda_2 \odot \mathbf{F}(\bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}), \text{ where } z = \bar{x} + th \\
= \ & \left[\min\left\{\lambda_1 \underline{f}(z) + \lambda_2 \underline{f}(\bar{x}) - \underline{f}(\bar{x}), \lambda_1 \overline{f}(z) + \lambda_2 \overline{f}(\bar{x}) - \overline{f}(\bar{x})\right\},\right. \\
& \qquad \left.\max\left\{\lambda_1 \underline{f}(z) + \lambda_2 \underline{f}(\bar{x}) - \underline{f}(\bar{x}), \lambda_1 \overline{f}(z) + \lambda_2 \overline{f}(\bar{x}) - \overline{f}(\bar{x})\right\}\right]
\end{aligned}$$

$$\begin{aligned}
&= \left[\min\left\{\lambda_1 \underline{f}(z) - \lambda_1 \underline{f}(\bar{x}), \lambda_1 \overline{f}(z) - \lambda_1 \overline{f}(\bar{x})\right\},\right.\\
&\qquad \left.\max\left\{\lambda_1 \underline{f}(z) - \lambda_1 \underline{f}(\bar{x}), \lambda_1 \overline{f}(z) - \lambda_1 \overline{f}(\bar{x})\right\}\right]\\
&= \lambda_1 \odot (\mathbf{F}(z) \ominus_{gH} \mathbf{F}(\bar{x})).
\end{aligned}$$

Therefore,

$$\frac{1}{s} \odot (\mathbf{F}(\bar{x} + sh) \ominus_{gH} \mathbf{F}(\bar{x})) \preceq \frac{1}{t} \odot (\mathbf{F}(\bar{x} + th) \ominus_{gH} \mathbf{F}(\bar{x})).$$

Consequently, we have $\mathbf{\Phi}(s) \preceq \mathbf{\Phi}(t)$. Thus, $\mathbf{\Phi}$ is a monotonic increasing function. □

Definition 2.4.3 (Bounded IVF)**.** *An IVF $\boldsymbol{F}$ from a nonempty subset $\mathcal{S}$ of $\mathbb{R}^n$ to $I(\mathbb{R})$ is said to be* bounded below *on $\mathcal{S}$ if there exists an interval $\boldsymbol{A} \in I(\mathbb{R})$ such that*

$$\boldsymbol{A} \preceq \boldsymbol{F}(x) \text{ for all } x \in \mathcal{S}.$$

The function $\boldsymbol{F}$ is said to be bounded above *on $\mathcal{S}$ if there exists an interval $\boldsymbol{A}' \in I(\mathbb{R})$ such that*

$$\boldsymbol{F}(x) \preceq \boldsymbol{A}' \text{ for all } x \in \mathcal{S}.$$

The function $\boldsymbol{F}$ is said to be bounded *on $\mathcal{S}$ if it is both bounded below and above.*

Remark 2.4.2. *It is easy to check that if the IVF $\boldsymbol{F}$ is bounded below (or bounded above) on $\mathcal{S} \subseteq \mathbb{R}^n$, then both the real-valued functions $\underline{f}$ and $\overline{f}$ are bounded below (or bounded above) on $\mathcal{S} \subseteq \mathbb{R}^n$ and vice-versa.*

Lemma 2.2. *Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a linear subspace of $\mathbb{R}^n$ and $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be a convex function on $\mathcal{S}$. Then, for each $\bar{x} \in \mathcal{S}$ and h in $\mathbb{R}^n$,*

$$\boldsymbol{F}(\bar{x}) \ominus_{gH} \boldsymbol{F}(\bar{x} - h) \preceq \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) \text{ for all } \lambda > 0.$$

Proof. Due to the convexity of $\mathbf{F}$ on $\mathcal{S}$, for any $\bar{x} \in \mathcal{S}$, $h \in \mathbb{R}^n$ and $\lambda > 0$ such that $\bar{x} + \lambda h \in \mathcal{S}$, we have

$$\begin{aligned} \mathbf{F}(\bar{x}) &= \mathbf{F}\left(\frac{1}{1+\lambda}(\bar{x}+\lambda h) + \frac{\lambda}{1+\lambda}(\bar{x}-h)\right) \\ &\preceq \frac{1}{1+\lambda} \odot \mathbf{F}(\bar{x}+\lambda h) \oplus \frac{\lambda}{1+\lambda} \odot \mathbf{F}(\bar{x}-h). \end{aligned}$$

This implies

$$\left[(1+\lambda)\underline{f}(\bar{x}), (1+\lambda)\overline{f}(\bar{x})\right] \preceq \left[\underline{f}(z) + \lambda\underline{f}(y), \overline{f}(z) + \lambda\overline{f}(y)\right],$$

where $z = \bar{x} + \lambda h$ and $y = \bar{x} - h$. Thus, we get

$$(1+\lambda)\underline{f}(\bar{x}) \leq \underline{f}(z) + \lambda\underline{f}(y)$$

$$\text{or,}\ \ \underline{f}(\bar{x}) - \underline{f}(y) \leq \frac{1}{\lambda}(\underline{f}(z) - \underline{f}(\bar{x})).$$

Similarly,

$$\overline{f}(\bar{x}) - \overline{f}(y) \leq \frac{1}{\lambda}(\overline{f}(z) - \overline{f}(\bar{x})).$$

Hence, in view of the last two inequalities, we obtain

$$\begin{aligned} &\left[\min\left\{\underline{f}(\bar{x}) - \underline{f}(y), \overline{f}(\bar{x}) - \overline{f}(y)\right\}, \max\left\{\underline{f}(\bar{x}) - \underline{f}(y), \overline{f}(\bar{x}) - \overline{f}(y)\right\}\right] \\ \preceq\ & \frac{1}{\lambda} \odot \left[\min\left\{\underline{f}(z) - \underline{f}(\bar{x}), \overline{f}(z) - \overline{f}(\bar{x})\right\}, \max\left\{\underline{f}(z) - \underline{f}(\bar{x}), \overline{f}(z) - \overline{f}(\bar{x})\right\}\right]. \end{aligned}$$

Thus, we get

$$\mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(y) \preceq \frac{1}{\lambda} \odot \left(\mathbf{F}(z) \ominus_{gH} \mathbf{F}(\bar{x})\right).$$

Hence,

$$\mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x} - h) \preceq \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) \quad \text{for all} \quad \lambda > 0.$$

□

For convex IVFs on a linear subspace, the existence of gH-directional derivative implies as the next theorem states.

Theorem 2.3. *Let $\mathcal{S}$ be a real linear subspace of $\mathbb{R}^n$ and $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be a convex function on $\mathcal{S}$. Then, at any $\bar{x} \in \mathcal{S}$, gH-directional derivative $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h)$ exists for every direction $h \in \mathbb{R}^n$.*

Proof. Let $\bar{x} \in \mathcal{S}$ and $h \in \mathbb{R}^n$. Define a function $\mathbf{\Phi} : \mathbb{R}^+ \setminus \{0\} \to I(\mathbb{R})$ by

$$\mathbf{\Phi}(\lambda) = \left[\underline{\phi}(\lambda), \overline{\phi}(\lambda)\right] = \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) \,.$$

Therefore, for all $\lambda > 0$, we obtain

$$\begin{aligned} \left[\underline{\phi}(\lambda), \overline{\phi}(\lambda)\right] \quad = \quad & \frac{1}{\lambda} \odot \big[\min\left\{\underline{f}(\bar{x} + \lambda h) - \underline{f}(\bar{x}), \overline{f}(\bar{x} + \lambda h) - \overline{f}(\bar{x})\right\}, \\ & \max\left\{\underline{f}(\bar{x} + \lambda h) - \underline{f}(\bar{x}), \overline{f}(\bar{x} + \lambda h) - \overline{f}(\bar{x})\right\} \big]. \end{aligned}$$

Since $\mathbf{F}$ is convex on $\mathcal{S}$, by Lemma 2.2, we have

$$\mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x} - h) \preceq \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) = \mathbf{\Phi}(\lambda) \text{ for all } \lambda \in \mathbb{R}^+.$$

Hence, the IVF $\mathbf{\Phi}$ is bounded below.

Further, by Lemma 2.1, $\mathbf{\Phi}$ is monotonically increasing. Thus, in view of Remark

2.4.1 and 2.4.2 both the real-valued functions $\underline{\phi}$ and $\overline{\phi}$ are monotonically increasing and bounded below. Therefore, the limits $\lim_{\lambda\to 0+} \underline{\phi}(\lambda)$ and $\lim_{\lambda\to 0+} \overline{\phi}(\lambda)$ exist and by Lemma 1.9 we see that the limit $\lim_{\lambda\to 0+} \phi(\lambda)$ exists.

Hence, the function $\mathbf{F}$ has a gH-directional derivative at $\bar{x} \in \mathcal{X}$, in the direction $h \in \mathbb{R}^n$. □

The following result characterize the efficient solutions of interval optimization problems with the help of better dominance relation of intervals.

