Chapter 1

Introduction

Optimization or mathematical programming is a collection of mathematical principles and techniques to find minima/maxima of a function or of a criterion over a set of constraints, which express as restrictions or rules on the problem. Now-a-days, classical optimization and its several branches have sound theoretical foundations and are featured by a vast collection of sophisticated algorithms and softwares. It has become an essential tool for powerful modeling and decision making in a wide range of applications in management science, industry, and engineering field.

Mathematically, an optimization problem comprises of three key ingredients—

- A set of *variables* or unknowns, called decision variables. Which designates a value that may vary within the scope of the given problem.
- A set of *constraints* which is a set allowable values or scopes for the variables. Constraints are typically determined by functional inequalities or equalities.
- An *objective function* that expresses the main criteria of the problem is either to be minimized or maximized satisfying the constraints.

Although there are many theories and optimization tools to obtain optimal solutions, it is not always possible to properly represent the real-world situations with classical mathematics due to the presence of uncertain events or environments. Most often, the recorded or collected data are inherently imprecise or inexact. The data may be affected by measurement errors or by random events. Sometimes the data may be roughly estimated, or it is more appropriate to assume that the data belongs to some uncertain set or interval. There are two major causes behind this imperfection or uncertainty of the information—imprecision and randomness.

Randomness is measured by the probability density function. An imprecise or ill-defined, or vaguely defined set is represented by a fuzzy set or set of intervals that provide upper and lower bounds of the imprecise variables. The optimization problems with interval coefficients are known as interval optimization problems. In this thesis, the focus is to develop the theories of smooth and nonsmooth analysis of interval-valued functions and to derive the optimality conditions of interval optimization problems.

1.1 Interval Analysis

It is known that digital computers have a limited numerical representation capability and work with rounded floating-point numbers. However, there are several numbers that have infinite digits in their numerical representation. For example, the numbers of kind π and $\sqrt{2}$ have infinite digits in their decimal places. Moreover, most rationals have rounded representation. Algebraic operations on floating points numbers can have accumulated errors which can be significant. For example, after the launching of a missile against the U.S. military in the Gulf War, a U.S. Patriot missile failed to intercept this attacking missile due to errors generated by approximations in numbers that were part of the algorithm implemented in the Patriot. The result of this was that twenty-eight people died, and ninety-eight were injured. A way to work with this type of error is to better understand real interval spaces and interval analysis. Since $\mathbb{R}$ is an unbounded totally ordered set that is endowed with the usual structure of order. Therefore, any number can be properly bounded by two adjacent numbers. When doing mathematical analysis with intervals, one comes face-to-face with the algebraic and analytical structures of spaces associated with intervals.

Interval analysis is a new and growing branch of applied mathematics that was formally developed by R. E. Moore in 1966 (see details in [57]). It gives an idea to computing that treats an interval as a new kind of number. The produced results by methods of interval analysis with properly-rounded interval arithmetic contain both ordinary machine arithmetic results as well as infinite precision arithmetic results. Thus, we have, at the outset, a completely general mechanism for bounding the accumulation of roundoff error in any machine computation. If roundoff is the only error present, then the widths of the interval results will go to zero as the length of the machine word increases.

In many real-world or mathematical problems such as static or dynamic, deterministic or probabilistic, linear or nonlinear, convex or nonconvex, etc., the knowledge about the underlying parameters which influence the behavior of mathematical problems, is imprecise or uncertain. Generally, one cannot measure the parameters affected by imprecision or uncertainties with exact values. In such situations, the parameters cannot be determined by a real number. We usually overcome this deficiency by using interval or stochastic values, which is a natural way of incorporating the uncertainties of parameters. The purpose of using interval is to provide upper and lower bounds of the parameters of such kind of mathematical problems.

Although interval analysis has not received the widespread acceptance that was expected by the creators or researchers. The common belief is that there are several faster and simpler methods that may be responsible for rounding and other types of errors. For example, in both single and double-precision floating points, it is common to calculate the results independently and compare the digits of the two results. If the significant digits of two results agree up to a certain number, then the matching digits are considered correct. However, we can easily find some examples where the significant digits of two results agree up to a certain number but the matching digits are not the correct result. One such classic example is given by Rump in 1988 as Rump's expression. The numerical evaluation with floating point of this expression give misleading and incorrect results. In the evaluation of Rump's expression with increasing numbers of digits, the results agreed in their first few significant digits. However, all the digits were incorrect and, although the computed solution was relatively far from zero, it failed to even find the correct sign. By using IEEE 754 floating-point arithmetic for Rump's expression:

$$h(y, z) = (333.75 - y^2)z^6 + y^2(11y^2z^2 - 121z^4 - 2) + 5.5z^8 + \frac{y}{2z}.$$

For $y = 77617$ and $z = 33096$, we will get the following results:

32 **bits:** $h(y, z) = 1.172604$

64 **bits:** $h(y, z) = 1.1726039400531786$

128 **bits:** $h(y, z) = 1.1726039400531786318588349045201838.$

In spite of their agreements in first digits, all three results are wrongs. The correct result is:

$$h(y, z) = -0.827396059946...$$

Evaluation using even the simplest method of interval analysis produces an interval that contains the correct value of the answer. Interval evaluation does not directly provide a better answer, it alerts us to the numerical instability of the expression and suggests that higher-accuracy methods to compute correct result. With the help of the developed theory of interval analysis, we can make interval bounds as narrow as possible. A major focus of interval analysis is to develop some interval algorithms which produce sharp (or nearly sharp) bounds on the solutions of mathematical problems with an uncertain environment.

1.2 Interval-valued Function

An interval-valued function is a mathematical function of one or more variables whose range is a set of closed and bounded intervals of real number which is denoted by $I(\mathbb{R})$. For the Euclidean space $\mathbb{R}^n$ and set of intervals vectors $I(\mathbb{R})^k$, the function $\mathbf{F}_{\mathbf{C}_v^k} : \mathcal{X} \rightarrow I(\mathbb{R})$, where $\mathcal{X}$ is a nonempty subset of $\mathbb{R}^n$, is known as an interval-valued function (IVF) that depends on k intervals in the interval vector $\mathbf{C}_v^k = (\mathbf{C}_1,\ \mathbf{C}_2, \ldots, \mathbf{C}_k)^T$, where $\mathbf{C}_j = [\underline{c_j},\ \overline{c_j}] \in I(\mathbb{R})$ for $j = 1, 2, \cdots, k$.

Parametrically, the vector $\mathbf{C}_v^k$ is observed by the following set

$$\Big\{c(t)\ \Big|\ c(t) = (c_1(t_1), c_2(t_2), \cdots, c_k(t_k))^T, c_j(t_j) = \underline{c_j} + t_j(\overline{c_j} - \underline{c_j}),$$
$$t = (t_1,\ t_2, \cdots, t_k)^T,\ 0 \leq t_j \leq 1,\ j = 1, 2, \cdots, k\Big\}.$$

Thus, the function $\mathbf{F}_{\mathbf{C}_v^k}$ can be presented as a collection of bunch of real-valued functions $f_{c(t)}$'s, i.e., for all $x \in \mathcal{X}$,

$$\mathbf{F}_{\mathbf{C}_v^k}(x) = \left\{ f_{c(t)}(x) | f_{c(t)} : \mathcal{X} \to \mathbb{R},\ c(t) \in \mathbf{C}_v^k,\ t \in [0,\ 1]^k \right\}.$$

The function $\mathbf{F}_{\mathbf{C}_v^k}$ can also be presented in another way. Let

$$\underline{f}(x) = \min_{t \in [0,1]^k} f_{c(t)}(x) \text{ and } \overline{f}(x) = \max_{t \in [0,1]^k} f_{c(t)}(x).$$

