CERTIFICATE

It is certified that the work contained in this thesis titled " ***Generalized Derivatives for Interval-valued Functions and Their Applications in Interval Optimization Problems*** " by ***Ram Surat Chauhan*** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy and SOTA for the award of Ph.D. degree.

24-June-2021

Dr. Debdas Ghosh
(Supervisor)
Assistant Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

DECLARATION BY THE CANDIDATE

I, ***Ram Surat Chauhan***, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of ***Dr. Debdas Ghosh*** from ***December, 2016*** to ***June, 2021*** at the ***Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi***. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not wilfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

Date: 12/06/2021

Place: Varanasi

(Ram Surat Chauhan)

CERTIFICATE BY THE SUPERVISOR

It is certified that the above statement made by the student is correct to the best of my/our knowledge.

12-June-2021

(Dr. Debdas Ghosh)
Assistant Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

14.6.2021

(Prof. T. Som)
Professor and Head
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

COPYRIGHT TRANSFER CERTIFICATE

Title of the Thesis: ***Generalized Derivatives for Interval-valued Functions and Their Applications in Interval Optimization Problems.***
Name of the Student: ***Ram Surat Chauhan***

Copyright Transfer

The undersigned hereby assigns to the Indian Institute of Technology (Banaras Hindu University), Varanasi all rights under copyright that may exist in and for the above thesis submitted for the award of the Ph.D. degree.

Date: 18/06/2021

Place: Varanasi

(Ram Surat Chauhan)

Note: However, the author may reproduce or authorize others to reproduce material extracted verbatim from the thesis or derivative of the thesis for author's personal use provided that the source and the Institute copyright notice are indicated.

ACKNOWLEDGEMENTS

Though only my name appears on the cover of this dissertation, so many great people have contributed to its production. I owe my gratitude to all those people who have made this thesis possible and because of whom my post graduate experience has been one that I will cherish forever.

First and foremost, I would like to express my heartfelt gratitude and everlasting indebtedness to my research supervisor, Dr. Debdas Ghosh, Assistant Professor, Department of Mathematical Sciences, IIT (BHU), Varanasi, for his immeasurable encouragement and insightful discussions. He bestowed upon me the greatest opportunity to achieve academic training par-excellence and learn communicative skills with absolute precision.

I would like to thank Prof. L. P. Singh and Prof. T. Som , our former and current Heads of the Department, respectively. They provided me with the requisite facilities to carry out my doctoral research during their tenure. Besides, I would also like to thank to the members of my doctoral scrutiny committee Prof. T. Som, Dr. Anuradha Banerjee of my Department, and Dr. Shyam Kamal of Department of Electrical Engineering, IIT (BHU), Varanasi. Their tireless encouragement and valuable comments during formal and informal seminars have strengthened and fertilized the ground of my research work. I wish to extend my warm and sincere gratitude to Prof. Subir Das, Convener, DPGC, of the Department of Mathematical Sciences and all faculty members of the department, especially Prof. K. N. Rai and Prof. Rekha Srivastav along with Prof. S. K. Pandey, Prof. S. Mukhopadhyay, Dr. Ashok Ji Gupta, Dr. V. K. Singh, Dr. R. K. Pandey, Dr. Sunil Kumar, Dr. Lavanya Sivakumar, Prof. Murali Krishna Vemuri, the other faculty members, teaching and non-teaching staffs of our department for their kind help during the period of my stay.

I am extremely grateful to Mr. Amit Kumar Debnath for his collaboration with me during my research work and his contribution to my research work. I thank my beloved senior Mr. Abhishek Singh for resolving most of the software problems. A special thanks go to Mr. Rakesh Kumar and Sumit Saini from my department for assisting me in preparing the thesis. I am also thankful to my friends and seniors: Mr. Harendra Kumar, Mr. Kushal Dhar Dwivedi, Mr. Ajay Kumar, Dr. Raj Kumar Gupta, Dr. Pappu Kumar, Mr. Kamlesh Kumar Pandey, Dr. Triloki Nath, Mr. Sanjeev Kumar Maurya, Mr. Jagdish Prasad Maurya, Dr. Vijay Kumar Yadav, Mr. Jitendra Pal Chaudhary, Dr. Neeraj Kumar Tripathi, Mr. Vijay Kumar Shukla, Mr. Abhishek Kumar, Mr. Mahesh Sharan, Mr. Pankaj Gautam, Avinash Dixit, and all the research scholars of the department for their moral supports.