Theorem 2.4 (Characterization of efficient points)**.** *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^n$, $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF, and $\bar{x} \in \mathcal{S}$ be an efficient point of the IOP (1.5). If the function $\boldsymbol{F}$ has a gH-directional derivative at $\bar{x}$ in the direction $x - \bar{x}$ for any $x \in \mathbb{R}^n$, then*

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(x - \bar{x}) \nprec \boldsymbol{0} \text{ for all } x \in \mathbb{R}^n. \tag{2.1}$$

The converse is true when $\mathcal{S}$ is convex and $\boldsymbol{F}$ is convex on $\mathcal{S}$.

Proof. Let $\bar{x} \in \mathcal{S}$ be an efficient point of the IVF $\mathbf{F}$. For any point $x \in \mathbb{R}^n$, the gH-directional derivative of $\mathbf{F}$ at $\bar{x}$ in the direction $x - \bar{x}$ is given by

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(x - \bar{x}) = \lim_{\lambda\to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda(x - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})).$$

Since the point $\bar{x}$ is an efficient point of the function $\mathbf{F}$, for any $x \in \mathbb{R}^n$ and $\lambda > 0$ with $\bar{x} + \lambda(x - \bar{x}) \in \mathcal{S}$, we get

$$\begin{aligned}
&\mathbf{F}(\bar{x} + \lambda(x - \bar{x})) \nprec \mathbf{F}(\bar{x})\\
\text{or, } &\mathbf{F}(\bar{x} + \lambda(x - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0}, \text{ by Lemma 1.4}\\
\text{or, } &\lim_{\lambda\to 0+} \tfrac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda(x - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})) \nprec \mathbf{0}.
\end{aligned}$$

This implies that

$$\max\left\{\underline{f}(\bar{x}+\lambda(x-\bar{x})-\underline{f}(\bar{x}), \overline{f}(\bar{x}+\lambda(x-\bar{x})-\overline{f}(\bar{x})\right\} \geq 0.$$

Since $\lambda > 0$, we obtain

$$\begin{aligned} &\lim_{\lambda\to 0+}\frac{1}{\lambda}\max\left\{\underline{f}(\bar{x}+\lambda(x-\bar{x})-\underline{f}(\bar{x}), \overline{f}(\bar{x}+\lambda(x-\bar{x})-\overline{f}(\bar{x})\right\} \geq 0 \\ \text{or,}\quad &\max\left\{\underline{f}_{\mathscr{D}}(\bar{x})(x-\bar{x}), \overline{f}_{\mathscr{D}}(\bar{x})(x-\bar{x})\right\} \geq 0. \end{aligned} \tag{2.2}$$

Therefore, $\mathbf{F}_{\mathscr{D}}(\bar{x})(x-\bar{x}) \nprec \mathbf{0}$ for all $x \in \mathbb{R}^n$.

To prove the latter part, we assume that $\mathcal{S}$ is convex and the function $\mathbf{F}$ is convex on $\mathcal{S}$. Then, by Theorem 2.3, for any $x \in \mathcal{S}$, $\mathbf{F}$ has a gH-directional derivative at $\bar{x} \in \mathcal{S}$ in every direction $x - \bar{x}$, where $x \in \mathbb{R}^n$. Let

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(x-\bar{x}) \nprec \mathbf{0} \text{ for all } x \in \mathbb{R}^n.$$

If possible let $\bar{x}$ be not an efficient point of $\mathbf{F}$. So there exists at least one $x' \in \mathcal{S}$ such that

$$\mathbf{F}(x') \preceq \mathbf{F}(\bar{x}).$$

Therefore, for any $\lambda \in (0, 1]$ we have

$$\begin{aligned} &\lambda \odot \mathbf{F}(x') \preceq \lambda \odot \mathbf{F}(\bar{x}) \\ \text{or, } &\lambda \odot \mathbf{F}(x') \oplus \lambda' \odot \mathbf{F}(\bar{x}) \preceq \lambda \odot \mathbf{F}(\bar{x}) \oplus \lambda' \odot \mathbf{F}(\bar{x}), \text{ where } \lambda' = 1 - \lambda \\ \text{or, } &\lambda \odot \mathbf{F}(x') \oplus \lambda' \odot \mathbf{F}(\bar{x}) \preceq (\lambda + \lambda') \odot \mathbf{F}(\bar{x}) = \mathbf{F}(\bar{x}). \end{aligned}$$

Due to the convexity of $\mathbf{F}$ on $\mathcal{S}$, we have

$$\mathbf{F}(\bar{x}+\lambda(x'-\bar{x})) = \mathbf{F}(\lambda x' + \lambda'\bar{x}) \preceq \lambda \odot \mathbf{F}(x') \oplus \lambda' \odot \mathbf{F}(\bar{x}) \preceq \mathbf{F}(\bar{x})$$

$$\text{or, } \mathbf{F}(\bar{x}+\lambda(x'-\bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \preceq \mathbf{0}$$

$$\text{or, } \lim_{\lambda \to 0+} \tfrac{1}{\lambda} \odot (\mathbf{F}(\bar{x}+\lambda(x'-\bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})) \preceq \mathbf{0}$$

$$\text{or, } \mathbf{F}_{\mathscr{D}}(\bar{x})(x'-\bar{x}) \preceq \mathbf{0}.$$

This is clearly contradictory the assumption that $\mathbf{F}_{\mathscr{D}}(\bar{x})(x-\bar{x}) \nprec \mathbf{0}$ for all $x \in \mathbb{R}^n$. Hence, $\bar{x}$ is the efficient point of $\mathbf{F}$. □

Example 2.2. *Consider the IOP:*

$$\min_{x \in \left[-\frac{2}{5}, \frac{3}{2}\right]} \boldsymbol{F}(x) = [x^2 - 2x + 1, x^2 + 2]. \tag{2.3}$$

The gH-directional derivative of $\boldsymbol{F}$ *at a point* $x \in \left[-\frac{2}{5}, \frac{3}{2}\right]$ *in a direction* $h \in \mathbb{R}$ *is*

$$\boldsymbol{F}_{\mathscr{D}}(x)(h) = \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(x+\lambda h) \ominus_{gH} \boldsymbol{F}(x))$$

$$= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\left[(x+\lambda h)^2 - 2(x+\lambda h) + 1, (x+\lambda h)^2 + 2\right] \ominus_{gH} \left[x^2 - 2x + 1, x^2 + 2\right]\right)$$

$$= \begin{cases} [2h(x-1),\ 2hx], & \text{if } h \geq 0 \\ [2hx,\ 2h(x-1)], & \text{otherwise.} \end{cases}$$

- Case *1. Let* $x \in [0, 1]$.

In this case, $2hx \geq 0$ *for* $h \geq 0$ *and* $2h(x-1) \geq 0$ *for* $h \leq 0$.

Hence,

$$\boldsymbol{F}_{\mathscr{D}}(x)(h) \nprec \boldsymbol{0} \text{ for all } h \in \mathbb{R}.$$

- Case *2. Let* $x \in \left(1, \frac{3}{2}\right]$.

Then, for $h < 0$ *we notice that*

$$\boldsymbol{F}_{\mathscr{D}}(x)(h) < \boldsymbol{0}.$$

- Case *3. Let* $x \in \left[-\frac{2}{5}, 0\right)$.

Then, for $h > 0$ *we see that*

$$\boldsymbol{F}_{\mathscr{D}}(x)(h) < \boldsymbol{0}.$$

Thus, the relation (2.1) of Theorem 2.4 holds only for $x \in [0, 1]$. *In Figure 2.1, the objective function* $\boldsymbol{F}$ *is depicted by the shaded region. From the Figure 2.1, we also see that each* $x \in [0, 1]$ *is an efficient point of the IOP (2.3).*

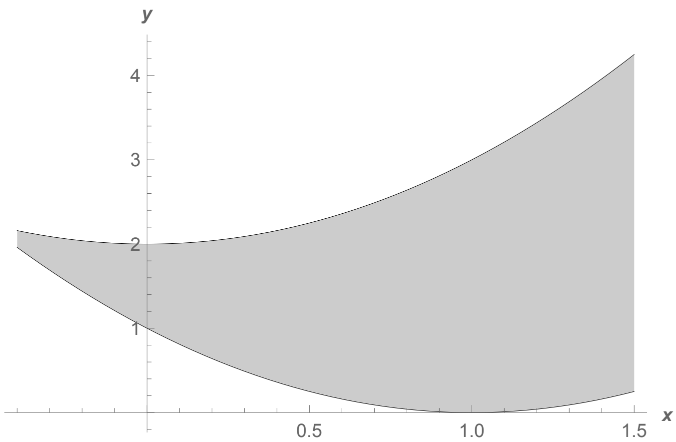

FIGURE 2.1: The IVF **F** of Example 2.2

2.5 Gâteaux Derivative of Interval-valued Functions

In this section, we define the linearity concept and gH-Gâteaux derivative for IVFs.