Then, for each argument point $x \in \mathcal{X}$, $\mathbf{F}_{\mathbf{C}_v^k}$ can be presented by

$$\mathbf{F}_{\mathbf{C}_v^k}(x) = \left[\underline{f}(x),\ \overline{f}(x) \right].$$

For instance, if we consider the IVF $\mathbf{F}_{\mathbf{C}_v^2}(x_1, x_2) = x_1 \odot \mathbf{C}_1 \oplus x_2^2 e^{x_1} \odot \mathbf{C}_2$ with $\mathbf{C}_1 = [1, 3]$ and $\mathbf{C}_2 = [-2, 1]$, then $\mathbf{F}_{\mathbf{C}_v^2}$ is the collection of functions

$$f_{(x_1(t_1), x_2(t_2))^T}(x_1, x_2) = c_1(t_1)x_1 + c_2(t_2)x_2^2 e^{x_1} = (1 + 2t_1)x_1 + (-2 + 3t_2)x_2^2 e^{x_1},$$

where t_1 and t_2 are in $[0, 1]$. Let

$$\underline{f}(x) = \min_{t \in [0,1]^k} f_{c(t)}(x) \text{ and } \overline{f}(x) = \max_{t \in [0,1]^k} f_{c(t)}(x).$$

Then,

$$\underline{f}(x_1, x_2) = \begin{cases} x_1 - 2x_2^2 e^{x_1} & \text{if } x_1 \geq 0, \\ 3x_1 - 2x_2^2 e^{x_1} & \text{if } x_1 < 0, \end{cases} \text{ and } \overline{f}(x_1, x_2) = \begin{cases} 3x_1 + x_2^2 e^{x_1} & \text{if } x_1 \geq 0, \\ x_1 + x_2^2 e^{x_1} & \text{if } x_1 < 0. \end{cases}$$

Hence, for each argument point $x \in \mathcal{X}$, $\mathbf{F}_{\mathbf{C}_v^k}$ can be presented by

$$\mathbf{F}_{\mathbf{C}_v^k}(x) = \left[\underline{f}(x),\ \overline{f}(x)\right].$$

1.3 Interval Optimization Problems

In real-world problems, the data collected by the decision-makers are always assumed to be real numbers with a certain value. In this case, the objective function of optimization problems is a real-valued function. However, there are some optimization problems where the objective may be uncertain due to inexact data. For example, suppose that a factory can produce two goods, say G_1 and G_2, in input quantities x_1 and x_2, subject to budget constraint $\mathcal{S} \subseteq \mathbb{R}^2$. For selling the goods G_1 and G_2 in the market, we assume that the factory can earn c_1 and c_2 dollars per units, respectively. In this case, the purpose is to maximize the objective function $c_1x_1+c_2x_2$ subject to the budget constraint set $\mathcal{S} \subseteq \mathbb{R}^2$. However, the prices of goods may fluctuate from time to time in the financial market. It seems more reasonable to assume the prices to be uncertain quantities. There are three kinds of methodologies that can model uncertain quantities: random variables, fuzzy numbers and intervals. If the coefficients c_1 and c_2 are assumed to be random variables, then the problem becomes a stochastic optimization problem. Birge and Louveaux in 1997 [9], Kall in 1976 [46], and Vajda in 1972 [81] explained the main stream of stochastic optimization problems and also introduced some useful methods to solve these optimization problems. If the coefficients c_1 and c_2 are assumed to be fuzzy, then the problem becomes a fuzzy optimization problem. On the other hand, if we assume the coefficients c_1 and c_2 to be compact intervals of real numbers, then although the prices may fluctuate from time to time, we can always make sure that the prices will fall into the corresponding intervals. In such cases the problem is classified as an *interval optimization problem* (IOP). It is well known that when fuzzy sets are used, they are described as families of their alpha-cuts. For each alpha-cuts the corresponding interval optimization problem could be easily solved and such partial solutions combined. Likewise for random variables, one could think of using confidence intervals. Due to this reason, IOPs are often thought of superior to fuzzy or stochastic optimization problems.

As we know that in most of the cases, the coefficients of the objective function in the stochastic optimization problems are considered as random variables with known distributions. However, the specifications of the distributions are very subjective. For example, many researchers invoke the Gaussian (normal) distributions with different parameters in stochastic optimization problems. These specifications do not completely match the real problems. Therefore, interval-valued optimization problems may provide an alternative choice to consider the uncertainty in the optimization problems. That is to say, the coefficients of the objective function in the interval-valued optimization problems are considered as compact intervals. Although the specifications of compact intervals may still be judged as subjective viewpoint, we can argue that the bounds of uncertain data (i.e., determining the compact intervals which give the upper and lower bounds of the possible observed data) are easier to be handled than specifying the Gaussian distributions in stochastic optimization problems.

Mathematically, an interval optimization problem is presented by

$$\min_{x \in \mathcal{X}} \mathbf{F}(x),$$

where $\mathcal{X} \subset \mathbb{R}^n$ and $\mathbf{F} : \mathcal{X} \to I(\mathbb{R})$. In the rigorous study on IOPs, we often need the following notions.

- ***Decision space:***

  A decision is characterized by the DM's choice between different possible courses of action, called alternatives. Set of all alternatives constitute the set $\mathcal{X}$ which is called as a feasible set. An alternative can be defined by a vector of real numbers $x = (x_1, x_2, \ldots, x_n)^t$. A vector for representing an alternative is said to be a decision vector. The components of this vector are called decision variables. Each decision variable is related to a particular aspect of the alternatives. The space of the decision vectors is called as decision space. The set $\mathcal{X}$ is also known as the decision feasible region. A point x in $\mathcal{X}$ is known as a decision feasible point.

- ***Objective space:***

  For any point x in the decision feasible region $\mathcal{X}$, an interval $\mathbf{F}(x)$ in $I(\mathbb{R})$ is obtained. The interval space in which the points $\mathbf{F}(x)$ lie is known as objective space. The set of all intervals $\mathbf{F}(x)$ where x in $\mathcal{X}$ is known as a feasible set in the objective space or objective feasible region. In solving IOPs, DM is always more interested in the objective space than the decision space because DM is often more interested in the objective values. We note that the image of the feasible set $\mathcal{X}$ under the interval-valued function $\mathbf{F}$ is the feasible set in the objective space.

In IOPs, the interval-valued objective function can be observed as a bunch of infinitely many real-valued objective functions. So, optimal solutions of IOPs behave like optimal solutions of multi-objective optimization problems. Unlike a single objective problem, there may not exist a unique solution of an IOP, since otherwise, there does not arise any conflict among all infinitely many real objectives of IOP, and it loses the essence of IOP. The basic difference between IOPs and single objective optimization problems is that the feasible region in the objective space of a single objective optimization problem is a totally ordered subset of the real line, whereas an IOP constitutes an infinite dimensional objective feasible region, which is not a totally ordered set in general. Thus, all solutions can be completely ordered according to the objective function in single objective optimization problems. In contrast, for an IOP, the solutions can only be ordered partially. As a consequence, in the case of single-objective problems, only one global optimum exists; but in the case of IOPs, conflicting real objectives can cause a situation where no solution is superior to the others. Thus, usually, there are many solutions to an IOP. The feasible solutions which can be improved without causing simultaneous deterioration in at least one criterion can not certainly be the optimal solution of the considered IOP. This concept leads to the foundation of non-dominated solutions.