Further, I would like to extend my thanks to my chamber mate Mr. Abhishek Singh, Mr. Gourav, Mr. Jauny, Mr. Ashutosh Upadhyay, Mr. Krishan Kumar, Ms. Anshika, and Ms. Suprova Ghosh for helping me a lot like a brother.

I am also grateful to my Institute, IIT (BHU), for providing necessary resources throughout my research. I express my thanks to all non-teaching staff members of the department for their supports. I feel very thankful to my close friends Mr. Deepak Tripathi, Mr. Vikash Srivastva, Mr. Vikas Maurya for their continued encouragement, stimulating help, and criticism.

Part of my capabilities and thinking process are due to my excellent undergraduate and postgraduate education at Dr. Ram Manohar Lohia Avadh University, Faizabad. My obligation does not fall short in extending its reverence to my post-graduate teachers Dr. S. K. Yadav, Prof. S. S. Mishra, Prof. C. K. Mishra, Prof.

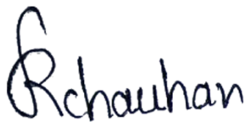
Lal Sahab Singh and to those friends of mine who have ensured a smooth transition and expedition for me into this world of mathematics.

I gratefully acknowledge the University Grants Commission, India for providing the fellowship in form of a Junior Research Fellowship and a Senior Research fellowship.

I express my sincere and cordial gratitude to my father Shri Bahraichi Chauhan, my mother Smt. Munni Devi, my elder brothers Shri Hirdayram Chauhan, Shri Kanikram Chauhan, my younger brother Mr. Arjun Chauhan and my elder sisters who always stood by my decisions and provided all kinds of supports, moral as well as financial. It was their love, care, and patience which encouraged me to move on. The person with the greatest indirect contribution to this work is my best friend Ms. Ragini Ranjan for her deepest love, endless patience, and continued support shown in the course of my research work.

This acknowledgment would be incomplete if the name of great visionary Pt. Madan Mohan Malaviya is not mentioned, who made this divine center of knowledge. Deepest regards to him.

Above all, praises and thanks to God, the Almighty, for His showers of blessings throughout my research work, who has made everything possible.



Ram Surat Chauhan

Contents

List of Figures

Abbreviations

IVF **I**nterval-**v**alued **F**unction

IOP **I**nterval **O**ptimization **P**roblem

DM **D**ecision **M**aker

Symbols

$\mathcal{X}$	real normed linear space with the norm $\|\cdot\|$
$\mathcal{S}$	nonempty subset of $\mathcal{X}$
$\mathcal{B}(\bar{x}, \delta)$	open ball centered at $\bar{x} \in \mathcal{X}$ with radius δ
$\overline{\mathcal{B}}(\bar{x}, \delta)$	$\mathcal{B}(\bar{x}, \delta) \setminus \{\bar{x}\}$
$\mathrm{cl}(\mathcal{S})$	closure of the nonempty set $\mathcal{S}$
$\mathcal{T}(\mathcal{S}, \bar{x})$	contingent cone to $\mathcal{S}$ at $\bar{x}$
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of nonnegative real numbers
$I(\mathbb{R})$	set of all compact intervals

PREFACE

Optimization is an approach that is used to characterizing, finding, and computing the maxima or minima of a function for a set of acceptable points or certain prespecified conditions. Its early stages were combined with ones of the differential calculus and mathematical analysis. The first idea of differential calculus and the rule for computing the maxima and minima was given by Fermat in 1638. Fermat introduced the optimality condition to obtain the extremum of a differentiable algebraic function f as $f'(x) = 0$. The rule of Fermat remains valid for the differentiable function of several variables and differentiable functions defined on topological and Hilbert vector spaces.

Optimization is not just mathematical analysis. Many decision-making problems in management, engineering, economics, computer sciences, and statistics are formulated as mathematical programs to find the maximization or minimization of an objective function subject to constraints and conditions. Such programs often have special structures: convex, nonconvex, linear, nonlinear, quadratic, semidefinite, dynamic, integer, stochastic programming, etc. This was the source of more theory and efficient algorithms to find the solutions. For these types of problems, there are vast mathematical principles and optimization techniques to handle them.