Definition 2.5.1 (Linear IVF)**.** *Let $\mathcal{S}$ be a linear subspace of $\mathbb{R}^n$. The function $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ is said to be linear if*

(i) $\boldsymbol{F}(\lambda x) = \lambda \odot \boldsymbol{F}(x)$ *for all* $x \in \mathcal{S}$ *and for all* $\lambda \in \mathbb{R}$ *and*

(ii) *for all* $x,\ y \in \mathcal{S}$,

$$\text{either } \boldsymbol{F}(x) \oplus \boldsymbol{F}(y) = \boldsymbol{F}(x+y)$$
$$\text{or none of } \boldsymbol{F}(x) \oplus \boldsymbol{F}(y) \text{ and } \boldsymbol{F}(x+y) \text{ dominates the other.}$$

Note 4. *Let $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be a linear IVF. If $\boldsymbol{F}(x) = \left[\underline{f}(x), \overline{f}(x)\right]$, then $\underline{f}(\lambda x) = \lambda \overline{f}(x)$ and $\overline{f}(\lambda x) = \lambda \underline{f}(x)$ for $\lambda < 0$.*

Lemma 2.5. *If an IVF $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ on a linear subspace $\mathcal{S}$ of $\mathbb{R}^n$ is linear, then*

$$\boldsymbol{F}(x) \nprec \boldsymbol{0} \Longrightarrow \boldsymbol{0} \nprec \boldsymbol{F}(-x).$$

Proof. Let $\mathbf{F}(x) \nprec \mathbf{0}$. Then,

$$\begin{aligned} & [\underline{f}(x), \overline{f}(x)] \nprec \mathbf{0} \\ \text{or,}\quad & \overline{f}(x) > 0 \\ \text{or,}\quad & -\overline{f}(x) < 0 \\ \text{or,}\quad & \underline{f}(-x) < 0, \text{ by Note 4} \\ \text{or,}\quad & \mathbf{0} \nprec [\underline{f}(-x), \overline{f}(-x)] \\ \text{or,}\quad & \mathbf{0} \nprec \mathbf{F}(-x). \end{aligned}$$

□

Definition 2.5.2 (Bounded linear operator)**.** *Let $\mathcal{X}$ be a normed linear space. A linear IVF $\mathbf{L} : \mathcal{X} \to I(\mathbb{R})$ is said to be a bounded linear operator if there exists $K > 0$*

such that

$$\|\mathbf{L}(h)\|_{I(\mathbb{R})} \leq K\|h\| \text{ for all } h \in \mathcal{X}.$$

Lemma 2.6. *Let $\mathcal{X}$ be a normed linear space. If the linear IVF $\mathbf{L} : \mathcal{X} \to I(\mathbb{R})$ is gH-continuous at the zero vector of $\mathcal{X}$, then it is a bounded linear operator.*

Proof. By the definition of gH-continuity of the IVF $\mathbf{L} : \mathcal{X} \to I(\mathbb{R})$ at the zero vector of $\mathcal{X}$, there exists $\delta > 0$ such that

$$\|\mathbf{L}(h) \ominus_{gH} \mathbf{L}(0)\|_{I(\mathbb{R})} \leq 1 \text{ for all } h \in \mathcal{X} \text{ with } \|h\| < \delta.$$

By Definition 2.5.1, we note that, $\mathbf{L}(0) = \mathbf{0}$. Hence,

$$\|\mathbf{L}(h)\|_{I(\mathbb{R})} \leq 1 \text{ for all } h \in \mathcal{X} \text{ with } \|h\| < \delta. \tag{2.4}$$

Thus, for all nonzero $h \in \mathcal{X}$, we have

$$\begin{aligned}
\|\mathbf{L}(h)\|_{I(\mathbb{R})} &= \left\| \frac{\|h\|}{\delta} \odot \mathbf{L}\left(\delta \frac{h}{\|h\|}\right) \right\|_{I(\mathbb{R})}, \text{ by the Definition 2.5.1} \\
&= \frac{\|h\|}{\delta} \left\| \mathbf{L}\left(\delta \frac{h}{\|h\|}\right) \right\|_{I(\mathbb{R})} \\
&\leq \frac{1}{\delta}\|h\|, \text{ by the inequality (2.4).}
\end{aligned}$$

Hence, $\mathbf{L}$ is a bounded linear operator. □

Lemma 2.7. *Let the IVF $\boldsymbol{F} : \mathbb{R}^2 \to I(\mathbb{R})$ be defined by*

$$\boldsymbol{F}(x_1, x_2) = x_1 \odot [\underline{a}, \overline{a}] \oplus x_2 \odot [\underline{b}, \overline{b}].$$

Then, $\boldsymbol{F}$ is a linear IVF.

Proof. See Appendix B.1. □

Note 5. *By a straightforward extension of the number of variables, the proof of the Lemma 2.7 shows that the IVF* $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ *which is defined by*

$$\boldsymbol{F}(x_1, x_2, \ldots, x_n) = x_1 \odot [\underline{a}_1, \overline{a}_1] \oplus x_2 \odot [\underline{a}_2, \overline{a}_2] \oplus \ldots \oplus x_n \odot [\underline{a}_n, \overline{a}_n]$$

is a linear IVF on $\mathbb{R}^n$*.*

Definition 2.5.3 (gH-Gâteaux derivative)**.** *Let* $\mathcal{S}$ *be a nonempty open subset of* $\mathbb{R}^n$ *and* $\boldsymbol{F}$ *be an IVF on* $\mathcal{S}$*. If at* $\bar{x} \in \mathcal{S}$*, the limit*

$$\boldsymbol{F}_{\mathscr{G}}(\bar{x})(h) = \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x}))$$

exists for all $h \in \mathbb{R}^n$ *and* $\boldsymbol{F}_{\mathscr{G}}(\bar{x})$ *is a* gH*-continuous linear IVF from* $\mathbb{R}^n$ *to* $I(\mathbb{R})$*, then* $\boldsymbol{F}_{\mathscr{G}}(\bar{x})$ *is said to be* gH*-Gâteaux derivative of* $\boldsymbol{F}$ *at* $\bar{x}$*. If* $\boldsymbol{F}$ *has a* gH*-Gâteaux derivative at* $\bar{x}$*, then* $\boldsymbol{F}$ *is said to be* gH*-Gâteaux differentiable at* $\bar{x}$*.*

Example 2.3. *Consider the IVF* $\boldsymbol{F}(x_1, x_2) = \boldsymbol{C}_1 \odot x_1 \oplus \boldsymbol{C}_2 \odot (x_2^2 e^{x_1})$ *with* $\boldsymbol{C}_1$*,* $\boldsymbol{C}_2 \in I(\mathbb{R})$*. For any* $(h_1, h_2) \in \mathbb{R}^2$*,*

$$\begin{aligned} \boldsymbol{F}_{\mathscr{G}}(0,0)(h_1,h_2) &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}((0,0) + \lambda(h_1,h_2)) \ominus_{gH} \boldsymbol{F}(0,0)) \\ &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\lambda \odot \boldsymbol{C}_1 \odot h_1 \oplus \lambda^2 \odot \boldsymbol{C}_2 \odot (h_2^2 e^{\lambda h_1})\right) \\ &= \boldsymbol{C}_1 \odot h_1. \end{aligned}$$

Clearly $\boldsymbol{F}_{\mathscr{G}}(0,0)$ *is linear and* gH*-continuous on* $\mathbb{R}^2$*. Therefore,* $\boldsymbol{F}$ *is a* gH*-Gâteaux differentiable function at* $(0,0)$*.*

Remark 2.5.1. *From Definitions 2.4.1 and 2.5.3 it is evident that if an IVF* $\boldsymbol{F}$ *on a nonempty open subset* $\mathcal{S}$ *of* $\mathbb{R}^n$ *has* gH*-Gâteaux derivative at* $\bar{x} \in \mathcal{S}$*, then* $\boldsymbol{F}$ *has*

gH-directional derivative at $\bar{x}$ in every direction $h \in \mathbb{R}^n$. However, the converse is not true. For instance, consider the IVF

$$\boldsymbol{F}(x_1, x_2) = \begin{cases} \boldsymbol{C}_1 \odot x_1 \oplus \boldsymbol{C}_2 \odot \left(x_2 e^{\frac{x_1}{x_2}}\right) & \text{if } x_2 \neq 0 \\ \boldsymbol{0} & \text{otherwise,} \end{cases}$$

where $\boldsymbol{C}_1,\ \boldsymbol{C}_2 \in I(\mathbb{R})$. The gH-directional derivative of $\boldsymbol{F}$ at $(0,0) \in \mathbb{R}^2$ in the direction $(h_1, h_2) \in \mathbb{R}^2$ is

$$\begin{aligned} \boldsymbol{F}_{\mathscr{D}}(0,0)(h_1,h_2) &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\boldsymbol{F}((0,0) + \lambda(h_1,h_2)) \ominus_{gH} \boldsymbol{F}(0,0)\right) \\ &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \boldsymbol{F}(\lambda h_1, \lambda h_2) \\ &= \begin{cases} \boldsymbol{C}_1 \odot h_1 \oplus \boldsymbol{C}_2 \odot \left(h_2 e^{\frac{h_1}{h_2}}\right), & \text{if } h_2 \neq 0 \\ \boldsymbol{0}, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, $\boldsymbol{F}$ has gH-directional derivative at $(0,0)$ in every direction (h_1, h_2). But $\boldsymbol{F}_{\mathscr{D}}(0,0)$ is not gH-continuous, and hence $\boldsymbol{F}$ has no gH-Gâteaux derivative at $(0,0)$.