- ***Non-dominated solution:***

  A non-dominated solution of IOP is a feasible point of interval objective space where any improvement in one criterion in a bunch of infinitely many real objectives of IOP can take place only through the worsening of at least one another criterion in that infinitely many real objectives. The concept of non-dominated solution of IOP is of primordial importance to recognize the conflicting nature of a bunch of infinitely many criteria of IOP. For a non-dominated point of IOP, there is no other feasible point in interval objective space that makes every criterion strictly better off. Each non-dominated point is equally acceptable as a solution to the IOP. Non-dominated point determines *efficient solution* from the entire feasible region.

1.4 Preliminaries

The following basic definitions and basic properties of intervals are used throughout this thesis.

1.4.1 Interval Arithmetic

Throughout the thesis, bold letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$, are used for denoting the elements of $I(\mathbb{R})$. An element $\mathbf{A}$ of $I(\mathbb{R})$ is presented by the corresponding small letter: $\mathbf{A} = [\underline{a}, \overline{a}]$.

In this section, we discuss Moore's interval arithmetic [57] followed by the concepts of gH-difference of two intervals and ordering of intervals [39].

Consider two intervals $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = \left[\underline{b}, \overline{b}\right]$. The *addition* of $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{A} \oplus \mathbf{B}$, is defined by

$$\mathbf{A} \oplus \mathbf{B} = \left[\ \underline{a} + \underline{b},\ \overline{a} + \overline{b}\ \right].$$

The *subtraction* of $\mathbf{B}$ from $\mathbf{A}$, denoted $\mathbf{A} \ominus \mathbf{B}$, is defined by

$$\mathbf{A} \ominus \mathbf{B} = \left[\ \underline{a} - \overline{b},\ \overline{a} - \underline{b}\ \right].$$

The *multiplication* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \odot \mathbf{B}$, is defined by

$$\mathbf{A} \odot \mathbf{B} = \left[\min\left\{\underline{a}\ \underline{b},\ \underline{a}\overline{b},\ \overline{a}\underline{b},\ \overline{a}\overline{b}\right\},\ \max\left\{\underline{a}\ \underline{b},\ \underline{a}\overline{b},\ \overline{a}\underline{b},\ \overline{a}\overline{b}\right\}\right].$$

The *multiplication* by a real number λ to $\mathbf{A}$, denoted $\lambda \odot \mathbf{A}$ or $\mathbf{A} \odot \lambda$, is defined by

$$\lambda \odot \mathbf{A} = \mathbf{A} \odot \lambda = \begin{cases} [\lambda \underline{a},\ \lambda \overline{a}], & \text{if } \lambda \geq 0 \\ [\lambda \overline{a},\ \lambda \underline{a}], & \text{if } \lambda < 0. \end{cases}$$

Notice that the definition of $\lambda \odot \mathbf{A}$ follows from the fact $\lambda = [\lambda, \lambda]$ and the definition of multiplication $\mathbf{A} \odot \mathbf{B}$.

Let $0 \notin \mathbf{B}$. The *division* of $\mathbf{A}$ by $\mathbf{B}$, denoted by $\mathbf{A} \oslash \mathbf{B}$, is defined by

$$\mathbf{A} \oslash \mathbf{B} = \left[\min\left\{\underline{a}/\underline{b},\ \underline{a}/\overline{b},\ \overline{a}/\underline{b},\ \overline{a}/\overline{b}\right\},\ \max\left\{\underline{a}/\underline{b},\ \underline{a}/\overline{b},\ \overline{a}/\underline{b},\ \overline{a}/\overline{b}\right\}\right].$$

We use the following definition for the difference between a pair of intervals since it is the most general definition of difference (see [31] for the details on why it is the most general).

Definition 1.4.1 (gH-difference of intervals [73])**.** *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two elements of $I(\mathbb{R})$. The gH-difference between $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted $\boldsymbol{A} \ominus_{gH} \boldsymbol{B}$, is defined by the interval $\boldsymbol{C}$ such that*

$$\boldsymbol{A} = \boldsymbol{B} \oplus \boldsymbol{C} \quad \text{or} \quad \boldsymbol{B} = \boldsymbol{A} \ominus \boldsymbol{C}.$$

Note 1 (See [73])**.** *It is to be noted that for two intervals $\boldsymbol{A} = [\underline{a},\ \overline{a}]$ and $\boldsymbol{B} = \left[\underline{b},\ \overline{b}\right]$,*

$$\boldsymbol{A} \ominus_{gH} \boldsymbol{B} = \left[\min\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\},\ \max\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\}\right]$$

and

$$(-1) \odot (\boldsymbol{A} \ominus_{gH} \boldsymbol{B}) = \boldsymbol{B} \ominus_{gH} \boldsymbol{A}.$$

As it is known that unlike the real numbers, intervals are not linearly ordered. Thus, in order to develop the analysis of interval-valued functions and interval optimization, we use the following order relation in this thesis.

Definition 1.4.2 (Dominance of intervals [7]). *Let* $\boldsymbol{A} = [\underline{a}, \overline{a}]$ *and* $\boldsymbol{B} = [\underline{b}, \overline{b}]$ *be two elements of* $I(\mathbb{R})$*. Note that* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *can be presented by*

$$\left.\begin{aligned} \boldsymbol{A} &= [\underline{a}, \overline{a}] = \{a(t) \mid a(t) = \underline{a} + t\,(\overline{a} - \underline{a}),\ 0 \leq t \leq 1\} \text{ and} \\ \boldsymbol{B} &= [\underline{b}, \overline{b}] = \{b(t) \mid b(t) = \underline{b} + t\,(\overline{b} - \underline{b}),\ 0 \leq t \leq 1\},\ \text{respectively.} \end{aligned}\right\} \tag{1.1}$$

Then,

(i) $\boldsymbol{B}$ *is said to be dominated by* $\boldsymbol{A}$ *if* $a(t) \leq b(t)$ *for all* $t \in [0, 1]$*, and then we write* $\boldsymbol{A} \preceq \boldsymbol{B}$*;*

(ii) $\boldsymbol{B}$ *is said to be strictly dominated by* $\boldsymbol{A}$ *if* $\boldsymbol{A} \preceq \boldsymbol{B}$ *and there exists a* $t_0 \in [0, 1]$ *such that* $a(t_0) \neq b(t_0)$*, and then we write* $\boldsymbol{A} \prec \boldsymbol{B}$*;*

(iii) *if* $\boldsymbol{B}$ *is not dominated by* $\boldsymbol{A}$*, then we write* $\boldsymbol{A} \npreceq \boldsymbol{B}$*; if* $\boldsymbol{B}$ *is not strictly dominated by* $\boldsymbol{A}$*, then we write* $\boldsymbol{A} \nprec \boldsymbol{B}$*;*

(iv) *if* $\boldsymbol{A}$ *is dominated by* $\boldsymbol{B}$ *or* $\boldsymbol{B}$ *is dominated by* $\boldsymbol{A}$*, then* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are said to be* comparable*,*

(v) *if* $\boldsymbol{A} \npreceq \boldsymbol{B}$ *and* $\boldsymbol{B} \npreceq \boldsymbol{A}$*, then we say that* none of **A** and **B** dominates the other*, or* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are* not comparable*.*

By using the fist and second inequalities of above definition, we have following properties of intervals.