Now-a-days, there are many optimization and real-world problems with imprecise or uncertain parameters that cannot be handled by only deterministic or probabilistic and linear or nonlinear models. To determine the imprecise or uncertain parameters in such problems, interval analysis was introduced by Moore [57]. Interval analysis is based on the representation of an uncertain variable by an interval, and it provides a natural way of incorporating the uncertainties of parameters. The importance of interval analysis from a theoretical and practical view point is explained by Moore

[57] in his book. Advances of interval analysis have been motivated by its wide application areas, such as control theory, dynamical theory, machine learning, artificial intelligence, etc.

Since its introduction almost 60 years ago, the subject has developed rapidly. It is serving as an impetus for research and rigor in numerical computations on machines. An element of interval analysis has a dual nature as both a number and a set of real numbers. Most of the algorithms for interval methods make use of this duality nature and combine set-theoretic operations such as set intersection with arithmetic operations. For the problem in which coefficients and initial data are acceptable as intervals values, the entire set of possible values of the solution will obtain in a single computation. The ability to compute with sets of values in interval arithmetic provides for some simple computational tests for existence, uniqueness, and convergence.

In this thesis, the author discusses some properties of interval analysis, smooth and nonsmooth analysis of interval-valued functions (IVFs), and optimality conditions for interval optimization problems (IOPs). The basic introduction of optimization, interval analysis with its origin, interval-valued functions, interval optimization problem with its origin are given at the starting of the first chapter. Interval arithmetic and some important properties of intervals are also explained. The definitions of continuity and convexity for IVFs along with their basic results, are explained in the last section of the first chapter.

The notions of directional, Gâteaux and Fréchet derivatives for IVFs are studied. Further, the conditions for the existence of these derivatives for IVFs are given. To observe the properties of these derivatives, the concepts of bounded, linear, monotonic, and Lipschitz continuity for IVFs are newly defined. It is also explained that the proposed derivatives are useful to check the convexity of IVFs. Further, it is observed that the efficient solutions of IOPs can find out with the help of these derivatives. The entire study on these derivatives for IVFs is supported by suitable illustrative examples.

The concepts of Clarke derivative, pseudoconvex and quasiconvex for IVFs are proposed. To describe the properties of Clarke derivative, the concepts of limit superior, limit inferior, and sublinear for IVFs are studied. Further, by using the derived concepts, the existence of Clarke derivative, the relation of Clarke derivative with directional derivative, the relation of convex with pseudoconvex, and relation of pseudoconvex with quasiconvex are shown for IVFs. With the help of the studied pseudoconvex, quasiconvex, and Lipschitz IVFs, we present a few results on characterizing efficient solutions to an interval optimization problem with upper Clarke and Fréchet differentiable IVF. The entire study on these concepts is supported by suitable illustrative examples.

Next, the notion of Hadamard semiderivative for IVFs is explained. In the presence of directonal derivative, continuity and Lipschitz continuity of IVFs, a necessary and sufficient condition for the existence of Hadamard semiderivative of IVFs are derived. The relation of Hadamard semiderivative with directional derivative and Gâteaux derivative for IVFs are shown. Further, the behavior of composition of two Hadamard semidifferentiable IVFs and maximum of Hadamard semidifferentiable IVFs are explained. Proposed semiderivative is observed to be useful to check the convexity of an IVF and also helpful to find out the efficient points of IOPs. For constraint IOPs, the Karush-Kuhn-Tucker sufficient condition to obtain efficient solutions are derived. The entire study on Hadamard semiderivative for IVFs is supported by suitable illustrative examples.

The Hadamard derivative for IVFs is studied subsequently. For an IVF, the relation of Hadamard derivative with Fréchet derivative and continuity are shown. Further, behavior of composition of two Hadamard differentiable IVFs and the maximum of Hadamard differentiable IVFs are explained. The proposed derivative is observed to be useful to check the convexity of an IVF and also helpful to characterize the efficient solutions of IOPs. For constraint IOPs, an extended Karush-Kuhn-Tucker necessary and sufficient condition by using the proposed derivative is derived. The entire study on Hadamard derivative for IVFs is supported by suitable illustrative examples.

Next, the notions of upper and lower Dini semiderivatives, upper and lower Hadamard semiderivatives for IVFs are studied. For an IVF, the relation of upper Dini semiderivative and upper Hadamard semiderivative with directional derivative, Clarke derivative, Hadamard semiderivative and continuity are shown. Proposed semiderivative is observed to be useful to characterize the efficient solutions of IOPs. The entire study on these semiderivatives for IVFs is supported by suitable illustrative examples.