Theorem 2.8. *Let $\mathcal{S}$ be a nonempty open convex subset of $\mathbb{R}^n$ and the function $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ has gH-Gâteaux derivative at every $\bar{x} \in \mathcal{S}$. If the function $\boldsymbol{F}$ is convex on $\mathcal{S}$, then*

$$\boldsymbol{F}(y) \nprec \boldsymbol{F}_{\mathscr{G}}(x)(y - x) \oplus \boldsymbol{F}(x) \text{ for all } x,\ y \in \mathcal{S}.$$

Proof. Let the function $\mathbf{F}$ be convex on $\mathcal{S}$. Then, for any $x,\ y \in \mathcal{S}$ and $\lambda,\ \lambda' \in (0,1]$ with $\lambda + \lambda' = 1$, we have

$$\mathbf{F}(x + \lambda(y - x)) = \mathbf{F}(\lambda y + \lambda' x) \preceq \lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(x).$$

Consequently,

$$\begin{aligned}
& \mathbf{F}(x+\lambda(y-x)) \ominus_{gH} \mathbf{F}(x) \\
\preceq\ & (\lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(x)) \ominus_{gH} \mathbf{F}(x) \\
=\ & \left[\lambda \underline{f}(y) + \lambda' \underline{f}(x), \lambda \overline{f}(y) + \lambda' \overline{f}(x)\right] \ominus_{gH} \left[\underline{f}(x), \overline{f}(x)\right] \\
=\ & \big[\min\{\lambda \underline{f}(y) + \lambda' \underline{f}(x) - \underline{f}(x), \lambda \overline{f}(y) + \lambda' \overline{f}(x) - \overline{f}(x)\}, \\
& \quad \max\{\lambda \underline{f}(y) + \lambda' \underline{f}(x) - \underline{f}(x), \lambda \overline{f}(y) + \lambda' \overline{f}(x) - \overline{f}(x)\}\big] \\
=\ & \big[\min\{\lambda \underline{f}(y) - \lambda \underline{f}(x), \lambda \overline{f}(y) - \lambda \overline{f}(x)\}, \\
& \quad \max\{\lambda \underline{f}(y) - \lambda \underline{f}(x), \lambda \overline{f}(y) - \lambda \overline{f}(x)\}\big] \\
=\ & \lambda \odot \big[\min\{\underline{f}(y) - \underline{f}(x), \overline{f}(y) - \overline{f}(x)\}, \\
& \qquad\qquad \max\{\underline{f}(y) - \underline{f}(x), \overline{f}(y) - \overline{f}(x)\}\big] \\
=\ & \lambda \odot (\mathbf{F}(y) \ominus_{gH} \mathbf{F}(x)),
\end{aligned}$$

which implies

$$\frac{1}{\lambda} \odot (\mathbf{F}(x+\lambda(y-x)) \ominus_{gH} \mathbf{F}(x)) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x).$$

As $\lambda \to 0+$, we obtain

$$\mathbf{F}_{\mathscr{G}}(x)(y-x) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x) \text{ for all } x,\ y \in \mathcal{S}. \tag{2.5}$$

If possible, let

$$\mathbf{F}(y') \prec \mathbf{F}(x') \oplus \mathbf{F}_{\mathscr{G}}(x')(y'-x') \text{ for some } x',\ y' \in \mathcal{S}.$$

Therefore,

$$\mathbf{F}(y') \ominus_{gH} \mathbf{F}(x') \prec \mathbf{F}_{\mathscr{G}}(x')(y'-x'),$$

which contradicts (5.3). Hence,

$$\mathbf{F}(y) \nprec \mathbf{F}_{\mathscr{G}}(x)(y-x) \oplus \mathbf{F}(x) \text{ for all } x,\ y \in \mathcal{S}.$$

□

Next result characterize the efficient points of IOPs.

Theorem 2.9 (Characterization of efficient points)**.** *Let $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF on a linear subspace $\mathcal{S}$ of $\mathbb{R}^n$ and $\bar{x} \in \mathcal{S}$ be an efficient point of the IOP (1.5). If the IVF $\boldsymbol{F}$ has a gH-Gâteaux derivative at $\bar{x}$ in every direction $h \in \mathbb{R}^n$, then*

$$0 \in \boldsymbol{F}_{\mathscr{G}}(\bar{x})(h) \textit{ for all } h \in \mathbb{R}^n.$$

The converse is true if $\mathcal{S}$ is convex and $\boldsymbol{F}$ is convex on $\mathcal{S}$.

Proof. Let the IVF $\mathbf{F}$ has a gH-Gâteaux derivative at $\bar{x}$. According to Theorem 2.4, we have

$$\mathbf{F}_{\mathscr{G}}(\bar{x})(x-\bar{x}) \nprec \mathbf{0} \text{ for all } x \in \mathbb{R}^n. \tag{2.6}$$

Let h be an arbitrary point in $\mathbb{R}^n$ such that $x = \bar{x} + h$. Then, by equation (2.6), we get

$$\mathbf{F}_{\mathscr{G}}(\bar{x})(h) \nsucc \mathbf{0}.$$

Again, if we take $x = \bar{x} - h$, then by equation (2.6) and Lemma 2.5 we obtain $\mathbf{0} \nprec \mathbf{F}_{\mathscr{G}}(\bar{x})(h)$.

Hence, by the last two relations, for all $h \in \mathbb{R}^n$, we have $0 \in \mathbf{F}_{\mathscr{G}}(\bar{x})(h)$.

Conversely, we consider that the function $\mathbf{F}$ is convex on $\mathcal{S}$ and $\mathbf{F}$ has a gH-Gâteaux derivative at $\bar{x}$ in every direction $h \in \mathbb{R}^n$. Let

$$0 \in \mathbf{F}_{\mathscr{G}}(\bar{x})(h) \text{ for all } h.$$

Then,

$$\mathbf{F}_{\mathscr{G}}(\bar{x})(h) \not\prec \mathbf{0} \text{ and } \mathbf{0} \not\prec \mathbf{F}_{\mathscr{G}}(\bar{x})(h) \text{ for all } h.$$

Since $\mathbf{F}_{\mathscr{G}}(\bar{x})(h) \not\prec \mathbf{0}$ for all h, by taking $h = x - \bar{x}$, we note that

$$\begin{aligned}
&\mathbf{F}_{\mathscr{G}}(\bar{x})(x - \bar{x}) \not\prec \mathbf{0} \text{ for all } x \in \mathbb{R}^n \\
\text{or, } &\mathbf{F}_{\mathscr{D}}(\bar{x})(x - \bar{x}) \not\prec \mathbf{0} \;\text{ for all } x \in \mathbb{R}^n, \text{ by Definition 2.5.3 and Remark 2.5.1} \\
\text{or, } &\mathbf{F}_{\mathscr{D}}(\bar{x})(x - \bar{x}) \nless \mathbf{0} \text{ for all } x \in \mathbb{R}^n
\end{aligned}$$

Hence, by Theorem 2.4, $\bar{x}$ is an efficient point of the IOP (1.5). □

Example 2.4. *Let us consider the IOP of Example 2.2. Here,*

$$\begin{aligned}
\boldsymbol{F}_{\mathscr{G}}(x)(h) &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(x + \lambda h) \ominus_{gH} \boldsymbol{F}(x)) \\
&= \begin{cases} [2h(x-1),\ 2hx] & \text{if } h \geq 0 \\ [2hx,\ 2h(x-1)] & \text{otherwise.} \end{cases} \\
&= 2h \odot [x - 1, x].
\end{aligned}$$

Clearly, $\boldsymbol{F}_{\mathscr{G}}(x)$ is linear (see Lemma 2.7) and gH-continuous in $h \in \mathbb{R}$ for each $x \in \left[-\frac{2}{5}, \frac{3}{2}\right]$. We see that at each $x \in [0, 1]$,

$$0 \in \boldsymbol{F}_{\mathscr{G}}(x)(h) \text{ for all } h \in \mathbb{R}.$$

Example 2.5. *In this example, we show that the condition '$0 \in \boldsymbol{F}_{\mathscr{G}}(\bar{x})(h)$ for all $h \in \mathbb{R}^n$' in Theorem 2.9 is necessary for an efficient point but not sufficient for nonconvex IVFs.*