Lemma 1.1. *Let* $\boldsymbol{A} = [\underline{a}, \overline{a}]$ *and* $\boldsymbol{B} = [\underline{b}, \overline{b}]$ *be two intervals in* $I(\mathbb{R})$*. Then,*

(i) $\boldsymbol{A} \preceq \boldsymbol{B}$ *if and only if* $\underline{a} \le \underline{b}$ *and* $\overline{a} \le \overline{b}$*, and*

(ii) $\boldsymbol{A} \prec \boldsymbol{B}$ *if and only if 'either* $\underline{a} \le \underline{b}$ *and* $\overline{a} < \overline{b}$ *or* $\underline{a} < \underline{b}$ *and* $\overline{a} \le \overline{b}$*'.*

Proof. We note that

$$\mathbf{A} = [\underline{a}, \bar{a}] = \{a(t) \ : \ a(t) = \underline{a} + t(\bar{a} - \underline{a}), 0 \le t \le 1\}$$
$$\text{and } \mathbf{B} = [\underline{b}, \bar{b}] = \{b(t) \ : \ b(t) = \underline{b} + t(\bar{b} - \underline{b}), 0 \le t \le 1\}.$$

(i) Let $\mathbf{A} \preceq \mathbf{B}$. Then, by Definition 1.4.2, we note that

$$\begin{aligned}
&\mathbf{A} \preceq \mathbf{B} \\
\Longrightarrow\ & \underline{a} + t(\bar{a} - \underline{a}) = a(t) \le b(t) = \underline{b} + t(\bar{b} - \underline{b}) \text{ for all } t \in [0,1] \\
\Longrightarrow\ & a(0) \le b(0) \text{ and } \ \bar{a}(1) \le \bar{b}(1) \\
\Longrightarrow\ & \underline{a} \le \underline{b} \text{ and } \bar{a} \le \bar{b}.
\end{aligned}$$

Conversely, if $\underline{a} \le \underline{b}$ and $\bar{a} \le \bar{b}$, then

$$\begin{aligned}
&(1-t)(\underline{a} - \underline{b}) + t(\overline{a} - \overline{b}) \le 0 \text{ for all } t \in [0,1] \\
\Longrightarrow\ & (\underline{a} + t(\bar{a} - \underline{a})) - (\underline{b} + t(\bar{b} - \underline{b})) \le 0 \text{ for all } t \in [0,1] \\
\Longrightarrow\ & a(t) - b(t) \le 0 \text{ for all } t \in [0,1] \\
\Longrightarrow\ & a(t) \le b(t) \text{ for all } t \in [0,1] \\
\Longrightarrow\ & \mathbf{A} \preceq \mathbf{B}.
\end{aligned}$$

(ii) If $\mathbf{A} \prec \mathbf{B}$, then by (ii) of Definition 1.4.2 and part (i), $\underline{a} \le \underline{b}$ and $\bar{a} \le \bar{b}$, and $\mathbf{A} \ne \mathbf{B}$.

If $\underline{a} = \underline{b}$ and $\overline{a} = \overline{b}$, then

$$\begin{aligned} & \underline{a} + t(\bar{a} - \underline{a}) = \underline{b} + t(\bar{b} - \underline{b}) \text{ for all } t \in [0, 1] \\ \implies & a(t) = b(t) \text{ for all } t \in [0, 1], \end{aligned}$$

which is contradictory to $\mathbf{A} \neq \mathbf{B}$.

Hence, either $\underline{a} \leq \underline{b}$ and $\overline{a} < \overline{b}$ or $\underline{a} < \underline{b}$ and $\overline{a} \leq \overline{b}$.

If $\underline{a} \leq \underline{b}$ and $\overline{a} < \overline{b}$, then $a(1) < b(1)$ and

$$\begin{aligned} & \underline{a} + t(\bar{a} - \underline{a}) \leq \underline{b} + t(\bar{b} - \underline{b}) \text{ for all } t \in [0, 1] \\ \implies & a(t) \leq b(t) \text{ for all } t \in [0, 1] \\ \implies & \mathbf{A} \preceq \mathbf{B}, \end{aligned}$$

which implies that $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$.

If $\underline{a} < \underline{b}$ and $\overline{a} \leq \overline{b}$, then $a(0) < b(0)$ and

$$\begin{aligned} & \underline{a} + t(\bar{a} - \underline{a}) \leq \underline{b} + t(\bar{b} - \underline{b}) \text{ for all } t \in [0, 1] \\ \implies & a(t) \leq b(t) \text{ for all } t \in [0, 1] \\ \implies & \mathbf{A} \preceq \mathbf{B}, \end{aligned}$$

which again implies that $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$.

Hence, $\mathbf{A} \prec \mathbf{B}$.

□

There are some most important results of interval optimization problems in this thesis which can not solved by using the dominance relation 1.4.2. So, an other dominance relations of intervals is defined as

Definition 1.4.3 (Better dominance relation of intervals). *Let* $\boldsymbol{A} = [\underline{a}, \overline{a}]$ *and* $\boldsymbol{B} = [\underline{b}, \overline{b}]$ *be two elements of* $I(\mathbb{R})$*. Note that* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *can be presented by*

$$\left.\begin{aligned} \boldsymbol{A} &= [\underline{a}, \overline{a}] = \{a(t) \mid a(t) = \underline{a} + t\,(\overline{a} - \underline{a})\,,\ 0 \leq t \leq 1\} \ \text{and} \\ \boldsymbol{B} &= [\underline{b}, \overline{b}] = \left\{b(t) \;\middle|\; b(t) = \underline{b} + t\left(\overline{b} - \underline{b}\right),\ 0 \leq t \leq 1\right\},\ \text{respectively.} \end{aligned}\right\} \tag{1.2}$$

Then,

(i) $\boldsymbol{B}$ *is said to be* better strictly dominated by *by* $\boldsymbol{A}$ *if* $a(t) < b(t)$ *for all* $t \in [0, 1]$*, and then we write* $\boldsymbol{A} < \boldsymbol{B}$*;*

(ii) *if* $\boldsymbol{B}$ *is not better strictly dominated by* $\boldsymbol{A}$*, then we write* $\boldsymbol{A} \nless \boldsymbol{B}$*.*

The first inequality of Definition 1.4.3 can be written as inequality of following lemma.

Lemma 1.2. *Let* $\boldsymbol{A} = [\underline{a}, \overline{a}]$ *and* $\boldsymbol{B} = [\underline{a}, \overline{a}]$ *be elements of* $I(\mathbb{R})$*. Then,* $\boldsymbol{A} < \boldsymbol{B}$ *if and only if* $\underline{a} < \underline{b}$ *and* $\overline{a} < \overline{b}$*.*

Proof. Let $\mathbf{A} < \mathbf{B}$. Then, by Definition 1.4.3, we note that

$$\begin{aligned} &\mathbf{A} < \mathbf{B} \\ \Longrightarrow\ &\underline{a} + t(\bar{a} - \underline{a}) = a(t) < b(t) = \underline{b} + t(\bar{b} - \underline{b}) \text{ for all } t \in [0, 1] \\ \Longrightarrow\ &a(0) \leq b(0) \text{ and } \ a(1) < b(1) \\ \Longrightarrow\ &\underline{a} < \underline{b} \text{ and } \ \bar{a} < \bar{b}. \end{aligned}$$

Conversely, if $\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$, then

$$
\begin{aligned}
& (1-t)(\underline{a}-\underline{b}) + t(\overline{a}-\overline{b}) < 0 \text{ for all } t \in [0,1] \\
\implies & (\underline{a} + t(\bar{a}-\underline{a})) - (\underline{b} + t(\bar{b}-\underline{b})) < 0 \text{ for all } t \in [0,1] \\
\implies & a(t) - b(t) < 0 \text{ for all } t \in [0,1] \\
\implies & a(t) < b(t) \text{ for all } t \in [0,1] \\
\implies & \mathbf{A} < \mathbf{B}.
\end{aligned}
$$

□

The set of all closed and bounded intervals $I(\mathbb{R})$ equipped with the norm $\|.\|_{I(\mathbb{R})}$ is a normed quasilinear space (see [53]) with respect to the operations $\{\oplus, \ominus_{gH}, \odot\}$, where $\|.\|_{I(\mathbb{R})}$ is defined as following.