Consider $\boldsymbol{F}(x) = [1, 2] \odot (-x^2) = [-2x^2, -x^2]$ and the IOP

$$\min_{x \in [-9,\ 0]} \boldsymbol{F}(x). \tag{2.7}$$

By a sketch of the graph of $\boldsymbol{F}(x)$, it is clear that $\boldsymbol{F}$ is not a convex IVF.
For $\bar{x} = 0$ and $h \in \mathbb{R}$,

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\bar{x} + \lambda h) \ominus_{gH} \boldsymbol{F}(\bar{x})) = \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\lambda h)) = \boldsymbol{0},$$

which is linear and gH-continuous. Therefore, $\boldsymbol{F}$ has a gH-Gâteaux derivative and at $\bar{x}$ and $\boldsymbol{F}_{\mathscr{G}}(\bar{x})(h) = \boldsymbol{0}$. Note that

$$0 \in \boldsymbol{F}_{\mathscr{G}}(\bar{x})(h) \text{ for all } h \in \mathbb{R},$$

but $\bar{x}$ is not an efficient point of the IOP (2.7) because

$$\boldsymbol{F}(x) \prec \boldsymbol{0} = \boldsymbol{F}(\bar{x}) \text{ for all } x \in [-9, 0).$$

2.6 Fréchet Derivatives of Interval-valued Functions

It is noteworthy that gH-Gâteaux derivative does not imply the gH-continuity of interval-valued function. For instance, consider the following example.

Example 2.6. *Let* $\boldsymbol{C} = [a, b]$ *and* $\boldsymbol{F} : \mathbb{R}^2 \to I(\mathbb{R})$ *be defined by*

$$\boldsymbol{F}(x_1, x_2) = \begin{cases} \left(\frac{x_1^8 x_2^2}{x_1^{16} + x_2^4}\right) \odot \boldsymbol{C} & \text{if } (x_1, x_2) \neq (0, 0) \\ \boldsymbol{0} & \text{otherwise.} \end{cases}$$

Then gH-directional derivative of $\boldsymbol{F}$ *at* $(0, 0) \in \mathbb{R}^2$ *in the direction* $(h_1, h_2) \in \mathbb{R}^2$ *is given by*

$$\boldsymbol{F}_{\mathscr{D}}(0,0)(h) \;=\; \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}((0,0) \oplus \lambda(h_1, h_2)) \ominus_{gH} \boldsymbol{F}(0,0)) \;=\; \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \boldsymbol{F}(\lambda h) \;=\; \boldsymbol{0}.$$

Clearly $\boldsymbol{F}_{\mathscr{D}}(0, 0)$ *is gH-continuous and linear in h. Therefore,* $\boldsymbol{F}$ *has gH-Gâteaux derivative at* $(0, 0)$*. But*

$$\lim_{\|h\| \to 0} (\boldsymbol{F}(h_1, h_2) \ominus_{gH} \boldsymbol{F}(0, 0)) \neq \boldsymbol{0}$$

Thus, $\boldsymbol{F}$ *is not gH-continuous at* $(0, 0)$*.*

In this section, we present a stronger concept of a derivative for an IVF from which gH-continuity is implied. In the following definition, we use the fact that $I(\mathbb{R})$ is a normed quasilinear space.

Definition 2.6.1 (gH-Fréchet derivative)**.** *Let* $\mathcal{S}$ *be a nonempty open subset of* $\mathbb{R}^n$ *and* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *be an IVF. For an* $\bar{x} \in \mathcal{S}$*, if there exists a gH-continuous and linear mapping* $\boldsymbol{G} : \mathbb{R}^n \to I(\mathbb{R})$ *with the following property*

$$\lim_{\|h\| \to 0} \frac{\|\boldsymbol{F}(\bar{x} + h) \ominus_{gH} \boldsymbol{F}(\bar{x}) \ominus_{gH} \boldsymbol{G}(h)\|_{I(\mathbb{R})}}{\|h\|} = 0,$$

then $\boldsymbol{F}$ *is said to have a gH-*Fréchet derivative *at* $\bar{x}$ *and we write* $\boldsymbol{G} = \boldsymbol{F}_{\mathscr{F}}(\bar{x})$*.*

Example 2.7. *Consider the IVF* $\boldsymbol{F}(x_1, x_2) = \boldsymbol{C}_1 \odot x_1 \oplus \boldsymbol{C}_2 \odot x_2^2$*, where* $\boldsymbol{C}_1$ *and* $\boldsymbol{C}_2$ *are any two closed intervals. For* $(0,0) \in \mathbb{R}^2$ *and any* $h = (h_1, h_2) \in \mathbb{R}^2$*, we have*

$$\begin{aligned}
\boldsymbol{F}_{\mathscr{G}}(0,0)(h) &:= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}((0,0) + \lambda(h_1, h_2)) \ominus_{gH} \boldsymbol{F}(0,0)) \\
&= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\lambda \odot \boldsymbol{C}_1 \odot h_1 \oplus \lambda^2 \odot \boldsymbol{C}_2 \odot h_2^2\right) \\
&= \boldsymbol{C}_1 \odot h_1.
\end{aligned}$$

We note that $\boldsymbol{F}_{\mathscr{G}}(0,0)$ *is a gH-continuous and linear mapping (by Note 5) from* $\mathbb{R}^2$ *to* $I(\mathbb{R})$*.*

Taking $\boldsymbol{G} = \boldsymbol{F}_{\mathscr{G}}(0,0)$*, we see that*

$$\begin{aligned}
& \lim_{\|h\| \to 0} \frac{\|\boldsymbol{F}(h) \ominus_{gH} \boldsymbol{F}(0,0) \ominus_{gH} \boldsymbol{G}(h)\|_{I(\mathbb{R})}}{\|h\|} \\
= & \lim_{\|h\| \to 0} \frac{\|\boldsymbol{C}_1 \odot h_1 \oplus \boldsymbol{C}_2 \odot h_2^2 \ominus_{gH} \boldsymbol{0} \ominus_{gH} \ \boldsymbol{C}_1 \odot h_1\|_{I(\mathbb{R})}}{\sqrt{h_1^2 + h_2^2}} \\
= & \lim_{\|h\| \to 0} \frac{\|\boldsymbol{C}_2 \odot h_2^2\|_{I(\mathbb{R})}}{\sqrt{h_1^2 + h_2^2}} \\
= & \lim_{\|h\| \to 0} \frac{h_2^2 \|\boldsymbol{C}_2\|_{I(\mathbb{R})}}{\sqrt{h_1^2 + h_2^2}} \\
= & \lim_{\|h\| \to 0} \frac{h_2^2 k}{\sqrt{h_1^2 + h_2^2}}, \text{ where } k = \|\boldsymbol{C}_2\|_{I(\mathbb{R})} \\
= & \ 0.
\end{aligned}$$

Hence, the Fréchet derivative of $\boldsymbol{F}$ *at* $(0,0)$ *is* $\boldsymbol{F}_{\mathscr{F}}(0,0)(h_1, h_2) = \boldsymbol{G}(h_1, h_2) = \boldsymbol{C}_1 \odot h_1$*.*

Theorem 2.10. *Let* $\mathcal{S}$ *be a normed linear subspace of* $\mathbb{R}^n$*. If the function* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *has a gH-Fréchet derivative at* $\bar{x} \in \mathcal{S}$*, then the function* $\boldsymbol{F}$ *is gH-continuous at* $\bar{x}$*.*

Proof. Let $\mathbf{F}_{\mathscr{F}}(\bar{x})$ be the Fréchet derivative of $\mathbf{F}$ at $\bar{x}$. Evidently, $\mathbf{F}_{\mathscr{F}}(\bar{x})$ is a gH-continuous and linear IVF. Hence, by Lemma 2.6, there exists an $\alpha > 0$ such that

$$\|\mathbf{F}_{\mathscr{F}}(\bar{x})(h)\|_{I(\mathbb{R})} \leq \alpha\|h\| \text{ for all } h \in \mathbb{R}^n.$$

As the function $\mathbf{F} : \mathcal{S} \to I(\mathbb{R})$ is gH-Fréchet differentiable at $\bar{x} \in \mathcal{S}$, for $\varepsilon > 0$ and $\bar{x} + h \in B(\bar{x}, \varepsilon)$, by Definition 2.6.1, we have

$$\|\mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{F}}(\bar{x})(h)\|_{I(\mathbb{R})} \leq \varepsilon\|h\|.$$

Hence, with the help of (i) of Lemma 1.6, for all h with $\bar{x} + h \in B(\bar{x}, \varepsilon)$,

$$\begin{aligned}
& \|\mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x})\|_{I(\mathbb{R})} \\
= \ & \|\mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x})\|_{I(\mathbb{R})} - \|\mathbf{F}_{\mathscr{F}}(\bar{x})(h)\|_{I(\mathbb{R})} + \|\mathbf{F}_{\mathscr{F}}(\bar{x})(h)\|_{I(\mathbb{R})} \\
\leq \ & \|\mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{F}}(\bar{x})(h)\|_{I(\mathbb{R})} + \|\mathbf{F}_{\mathscr{F}}(\bar{x})(h)\|_{I(\mathbb{R})} \\
\leq \ & \varepsilon\|h\| + \alpha\|h\| \\
= \ & (\varepsilon + \alpha)\|h\|.
\end{aligned}$$

Thus, $\lim_{\|h\| \to 0} (\mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x})) = \mathbf{0}$, and hence the function $\mathbf{F}$ is gH-continuous at $\bar{x}$. □

Theorem 2.11. *Let $\mathcal{S}$ be a nonempty open subset of $\mathbb{R}^n$. If the gH-Fréchet derivative of the function $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ at some $\bar{x} \in \mathcal{S}$ exists, then the gH-Gâteaux derivative of $\boldsymbol{F}$ at $\bar{x}$ along any $h \in \mathbb{R}^n$ exists and both the derivative values are equal.*

Proof. Let $\mathbf{F}_{\mathscr{F}}(\bar{x})$ be the gH-Fréchet derivative of $\mathbf{F}$ at $\bar{x} \in \mathcal{S}$. Then, we have

$$\lim_{\lambda \to 0+} \frac{\|\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{F}}(\bar{x})(\lambda h)\|_{I(\mathbb{R})}}{\|\lambda h\|} = 0 \text{ for all } h \in \mathbb{R}^n \backslash \{0\}$$

$$\text{or, } \lim_{\lambda \to 0+} \frac{1}{\lambda} \|\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{F}}(\bar{x})(\lambda h)\|_{I(\mathbb{R})} = 0 \text{ for all } h \in \mathbb{R}^n \backslash \{0\}. \tag{2.8}$$

Since $\mathbf{F}_{\mathscr{F}}(\bar{x})$ is linear, and thus $\mathbf{F}_{\mathscr{F}}(\bar{x})(\lambda\, h) = \lambda \odot \mathbf{F}_{\mathscr{F}}(\bar{x})(h)$, the equation (5.1) gives

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{F}}(\bar{x})(h)) = \mathbf{0} \text{ for all } h \in \mathbb{R}^n \backslash \{0\}$$

$$\text{or, } \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot [\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x})] = \mathbf{F}_{\mathscr{F}}(\bar{x})(h) \text{ for all } h \in \mathbb{R}^n \backslash \{0\}.$$

Therefore, $\mathbf{F}$ is gH-Gâteaux differentiable at $\bar{x}$ and $\mathbf{F}_{\mathscr{G}}(\bar{x}) = \mathbf{F}_{\mathscr{F}}(\bar{x})$. □

Example 2.8. *There are some IVFs which are gH-Gâteaux differentiable but not gH-Fréchet differentiable. For instance, for* $\boldsymbol{C} = [a, b]$*, consider the IVF* $\boldsymbol{F} : \mathbb{R}^2 \to I(\mathbb{R})$ *which is defined by*

$$\boldsymbol{F}(x_1, x_2) = \begin{cases} \left(\frac{x_1^2 x_2}{x_1^4 + x_2^2} \sqrt{x_1^2 + x_2^2}\right) \odot \boldsymbol{C}, & \text{if } (x_1, x_2) \neq (0,0) \\ \boldsymbol{0}, & \text{otherwise.} \end{cases}$$

At $(0,0) \in \mathbb{R}^2$*, for* $h = (h_1, h_2) \in \mathbb{R}^2$*, we note that*

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}((0,0) + \lambda(h_1, h_2)) \ominus_{gH} \boldsymbol{F}(0,0)) = \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \boldsymbol{F}(\lambda h_1, \lambda h_2) = \boldsymbol{0}.$$

Since the function $\boldsymbol{G}(h) = \boldsymbol{0}$ *is gH-continuous and linear in h,* $\boldsymbol{F}$ *has gH-Gâteaux derivative at* $(0,0)$ *and* $\boldsymbol{F}_{\mathscr{G}}(0,0)(h) = \boldsymbol{0}$ *for all* $h \in \mathbb{R}^2$.

If $\boldsymbol{F}$ *is Fréchet differentiable at* $\bar{x}$*, then by Theorem 2.11,* $\boldsymbol{F}_{\mathscr{F}}(0,0)(h) = \boldsymbol{F}_{\mathscr{G}}(0,0)(h) = \boldsymbol{0}$.

But we notice that

$$\begin{aligned}
&\lim_{\|h\|\to 0} \frac{\|\boldsymbol{F}(h) \ominus_{gH} \boldsymbol{F}(0,0) \ominus_{gH} \boldsymbol{F}_{\mathscr{G}}(0,0)(h)\|_{I(\mathbb{R})}}{\|h\|} \\
&= \lim_{\|h\|\to 0} \frac{\|\boldsymbol{F}(h)\|_{I(\mathbb{R})}}{\|h\|} \\
&= \lim_{\|h\|\to 0} \frac{|\ h_1^2 h_2 \sqrt{h_1^2+h_2^2}\ |\ \|\boldsymbol{C}\|_{I(\mathbb{R})}}{(h_1^4+h_2^2)\sqrt{h_1^2+h_2^2}} \\
&= \lim_{\|h\|\to 0} \frac{|\ h_1^2 h_2 \sqrt{h_1^2+h_2^2}\ |\ \|\boldsymbol{C}\|_{I(\mathbb{R})}}{(h_1^4+h_2^2)\sqrt{h_1^2+h_2^2}} \\
&= \lim_{\|h\|\to 0} \frac{|\ h_1^2 h_2\ |\ k}{h_1^4+h_2^2}, \ \textit{where} \ \ k = \|\boldsymbol{C}_2\|_{I(\mathbb{R})}
\end{aligned}$$

which does not exist. Thus, $\boldsymbol{F}$ *is not gH-Fréchet differentiable at* $(0,0)$.

Definition 2.6.2 (gH-Lipschitz continuous IVF)**.** *An IVF* $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ *is said to be gH-Lipschitz continuous on* $\mathcal{S} \subseteq \mathbb{R}^n$ *if there exists* $M > 0$ *such that*

$$\|\boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(y)\|_{I(\mathbb{R})} \leq M\|x-y\| \text{ for all } x, y \in \mathcal{S}.$$

The constant M *is called a Lipschitz constant.*

Theorem 2.12. *Let the IVF* $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ *be gH-Lipschitz continuous on* $\mathbb{R}^n$ *and gH-Gâteaux differentiable at* $\bar{x} \in \mathbb{R}^n$*. Then,* $\boldsymbol{F}$ *is gH-Fréchet differentiable at* $\bar{x}$.

Proof. Let $\mathcal{S}$ be the closed unit sphere of $\mathbb{R}^n$.

Since $\mathcal{S}$ is totally bounded, for any given $\varepsilon > 0$, there exists a finite set $\mathcal{A} = \{a_1, a_2, \cdots, a_k\} \subset \mathbb{R}^n$ such that $\mathcal{S} \subset \bigcup_{a_i \in \mathcal{A}} B(a_i, \varepsilon)$, where $B(a_i, \varepsilon)$ is the open ball of radius ε centered at a_i.

Thus, for any $s \in \mathcal{S}$ there exists $a_i \in \mathcal{A}$ such that $\|s - a_i\| < \varepsilon$.