Definition 1.4.4 (Norm on $I(\mathbb{R})$ [57])**.** *For an $\boldsymbol{A} = [\underline{a}, \bar{a}]$ in $I(\mathbb{R})$, the function $\|.\|_{I(\mathbb{R})} : I(\mathbb{R}) \to \mathbb{R}^+$, defined by*

$$\|\boldsymbol{A}\|_{I(\mathbb{R})} = \max\{|\underline{a}|, |\bar{a}|\},$$

is a norm on $I(\mathbb{R})$. In rest of thesis, we simply use the symbol '$\|\cdot\|$' to denote the usual Euclidean norm on $\mathbb{R}^n$.

Definition 1.4.5. *(Maximum and minimum of intervals). Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_m$ be the elements of $I(\mathbb{R})$ with $\boldsymbol{A}_1 \preceq \boldsymbol{A}_2 \preceq \cdots \preceq \boldsymbol{A}_m$. Then,*

$$\max\{\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_m\} = \boldsymbol{A}_m \textit{ and } \min\{\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_m\} = \boldsymbol{A}_1.$$

Remark 1.3. It can be easily observe that the maximum of any finite set $\mathbf{S}$ of comparable intervals always belong to the set $\mathbf{S}$.

1.4.2 Basic Properties of Interval Analysis

The following basic properties of interval analysis with the help of dominance relation of intervals, norm of a interval, and gH-difference of two intervals, are used throughout the thesis.

Lemma 1.4. *For two intervals* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *in* $I(\mathbb{R})$*,*

(i) $\boldsymbol{A} \preceq \boldsymbol{B} \iff \boldsymbol{A} \ominus_{gH} \boldsymbol{B} \preceq \boldsymbol{0}$,

(ii) $\boldsymbol{A} \not\prec \boldsymbol{B} \iff \boldsymbol{A} \ominus_{gH} \boldsymbol{B} \not\prec \boldsymbol{0}$.

Proof. See Appendix A.1. □

Note 2 (See [53]). *For RDM interval arithmetic lemma 1.4 does not hold.*
For instance, let $\boldsymbol{A} = [1, 3]$ *and* $\boldsymbol{B} = [2, 5]$*. Then,* $\boldsymbol{A} \prec \boldsymbol{B}$*.*
But according to RDM interval arithmetic

$$[1, 3] = 1 + \alpha_1(3 - 1) = 1 + 2\alpha_1, \ \ \alpha_1 \in [0, 1]$$

and

$$[2, 5] = 2 + \alpha_2(5 - 2) = 2 + 3\alpha_2, \ \ \alpha_2 \in [0, 1].$$

Now

$$\begin{aligned} [1, 3] - [2, 5] &= 1 + 2\alpha_1 - 2 - 3\alpha_2, \quad \alpha_1, \alpha_2 \in [0, 1] \\ &= -1 + 2\alpha_1 - 3\alpha_2 \\ &= [-4, 1] \not\prec \boldsymbol{0}. \end{aligned}$$

Hence, $\boldsymbol{A} \prec \boldsymbol{B} \not\Rightarrow \boldsymbol{A} - \boldsymbol{B} \prec \boldsymbol{0}$.
Similarly for $\boldsymbol{A} = [1, 3]$ *and* $\boldsymbol{B} = [2, 4]$*,* $\boldsymbol{A} \ominus_{gH} \boldsymbol{B} \not\prec \boldsymbol{0}$ *but* $\boldsymbol{A} \prec \boldsymbol{B}$.

Lemma 1.5. *For all* $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} \in I(\mathbb{R})$,

(i) $\boldsymbol{B} \nprec \boldsymbol{A} \ominus_{gH} (\boldsymbol{A} \ominus_{gH} \boldsymbol{B})$,

(ii) *if* $\boldsymbol{B} \nprec \boldsymbol{A} \ominus_{gH} \boldsymbol{C}$ *and* $\boldsymbol{0} \prec \boldsymbol{A}$, *then* $\boldsymbol{B} \nprec (-1) \odot \boldsymbol{C}$,

(iii) $(\boldsymbol{A} \ominus_{gH} \boldsymbol{C}) \oplus (\boldsymbol{C} \ominus_{gH} \boldsymbol{B}) \nprec \boldsymbol{A} \ominus_{gH} \boldsymbol{B}$,

(iv) $\boldsymbol{A} \nprec \boldsymbol{0}$ *and* $\boldsymbol{A} \preceq \boldsymbol{B} \implies \boldsymbol{B} \nprec \boldsymbol{0}$,

(v) $\boldsymbol{A} \ominus_{gH} \boldsymbol{B} \nprec \boldsymbol{0}$ *and* $\boldsymbol{C} \preceq \boldsymbol{B} \implies \boldsymbol{A} \ominus_{gH} \boldsymbol{C} \nprec \boldsymbol{0}$,

(vi) *if* $\boldsymbol{C} \preceq \boldsymbol{B}$, *then* $\boldsymbol{A} \ominus_{gH} \boldsymbol{B} \preceq \boldsymbol{A} \ominus_{gH} \boldsymbol{C}$.

Proof. See Appendix A.2. □

Lemma 1.6. *For all* $\boldsymbol{A},\ \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} \in I(\mathbb{R})$,

(i) $\|\boldsymbol{A}\|_{I(\mathbb{R})} - \|\boldsymbol{B}\|_{I(\mathbb{R})} \leq \|\boldsymbol{A} \ominus_{gH} \boldsymbol{B}\|_{I(\mathbb{R})}$,

(ii) $\|\boldsymbol{A} \ominus_{gH} \boldsymbol{B}\|_{I(\mathbb{R})} \leq \|(\boldsymbol{A} \ominus_{gH} \boldsymbol{C}) \oplus (\boldsymbol{C} \ominus_{gH} \boldsymbol{B})\|_{I(\mathbb{R})}$,

(iii) $\boldsymbol{B} \preceq \boldsymbol{A} \oplus [L, L]$, *where* $L = \|\boldsymbol{B} \ominus_{gH} \boldsymbol{A}\|_{I(\mathbb{R})}$, *and*

(iv) $\|(\boldsymbol{A} \ominus_{gH} \boldsymbol{B}) \ominus_{gH} (\boldsymbol{C} \ominus_{gH} \boldsymbol{D})\|_{I(\mathbb{R})} \leq \|\boldsymbol{A} \ominus_{gH} \boldsymbol{C}\|_{I(\mathbb{R})} \oplus \|\boldsymbol{B} \ominus_{gH} \boldsymbol{D}\|_{I(\mathbb{R})}$.

Proof. See Appendix A.3. □

Remark 1.4.1. *The following two points are noticeable.*

(i) *It is noteworthy that although* $\oplus$ *is associative in* $I(\mathbb{R})$, *for two intervals* $\boldsymbol{A}$ *and* $\boldsymbol{C}$ *in* $I(\mathbb{R})$, *the interval* $((\boldsymbol{A} \ominus_{gH} \boldsymbol{C}) \oplus \boldsymbol{C})$ *is not always equal to* $\boldsymbol{A}$. *For instance, consider* $\boldsymbol{A} = [6, 9]$ *and* $\boldsymbol{C} = [3, 7]$. *Then,* $(\boldsymbol{A} \ominus_{gH} \boldsymbol{C}) \oplus \boldsymbol{C} = [2, 3] \oplus [3, 7] = [5, 10] \neq \boldsymbol{A}$. *Thus, (ii) of Lemma 1.6 is* not an obvious property *of* $\| \cdot \|_{I(\mathbb{R})}$ *on the elements in* $I(\mathbb{R})$.