Since $\mathbf{F}$ is gH-Lipschitz continuous, there exists $M > 0$ such that

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq M\|x - y\| \text{ for all } x, y \in \mathbb{R}^n.$$

As $\mathbf{F}$ is gH-Gâteaux differentiable at an $\bar{x}$, for any $i = 1, 2, \ldots, k$, we have a continuous linear mapping $\mathbf{F}_{\mathscr{G}}(\bar{x})$ such that

$$\begin{aligned}\mathbf{F}_{\mathscr{G}}(\bar{x})(a_i) &= \lim_{\lambda\to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda a_i) \ominus_{gH} \mathbf{F}(\bar{x}))\\ \text{i.e., } \lim_{\lambda\to 0+} \frac{1}{\lambda} &\odot (\mathbf{F}(\bar{x} + \lambda a_i) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda \odot \mathbf{F}_{\mathscr{G}}(\bar{x})(a_i)) = \mathbf{0}.\end{aligned}$$

Thus, for the chosen $\varepsilon > 0$, for each $i = 1, 2, \ldots, k$, there exists $\delta_i > 0$ such that for $|\lambda| < \delta_i$,

$$\|\mathbf{F}(\bar{x} + \lambda a_i) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda \odot \mathbf{F}_{\mathscr{G}}(\bar{x})(a_i)\|_{I(\mathbb{R})} < \varepsilon\ |\lambda|. \tag{2.9}$$

Hence, with the help of (iv) Lemma 1.6 and Corollary 1.4.1, for any $s \in \mathcal{S}$ and $|\lambda| < \delta = \min\{\delta_1, \delta_2, \ldots, \delta_k\}$, we get

$$\begin{aligned}&\|\mathbf{F}(\bar{x} + \lambda s) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda \odot \mathbf{F}_{\mathscr{G}}(\bar{x})(s)\|_{I(\mathbb{R})}\\ \leq\ & \|\mathbf{F}(\bar{x} + \lambda s) \ominus_{gH} \mathbf{F}(\bar{x} + \lambda a_i)\|_{I(\mathbb{R})} + \|\mathbf{F}(\bar{x} + \lambda a_i) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{G}}(\bar{x})(s)\|_{I(\mathbb{R})}\\ & + \|\lambda \odot (\mathbf{F}_{\mathscr{G}}(\bar{x})(s) \ominus_{gH} \mathbf{F}_{\mathscr{G}}(\bar{x})(a_i))\|_{I(\mathbb{R})}\\ \leq\ & \big(M + M + B\big)|\lambda|\varepsilon, \text{ for some } B > 0, \text{ by (2.9) and Lemma 2.6}\\ \leq\ & \big(2M + B\big)|\lambda|\varepsilon\\ =\ & \big(2M + B\big)\|\lambda\ s\|\ \varepsilon, \text{ as } \|s\| = 1.\end{aligned}$$

Hence,

$$\lim_{\lambda\to 0} \frac{\|\mathbf{F}(\bar{x} + \lambda s) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda \odot \mathbf{F}_{\mathscr{G}}(\bar{x})(s)\|_{I(\mathbb{R})}}{\|\lambda\ s\|} = 0,$$

and so $\mathbf{F}$ is gH-Fréchet differentiable at $\bar{x}$. □

Theorem 2.13 (Chain rule)**.** *Let $\mathcal{W}$ and $\mathcal{S}$ be nonempty open subsets of $\mathbb{R}^n$ and $\mathbb{R}$, respectively. If the function $f : \mathcal{W} \to \mathcal{S}$ is differentiable at $x \in \mathcal{X}$ and $\boldsymbol{G} : \mathcal{S} \to I(\mathbb{R})$ has a gH-Fréchet derivative at $f(x)$. Suppose there exists $\boldsymbol{A} \in I(\mathbb{R})$ such that for all $k \in \mathbb{R}$, $\boldsymbol{G}_{\mathscr{F}}(y)(k) = k \odot \boldsymbol{A}$. Then, the composite function $\boldsymbol{G} \circ f : \mathcal{W} \to I(\mathbb{R})$ has a gH-Fréchet derivative at $x \in \mathcal{W}$, and $(\boldsymbol{G} \circ f)_{\mathscr{F}}(x) = \nabla f(x) \odot \boldsymbol{G}_{\mathscr{F}}(f(x))$.*

Proof. Let $\mathbf{F} = \mathbf{G} \circ f$ and $y = f(x)$. By the definitions of differentiability of f at x and gH-Fréchet derivative of $\mathbf{G}$ at $y = f(x)$, we have

$$
\begin{aligned}
& f(x+h) - f(x) = h^T\ \nabla f(x) + \|h\|\xi(h), \text{ where } \lim_{\|h\|\to 0} \xi(h) = 0 \\
\text{and} \quad & \mathbf{G}(y+k) \ominus_{gH} \mathbf{G}(y) = \mathbf{G}_{\mathscr{F}}(y)(k) \oplus k \odot \mathbf{E}(k), \text{ where } \lim_{k\to 0} \mathbf{E}(k) = \mathbf{0}.
\end{aligned}
$$

Let $\phi(h) = h^T\ \nabla f(x) + \|h\|\ \xi(h)$. To prove the theorem, we show that $\mathbf{F}_{\mathscr{F}}(x) = \nabla f(x) \odot \mathbf{G}_{\mathscr{F}}(y)$. We note that

$$
\begin{aligned}
\mathbf{F}(x+h) \ominus_{gH} \mathbf{F}(x) &= \mathbf{G}(f(x+h)) \ominus_{gH} \mathbf{G}(f(x)) \\
&= \mathbf{G}\left(f(x) + h^T \nabla f(x) + \|h\|\ \xi(h)\right) \ominus_{gH} \mathbf{G}(y) \\
&= \mathbf{G}(y + \phi(h)) \ominus_{gH} \mathbf{G}(y) \\
&= \mathbf{G}_{\mathscr{F}}(y)(\phi(h)) \oplus \phi(h) \odot \mathbf{E}(\phi(h)) \\
&= \phi(h) \odot \mathbf{A} \oplus \phi(h) \odot \mathbf{E}(\phi(h)) \\
&= \left(h^T\ \nabla f(x) + \|h\|\ \xi(h)\right) \odot \mathbf{A} \oplus \phi(h) \odot \mathbf{E}(\phi(h)) \\
&= h^T\ \nabla f(x) \odot \mathbf{A} + \|h\|\ \xi(h) \odot \mathbf{A} \oplus \phi(h) \odot \mathbf{E}(\phi(h)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \lim_{\|h\|\to 0} \frac{1}{\|h\|} \left\|\mathbf{F}(x+h) \ominus_{gH} \mathbf{F}(x) \ominus_{gH} h^T \odot (\nabla f(x) \odot \mathbf{A})\right\| \\
= & \lim_{\|h\|\to 0} \left(\xi(h) \odot \mathbf{A} \oplus \frac{\phi(h)}{\|h\|} \odot \mathbf{E}(\phi(h)\right)
\end{aligned}
$$

$$= \lim_{\|h\|\to 0}\left(\xi(h)\odot \mathbf{A}\oplus\left(\frac{h^T\nabla f(x)}{\|h\|}+\xi(h)\right)\odot \mathbf{E}(\phi(h)\right)$$
$$= \mathbf{0},\ \text{since} \lim_{\|h\|\to 0}\phi(h)=0.$$

As the IVF $h^T\odot(\nabla f(x)\odot \mathbf{A})$ is linear in h and gH-continuous, the function $\mathbf{F}$ is gH-Fréchet differentiable at $x\in\mathcal{W}$ and $\mathbf{F}_{\mathscr{F}}(x)=\nabla f(x)\odot\mathbf{A}=\nabla f(x)\odot \mathbf{G}_{\mathscr{F}}(f(x))$. □

Example 2.9. *Let $f:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ be defined by $f(x_1,x_2)=\sin(x_1)+e^{x_2}$ and $\boldsymbol{G}:\mathbb{R}^+\to I(\mathbb{R})$ be the IVF*

$$\boldsymbol{G}(x)=x^2\odot[1,2]\oplus x\odot[-1,0].$$

Then, $\boldsymbol{G}\circ f:\mathbb{R}^+\times\mathbb{R}^+\to I(\mathbb{R})$ has the expression

$$(\boldsymbol{G}\circ f)(x_1,x_2)=(\sin(x_1)+e^{x_2})^2\odot[1,2]\oplus(\sin(x_1)+e^{x_2})[-1,0]$$

and $\nabla f(\bar{x}_1,\bar{x}_2)=(\cos(\bar{x}_1),e^{\bar{x}_2})$.
Let $\bar{x}=f(\bar{x}_1,\bar{x}_2)=\sin(\bar{x}_1)+e^{\bar{x}_2}$.

Then, for every $h\in\mathbb{R}^+$, we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\frac{1}{\lambda}\odot\Big(\boldsymbol{G}(\bar{x}+\lambda h)\ominus_{gH}\boldsymbol{G}(\bar{x})\Big)\\
=&\lim_{\lambda\to 0}\frac{1}{\lambda}\odot\Big(\big((\bar{x}+\lambda h)^2\odot[1,2]\oplus(\bar{x}+\lambda h)\odot[-1,0]\big)\ominus_{gH}\big(\bar{x}^2\odot[1,2]\oplus\bar{x}\odot[-1,0]\big)\Big)\\
=&\lim_{\lambda\to 0}\frac{1}{\lambda}\odot\Big(\big[(\bar{x}+\lambda h)^2-(\bar{x}+\lambda h),2(\bar{x}+\lambda h)^2\big]\ominus_{gH}\big[\bar{x}^2-\bar{x},2\bar{x}^2\big]\Big)\\
=&\lim_{\lambda\to 0}\frac{1}{\lambda}\odot\big[\min\{(\bar{x}+\lambda h)^2-(\bar{x}+\lambda h)-\bar{x}^2+\bar{x},2(\bar{x}+\lambda h)^2-2\bar{x}^2\},\\
&\qquad\max\{(\bar{x}+\lambda h)^2-(\bar{x}+\lambda h)-\bar{x}^2+\bar{x},2(\bar{x}+\lambda h)^2-2\bar{x}^2\}\big]\\
=&\big[\min\big\{\lim_{\lambda\to 0}\tfrac{1}{\lambda}\big((\bar{x}+\lambda h)^2-(\bar{x}+\lambda h)-\bar{x}^2+\bar{x}\big),\lim_{\lambda\to 0}\tfrac{1}{\lambda}\big(2(\bar{x}+\lambda h)^2-2\bar{x}^2\big)\big\},\\
&\max\big\{\lim_{\lambda\to 0}\tfrac{1}{\lambda}\big((\bar{x}+\lambda h)^2-(\bar{x}+\lambda h)-\bar{x}^2+\bar{x}\big),\lim_{\lambda\to 0}\tfrac{1}{\lambda}\big(2(\bar{x}+\lambda h)^2-2\bar{x}^2\big)\big\}\big]
\end{aligned}$$