(ii) *For two elements* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *of* $I(\mathbb{R})$*, if* $\boldsymbol{B} = \boldsymbol{A} \oplus (\boldsymbol{B} \ominus_{gH} \boldsymbol{A})$*, then (iii) of Lemma 1.6 is an obvious property since* $\boldsymbol{B} \ominus_{gH} \boldsymbol{A} \preceq [L, L]$*. However,* $(\boldsymbol{A} \oplus (\boldsymbol{B} \ominus_{gH} \boldsymbol{A}))$ *is not always equal to* $\boldsymbol{B}$*. For instance, for* $\boldsymbol{A} = [4, 10]$ *and* $\boldsymbol{B} = [-3, 2]$*,*

$$\boldsymbol{A} \oplus (\boldsymbol{B} \ominus_{gH} \boldsymbol{A}) = [4, 10] \oplus [-8, -7] = [-4, 3] \neq \boldsymbol{B}.$$

Therefore, (iii) of Lemma 1.6 is not a trivial property.

(iii) *For any* $\boldsymbol{A}$*,* $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *in* $I(\mathbb{R})$*, if*

$$\boldsymbol{B} \ominus_{gH} \boldsymbol{A} \preceq \boldsymbol{C} \implies \boldsymbol{B} \preceq \boldsymbol{A} \oplus \boldsymbol{C}, \tag{1.3}$$

then replacing $\boldsymbol{C}$ *by* $[L, L]$*, we see that (iii) of Lemma 1.6 is an obvious property. However, (1.3) is not always true. For instance, if* $\boldsymbol{B} = [-3, 2]$*,* $\boldsymbol{A} = [4, 10]$ *and* $\boldsymbol{C} = [-7.5, -6]$*, then*

$$\boldsymbol{B} \ominus_{gH} \boldsymbol{A} = [-8, -7] \text{ and } \boldsymbol{A} \oplus \boldsymbol{C} = [-3.5, 4].$$

Hence, $\boldsymbol{B} \ominus_{gH} \boldsymbol{A} \preceq \boldsymbol{C}$*, but* $\boldsymbol{B}$ *and* $\boldsymbol{A} \oplus \boldsymbol{C}$ *are not comparable. Thus, (iii) of Lemma 1.6 is not an obvious property.*

(iv) *For elements* $\boldsymbol{A}$*,* $\boldsymbol{B}$*,* $\boldsymbol{C}$ *and* $\boldsymbol{D}$ *of* $I(\mathbb{R})$*, if*

$$(\boldsymbol{A} \ominus_{gH} \boldsymbol{B}) \ominus_{gH} (\boldsymbol{C} \ominus_{gH} \boldsymbol{D}) = (\boldsymbol{A} \ominus_{gH} \boldsymbol{C}) \oplus (\boldsymbol{D} \ominus_{gH} \boldsymbol{B}),$$

then (iv) of Lemma 1.6 is an obvious property since $\|(\boldsymbol{P} \oplus \boldsymbol{Q}\|_{I(\mathbb{R})} \leq \|\boldsymbol{P}\|_{I(\mathbb{R})} \oplus \|\boldsymbol{Q}\|_{I(\mathbb{R})}$ *for all* $\boldsymbol{P}$*,* $\boldsymbol{Q} \in I(\mathbb{R})$*. However,* $(\boldsymbol{A} \ominus_{gH} \boldsymbol{B}) \ominus_{gH} (\boldsymbol{C} \ominus_{gH} \boldsymbol{D})$ *is not always equal to* $(\boldsymbol{A} \ominus_{gH} \boldsymbol{C}) \oplus (\boldsymbol{D} \ominus_{gH} \boldsymbol{B})$*. For instance, for* $\boldsymbol{A} = [0, 1]$*,* $\boldsymbol{B} = [-3, 2]$*,* $\boldsymbol{C}$

= [-1, 1] and $\boldsymbol{D}$ *= [-2, 5]*

$$(\boldsymbol{A} \ominus_{gH} \boldsymbol{B}) \ominus_{gH} (\boldsymbol{C} \ominus_{gH} \boldsymbol{D}) = [-1,3] \ominus_{gH} [-4,1] = [2,3],$$

and

$$(\boldsymbol{A} \ominus_{gH} \boldsymbol{C}) \oplus (\boldsymbol{D} \ominus_{gH} \boldsymbol{B}) = [0,1] \oplus [1,3] = [1,4].$$

Hence, $(\boldsymbol{A} \ominus_{gH} \boldsymbol{B}) \ominus_{gH} (\boldsymbol{C} \ominus_{gH} \boldsymbol{D}) \neq (\boldsymbol{A} \ominus_{gH} \boldsymbol{C}) \oplus (\boldsymbol{D} \ominus_{gH} \boldsymbol{B})$ *as* $[2,3] \neq [1,4]$. *Therefore, (iv) of Lemma 1.6 is not a trivial property.*

Corollary 1.4.1. *For all* $\boldsymbol{A}, \boldsymbol{B} \in I(\mathbb{R})$,

$$\|\boldsymbol{A} \ominus_{gH} \boldsymbol{B}\|_{I(\mathbb{R})} \leq \|\boldsymbol{A}\|_{I(\mathbb{R})} + \|\boldsymbol{B}\|_{I(\mathbb{R})}. \tag{1.4}$$

Proof. In (ii) Lemma 1.6, by taking $\mathbf{C} = \mathbf{0}$ we get

$$\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} \leq \|\mathbf{A} \ominus_{gH} \mathbf{0}\|_{I(\mathbb{R})} + \|\mathbf{0} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} = \|\mathbf{A}\|_{I(\mathbb{R})} + \|\mathbf{B}\|_{I(\mathbb{R})}.$$

□

Lemma 1.7. *For all* $x,\ y \in \mathbb{R}$ *and* $\boldsymbol{C} \in I(\mathbb{R})$,

(i) *if* $\boldsymbol{C} \succeq \boldsymbol{0}$, *then* $|x+y| \odot \boldsymbol{C} \preceq |x| \odot \boldsymbol{C} \oplus |y| \odot \boldsymbol{C}$,

(ii) *if* $\boldsymbol{C} \preceq \boldsymbol{0}$, *then* $|x+y| \odot \boldsymbol{C} \succeq |x| \odot \boldsymbol{C} \oplus |y| \odot \boldsymbol{C}$, *and*

(iii) $\boldsymbol{C} \not\preceq \boldsymbol{0} \implies |x+y| \odot \boldsymbol{C} \not\succeq |x| \odot \boldsymbol{C} \oplus |y| \odot \boldsymbol{C}$.

Proof. See Appendix A.4. □

1.4.3 Few Elements of Convex Analysis

A nonempty subset $\mathcal{C}$ of $\mathcal{X}$ is called *cone* [64] if

$$x \in \mathcal{C}, \lambda \geq 0 \implies \lambda x \in \mathcal{C}.$$

Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$. The set

$$\text{cone}(\mathcal{S}) = \{\lambda s : \lambda \geq 0 \text{ and } s \in \mathcal{S}\}$$

is called the *cone generated by* $\mathcal{S}$ [64]. Besides, a vector $h \in \mathcal{X}$ is called a *tangent vector* [42] to $\mathcal{S}$ at $\bar{x} \in \text{cl}(\mathcal{S})$ if there are two sequences $\{x_n\}$ in $\mathcal{S}$ and $\{\lambda_n\}$ in positive real numbers with

$$\bar{x} = \lim_{n \to +\infty} x_n \text{ and } h = \lim_{n \to +\infty} \lambda_n (x_n - \bar{x}).$$

The set of all tangent vectors to $\mathcal{S}$ at $\bar{x}$ is called *tangent cone* [42] to $\mathcal{S}$ at $\bar{x}$ and is denoted by $\mathcal{T}(\mathcal{S}, \bar{x})$. Further, $\mathcal{S}$ of $\mathcal{X}$ is called *star-shaped* [42] with respect to some $\bar{x} \in \mathcal{S}$ if for all $x \in \mathcal{S}$,

$$\lambda x + (1 - \lambda)\bar{x} \in \mathcal{S} \text{ for all } \lambda \in [0, 1].$$

1.4.4 Some Basic Definitions and Properties of Interval-valued Functions

In this section, we define some basic definitions for interval-valued functions which are used throughout the thesis.