$$\begin{aligned}
&= [\min\{2\bar{x}h - h, 4\bar{x}h\}, \ \max\{2\bar{x}h - h, 4\bar{x}h\}] \\
&= [2\bar{x}h - h, 4\bar{x}h] \\
&= [2\bar{x} - 1, 4\bar{x}] \odot h,
\end{aligned}$$

which is a linear gH-continuous IVF (see Note 5). Thus, the gH-Gâteaux derivative of $\boldsymbol{G}$ *at* $\bar{x} \in \mathbb{R}$ *is* $\boldsymbol{G}_{\mathscr{G}}(\bar{x}) := [2\bar{x} - 1, 4\bar{x}]$*. One can check that*

$$\lim_{|h| \to 0} \frac{\| \boldsymbol{G}(\bar{x} + h) \ominus_{gH} \boldsymbol{G}(\bar{x}) \ominus_{gH} \boldsymbol{G}_{\mathscr{G}}(\bar{x})(h) \|_{I(\mathbb{R})}}{|h|} = 0.$$

Hence, $\boldsymbol{G}_{\mathscr{F}}(\bar{x}) = \boldsymbol{G}_{\mathscr{G}}(\bar{x}) = [2\bar{x} - 1, 4\bar{x}]$ *and thus*

$$\boldsymbol{G}_{\mathscr{F}}(f(\bar{x}_1, \bar{x}_2)) = [2\sin(\bar{x}_1) + 2e^{\bar{x}_2} - 1, 4\sin(\bar{x}_1) + 4e^{\bar{x}_2}].$$

Again, for every $(h_1, h_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ *we have*

$$\begin{aligned}
&\lim_{\lambda \to 0} \frac{1}{\lambda} \odot \Big(\boldsymbol{G} \circ f((\bar{x}_1, \bar{x}_2) + \lambda(h_1, h_2)) \ominus_{gH} \boldsymbol{G} \circ f(\bar{x}_1, \bar{x}_2) \Big) \\
= \ &\lim_{\lambda \to 0} \frac{1}{\lambda} \odot \Big(\big((\sin(\bar{x}_1 + \lambda h_1) + e^{\bar{x}_2 + \lambda h_2})^2 \odot [1, 2] \\
&\qquad \oplus (\sin(\bar{x}_1 + \lambda h_1) + e^{\bar{x}_2 + \lambda h_2}) \odot [-1, 0] \big) \\
&\qquad \ominus_{gH} \big((\sin(\bar{x}_1) + e^{\bar{x}_2})^2 \odot [1, 2] \oplus (\sin(\bar{x}_1) + e^{\bar{x}_2}) \odot [-1, 0] \big) \Big) \\
= \ &\lim_{\lambda \to 0} \frac{1}{\lambda} \odot \Big(\big[(\sin(\bar{x}_1 + \lambda h_1) + e^{\bar{x}_2 + \lambda h_2})^2 - (\sin(\bar{x}_1 + \lambda h_1) + e^{\bar{x}_2 + \lambda h_2}), \\
&\qquad 2(\sin(\bar{x}_1 + \lambda h_1) + e^{\bar{x}_2 + \lambda h_2})^2 \big] \\
&\qquad \ominus_{gH} \big[(\sin(\bar{x}_1) + e^{\bar{x}_2})^2 - (\sin(\bar{x}_1) + e^{\bar{x}}_2), 2(\sin(\bar{x}_1) + e^{\bar{x}_2})^2 \big] \Big)
\end{aligned}$$

$$\begin{aligned}
= \ &\lim_{\lambda \to 0} \frac{1}{\lambda} \odot \big[\min \big\{ (\sin(\bar{x}_1 + \lambda h_1) + e^{\bar{x}_2 + \lambda h_2})^2 - (\sin(\bar{x}_1 + \lambda h_1) + e^{\bar{x}_2 + \lambda h_2}) \\
&\qquad - (\sin(\bar{x}_1 + e^{\bar{x}_2})^2 + (\sin(\bar{x}_1 + e^{\bar{x}_2}), \\
&\qquad 2(\sin(\bar{x}_1 + \lambda h_1) + e^{\bar{x}_2 + \lambda h_2})^2 - 2(\sin(\bar{x}_1 + e^{\bar{x}_2}) \big\},
\end{aligned}$$

$$
\begin{aligned}
&\max\{(\sin(\bar{x}_1+\lambda h_1)+e^{\bar{x}_2+\lambda h_2})^2-(\sin(\bar{x}_1+\lambda h_1)+e^{\bar{x}_2+\lambda h_2})\\
&\quad-(\sin(\bar{x}_1+e^{\bar{x}_2})^2+(\sin(\bar{x}_1)+e^{\bar{x}_2}),\\
&\quad 2(\sin(\bar{x}_1+\lambda h_1)+e^{\bar{x}_2+\lambda h_2})^2-2(\sin(\bar{x}_1)+e^{\bar{x}_2})\big\}\big]\\
=&\ \big[\min\big\{2h_1\cos(\bar{x}_1)(\sin(\bar{x}_1)+e^{\bar{x}_2})+2h_2e^{\bar{x}_2}(\sin(\bar{x_1})+e^{\bar{x}_2})\\
&\quad-(h_1\cos(\bar{x}_1+h_2e^{\bar{x}_2}),\\
&\quad 4h_1\cos(\bar{x}_1)(\sin(\bar{x}_1)+e^{\bar{x}_2})+4h_2e^{\bar{x}_2}(\sin(\bar{x}_1+e^{\bar{x}_2})\big\},\\
&\max\big\{2h_1\cos(\bar{x}_1)(\sin(\bar{x}_1)+e^{\bar{x}_2})+2h_2e^{\bar{x}_2}(\sin(\bar{x_1})+e^{\bar{x}_2})\\
&\quad-(h_1\cos(\bar{x}_1+h_2e^{\bar{x}_2}),\\
&\quad 4h_1\cos(\bar{x}_1)(\sin(\bar{x}_1)+e^{\bar{x}_2})+4h_2e^{\bar{x}_2}(\sin(\bar{x}_1+e^{\bar{x}_2})\big\}\big]\\
=&\ \big[(h_1,h_2)(\cos(\bar{x_1}),e^{\bar{x}_2})(2\sin(\bar{x_1})+2e^{\bar{x_2}}-1),\\
&\quad(h_1,h_2)(\cos(\bar{x_1}),e^{\bar{x}_2})(4\sin(\bar{x_1})+4e^{\bar{x_2}})\big]\\
=&\ (h_1,h_2)(\cos(\bar{x_1}),e^{\bar{x}_2})\odot[2\sin(\bar{x_1})+2e^{\bar{x_2}}-1,4\sin(\bar{x_1})+4e^{\bar{x_2}}].
\end{aligned}
$$

Thus, the gH-Fréchet derivative of $\boldsymbol{G}\circ f$ *at the point* $(\bar{x}_1,\bar{x}_2)$ *is*

$$
\begin{aligned}
(\boldsymbol{G}\circ f)_{\mathscr{F}}(\bar{x}_1,\bar{x}_2) &= (\cos(\bar{x_1}),e^{\bar{x}_2})\odot[2\sin(\bar{x_1})+2e^{\bar{x_2}}-1,4\sin(\bar{x_1})+4e^{\bar{x_2}}]\\
&= \boldsymbol{G}_{\mathscr{F}}(f(\bar{x}_1,\bar{x}_2))\odot\nabla f(\bar{x}_1,\bar{x}_2).
\end{aligned}
$$

2.7 Concluding remarks

This chapter introduced the notions of directional, Gâteaux and Fréchet derivatives for IVFs. To define these derivatives and to study their properties, the concepts of gH-Lipschitz continuity, linear IVF, bounded linear IVF, monotonic, and bounded IVFs have been proposed. By using proposed concepts, the efficient solutions of IOPs have been characterized. All the ideas can easily be used in control systems and differential equations in a noisy or uncertain environment. Future research can focus on solving differential equations in a noisy or uncertain environment.

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