Definition 1.4.6 (Interval-valued convex function [88]). *Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. An IVF $\boldsymbol{F} : \mathcal{X} \to I(\mathbb{R})$ is said to be convex on $\mathcal{X}$ if*

$$\boldsymbol{F}(\lambda x_1 + \lambda' x_2) \preceq \lambda \odot \boldsymbol{F}(x_1) \oplus \lambda' \odot \boldsymbol{F}(x_2) \text{ for all } x_1,\ x_2 \in \mathcal{X} \text{ and for all } \lambda,\ \lambda' \in [0,\ 1],$$

where $\lambda + \lambda' = 1$.

Lemma 1.8 (See [88]). *$\boldsymbol{F}$ is a convex IVF on a convex set $\mathcal{S} \subseteq \mathcal{X}$ if and only if $\underline{f}$ and $\overline{f}$ are convex on $\mathcal{S}$.*

Lemma 1.9 (See [88]). *Let $\boldsymbol{F}(x) = \left[\underline{f}(x), \overline{f}(x)\right]$ be an IVF on a nonempty subset $\mathcal{X}$ of $\mathbb{R}^n$. Then, $\lim_{x \to x_0} \boldsymbol{F}(\lambda)$ exists if and only if $\lim_{x \to x_0} \underline{f}(\lambda)$ and $\lim_{x \to x_0} \overline{f}(\lambda)$ exist and*

$$\lim_{x \to x_0} \boldsymbol{F}(\lambda) = \left[\lim_{x \to x_0} \underline{f}(\lambda), \lim_{x \to x_0} \overline{f}(\lambda)\right].$$

Definition 1.4.7 (gH-continuity [31]). *Let $\boldsymbol{F}$ be an IVF on a nonempty subset $\mathcal{X}$ of $\mathbb{R}^n$. The function $\boldsymbol{F}$ is said to be gH-continuous at $\bar{x} \in \mathcal{X}$ if for any $h \in \mathbb{R}^n$ with $\bar{x} + h \in \mathcal{X}$,*

$$\lim_{\|h\| \to 0} (\boldsymbol{F}(\bar{x} + h) \ominus_{gH} \boldsymbol{F}(\bar{x})) = \boldsymbol{0}.$$

Definition 1.4.8 (Efficient point [31]). *Let $\mathcal{X}$ be a nonempty subset of $\mathbb{R}^n$ and $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R})$ be an IVF. A point $\bar{x} \in \mathcal{X}$ is said to be an* efficient point *of an IOP:*

$$\min_{x \in \mathcal{X}} \boldsymbol{F}(x) \tag{1.5}$$

if $\boldsymbol{F}(x) \nprec \boldsymbol{F}(\bar{x})$ for all $x \in \mathcal{X}$.

Lemma 1.10. *Let $\mathcal{S}$ be a linear space of $\mathcal{X}$ and $\boldsymbol{F} : \mathcal{S} \to I(\mathbb{R})$ be a linear IVF. Then, the following results hold.*

(i) *If $\boldsymbol{F}(x) \nprec \boldsymbol{0}$ for all $x \in \mathcal{S}$, then $\boldsymbol{0}$ and $\boldsymbol{F}(x)$ are not comparable.*

(ii) *If* $\boldsymbol{F}(x) \preceq \boldsymbol{0}$ *for all* $x \in \mathcal{S}$*, then* $\boldsymbol{F}(x) = \boldsymbol{0}$*.*

Proof. See Appendix A.5. □

1.5 Literature Survey

1.5.1 Literature on Interval Analysis

In the literature of interval analysis, initially, there are three people (Warmus in 1956 [83], Sunaga in 1958 [68], and Moore in 1966 [57]) who independently developed interval arithmetic. Although, Warmus and Sunaga were the first to create interval arithmetic, but Moore is the prime mover of interval analysis and considered the father of its development. His book on interval analysis was published in 1966 [57]. Further, basic contributions in interval analysis are given by Apostolatos and Kulisch in 1967 [4], Hansen in 1965 [34], Kruckeberg in 1966 [47], Nickel in 1966 [60], and others. By using Moore's interval arithmetic, Mayer in 1970 [56] described the concept of quasilinear space for compact intervals. In Moore's interval arithmetic, there are a few limitations (see [28] for details), such as, the additive inverse of a degenerate intervals, i.e., an interval whose upper and lower limits are equal, exist only. For the same reason, many conventional properties for real numbers are not true for compact intervals, for instance, for two compact intervals $\mathbf{A}$ and $\mathbf{C}$, $(\mathbf{A} \ominus_{gH} (\mathbf{C}) \oplus \mathbf{C} \neq \mathbf{A}$ (see Remark 2.3.1 of [28]). Thus, to develop a theoretical framework of the calculus of interval analysis, Hukuhara in 1967 [37] introduced a new concept for the difference of compact intervals, known as Hukuhara difference (H-difference) of intervals and derive several properties of compact intervals by using this difference. Although H-difference provides the additive inverse of compact intervals, this difference of a compact interval **B** from a compact interval **A** can be calculated only when the width of **A** is greater than or equal to that of **B** (see details in [12]). To overcome this difficulty, the nonstandard difference of intervals is introduced by Markov in 1979 [55], and for the same reason, Stefanini in 2008 [72] introduced a strong concept of difference of two intervals as generalized Hukuhara difference (gH-difference) of intervals. The gH-difference provides an additive inverse of any compact interval and is applicable for all pairs of compact intervals. By using the gH-difference of intervals and Moore's interval arithmetic, Stefanini proved the cancellation law for the addition of intervals and the distributive law for subtraction of intervals by the scalar. Further, in [53], with the help of norm of an interval defined by Moore and gH-difference of intervals, it is explained that set of compact intervals is quasi normed linear space. In 2016, Tao proved some results of interval arithmetic and semi-linear interval differential

equations under the gH-difference. Recently, to generalize the concepts of smooth and nonsmooth analysis for interval-valued function and to derive interval variational inequalities, Ghosh in 2019 [28] and Gaurav in 2020 [48] proved some inequalities of intervals by using dominance relation and gH-difference of intervals. Apart from Moore's interval arithmetic, another concept of interval arithmetic has been developed by Piegat and Landowski [49], namely RDM interval arithmetic, which also ensures the existence of an additive inverse for any compact interval. Generally, all the properties of RDM interval arithmetic are similar to Moore's interval arithmetic except the subtraction of an interval from itself. In this thesis, we use Moore's interval arithmetic with gH-difference instead of RDM interval arithmetic (see Note 2 in [28] for the reason).

1.5.2 Literature on Calculus of Interval-valued Functions

To observe the properties of an IVF, calculus plays an essential role. Initially, to develop the calculus of IVFs, Hukuhara in 1967 [37] introduced the concept of differentiability of IVFs with the help of H-difference of intervals. However, the definition of Hukuhara differentiability (H-differentiability) is found to be restrictive (see [12]). To remove the deficiencies of H-differentiability, Bede and Gal in 2005 [6] defined strongly generalized derivative (G-derivative) for IVFs and derived a Newton-Leibnitz-type formula. In order to formulate the mean-value theorem for IVFs, Markov in 1979 [55] introduced a new concept of difference of intervals and defined differentiability of IVFs by using this difference. In 2009 [73], Stefanini and Bede defined the generalized Hukuhara differentiability (gH-differentiability) of IVFs by using the concept of generalized Hukuhara difference. In defining the calculus of IVFs, the concepts of gH-derivative, gH-partial derivative, gH-gradient, and gH-differentiability for IVFs have been developed in [31, 73, 74].

To derive a Karush-Kuhn-Tucker (KKT) condition for IOPs, Guo et al. in 2019 [33] defined gH-symmetric derivative for IVFs. Ghosh in 2016 [30], analyzed the notion of gH-differentiability of multi-variable IVFs to propose the Newton method for IOPs. The concept of second-order differentiability of IVFs is introduced by Van [80] to study the existence of a unique solution of interval differential equations. Lupulescu [52] defined delta generalized Hukuhara differentiability on time scales by using gH-difference. Chalco et al. [13] introduced the concept of π-derivative for IVFs that generalizes Hukuhara derivative and G-derivative, and proved that this derivative is equivalent to gH-derivative. In [69], Stefanini and Bede defined level-wise gH-differentiability and generalized fuzzy differentiability by LU-parametric representation for fuzzy-valued functions. Kalani et al. [45] analyzed the concept of interval-valued fuzzy derivative for perfect and semi-perfect interval-valued fuzzy mappings to derive a method for solving interval-valued fuzzy differential equations using the extension principle. Recently, Ghosh et al. [28] have provided the idea of gH-directional derivative, gH-Gâteaux derivative, and gH-Fréchet derivative of IVFs to derive the optimality conditions for IOPs.

1.5.3 Literature on Interval Optimization Problem

In recent years, the interval analysis method was developed to model the uncertainty in uncertain optimization problems, in which the bounds of the uncertain coefficients are only required, not necessarily knowing the probability distributions or membership functions. Tanaka et al. in 1984 [77] and Rommelfanger in 1989 [65] discussed the linear programming problem with interval coefficients in the objective function. Chanas and Kuchta in 1996 [15, 16] suggested an approach based on an order relation of interval number to convert the linear optimization problem with uncertainty into a deterministic optimization problem. Tong in 1994 [79] investigated the problems in which the coefficients of the objective function and the constraints are all interval numbers. He obtained the possible interval of the solution by taking the maximum value range and minimum value rangeminequalities as constraint conditions. Liu and Da in 1999 [19] proposed an interval number optimization method based on a fuzzy constraint satisfactory degree to deal with the linear problems. Sengupta et al. in 2001 [67] studied the linear interval number programming problems in which the coefficients of the objective function and inequality constraints are all interval numbers. They proposed the concept of "acceptability index" and gave one solution for the uncertain linear programming. Zhang et al. in 1999 [63] assumed interval numbers as random variables with uniform distributions and constructed a possibility degree to solve the multi-criteria decision problem. The above methods point out a fine way for uncertain optimization. The reference Ma in 2002 [54] seems the first publication on nonlinear interval number programming (NINP). In this reference, a deterministic optimization method is used to obtain the interval of the nonlinear objective function, and a three-objective optimization problem is formulated. Wu [88], used the concept of Hukuhara differentiability to study KKT conditions of optimization problems with IVFs. Wu [87], has also illustrated the solution concept of optimization problems with IVFs by imposing a partial ordering on the set of all closed intervals and applying the existing calculus of IVFs. In 2013, the KKT conditions, based on gH-differentiability, of optimization problems with IVFs have been illustrated by Chalco-Cano and others [11]. After that, Bhurji and Panda [8] have defined interval-valued function in the parametric form and studied its properties, and developed a methodology to study the existence of the efficient solution of an optimization problem with IVFs. Ghosh [31] has introduced a new definition of gH-differentiability and proposed a Newton method [31] and an updated Newton method [32] to capture the efficient solution of an optimization problem with IVFs. Recently, Ghosh et al. [26] have proposed the theory of alternatives and hence the KKT optimality condition for IOPs. Importantly, it is shown in [26] that KKT optimality conditions appear with the inclusion relations instead of equations.

1.6 Objective of the Thesis

The objectives of the thesis are:

- To define and analyze the notions of generalized derivative and semiderivatives for IVFs, like directional derivative, Gâteaux derivative, Fréchet derivative, Clarke derivative, Hadamard derivative, Hadamard semiderivative, Dini semiderivative, etc.,

- to view the relations among these derivatives and semiderivatives for IVFS,
- to explore and characterize the efficient of IOPs by using these derivatives and semiderivatives , and
- to find a methodology to capture the complete non-dominated set of an IOPs.

1.7 Organization of the Thesis

This thesis consists of seven chapters including an introductory chapter and a chapter comprised of conclusion and future scopes. In this chapter, which is the introductory chapter, a concise but adequate literature of these topics has been discussed. It also defines the objective of this thesis.

In **Chapter 2**, three new concepts—directional, Gâteaux and Fréchet derivatives for IVFs— are studied. Further, the conditions for the existence of these derivatives for IVFs are given. To explain the properties of these derivatives, bounded, linear, monotonic, and Lipschitz continuity for IVFs are newly defined. The idea about to find the optimal solutions of IOPs is described.

Chapter 3 analyzes the concepts of Clarke derivative, pseudoconvex and quasiconvex for IVFs. To describe the properties of Clarke derivative, the concepts of limit superior, limit inferior, and sublinear for IVFs are studied. Further, by using the derived concepts, the existence of Clarke derivative, the relation of Clarke derivative with directional derivative, the relation of convex with pseudoconvex, and relation of pseudoconvex with quasiconvex are shown for IVFs. With the help of the studied concepts, a few results on characterizing efficient solutions to an IOP are derived.

In **Chapter 4**, the notion of Hadamard semiderivative for IVFs is explained. In the presence of directonal derivative, a necessary and sufficient condition for the existence of Hadamard semiderivative of IVFs are derived. The relation of Hadamard semiderivative with directional derivative and Gâteaux derivative for IVFs are shown. Further, the behavior of composition of two Hadamard semidifferentiable IVFs and maximum of Hadamard semidifferentiable IVFs are explained. It is obseved that the proposed concepts is useful to find out the efficient solutions of IOPs. For constraint IOPs, the Karush-Kuhn-Tucker sufficient condition to obtain efficient solutions are derived.

Chapter 5 describes the idea of Hadamard derivative for IVFs. For an IVF, the relation of Hadamard derivative with Fréchet derivative and continuity are shown. Further, behavior of composition of two Hadamard differentiable IVFs and the maximum of Hadamard differentiable IVFs are explained. The proposed derivative is observed to be useful to check the convexity of an IVF and also helpful to characterize the efficient solutions of IOPs. For constraint IOPs, an extended Karush-Kuhn-Tucker necessary and sufficient condition by using the proposed derivative is derived.

In **Chapter 6**, the concepts of upper and lower Dini semiderivatives, upper and lower Hadamard semiderivatives for IVFs are studied. For an IVF, the relation of upper Dini semiderivative and upper Hadamard semiderivative with directional derivative, Clarke derivative, Hadamard semiderivative and continuity are shown. Proposed semiderivative is observed to be useful to characterize the efficient solutions of IOPs.

Finally, **Chapter 7** completes this work by summarizing the concluding remarks and forecasting potential avenues for the future researchers.

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