Chapter 1

Introduction

1.1 Singularly perturbed problems

Among boundary value problems, a significantly studied class includes the problems for singularly perturbed differential equations. The main reason is that in diverse areas of engineering and applied mathematics the modelling of several physical phenomena is done through singularly perturbed differential equations. For example, in computational fluid dynamics, hydrodynamics, chemical reactor theory, financial modelling, gas porous electrodes theory, mathematical biology, heat and mass transfer processes in composite materials with small diffusion coefficients. Mathematical models involving singularly perturbed differential equations appear, for example, in modelling of semiconductor devices, viscous fluid flow problems with large Reynolds numbers, and convective heat transfer problems with large Péclet numbers.

The first time singularly perturbed problems came in light in 1904 when Prandtl presented his article [1] on boundary layer phenomena in the Third International Congress of Mathematicians. This ingenious research explained the substantial effect of small viscosity in a fluid flow about the body. Due to the effect of viscosity in a very thin region near the body surface fluid velocity decreases rapidly from its steady value far from the body. This results in two non-uniform velocity profiles. This shows that multiscale behavior may arise, even in simple flow problems, and it is also important to see how erroneous it may be if we neglect the same. However, the term "Singular perturbation" first came in literature in 1946 in the paper [2] by Freidrichs and Wasow.

Singular perturbation theory completely changed the foundation of modern fluid dynamics. In many fields of science and engineering the physical phenomena are modelled by singularly perturbed problems, yet it is strange to see that in the earlier stage not much interest was shown in solving them. A substantial work on singularly perturbed problems was first recorded in the form of the Ph.D. thesis of Wasow [3] in 1941. The theory of singular perturbation gained its greatest significance through Freidrichs and Wasow's groundbreaking works [2, 4]. After that numerous mathematicians started working on this branch of mathematics. In the last few decades this branch of mathematics has flourished to a good level. Although many useful techniques have been developed, important developments are continuing and the advantageous research is on.

A formal definition of singularly perturbed problems can be found in [5].

Definition 1.1.1. [5] Let $(\mathcal{P}_\varepsilon)$ be a problem with solution $u_\varepsilon \in S$ for all $\varepsilon \in G$, where S a is function space with norm $\|.\|_S$ and $G \subset \mathbb{R}^n$ is a parameter domain. The continuous function $u : G \to S, \varepsilon \mapsto u_\varepsilon$ is said to be regular for $\varepsilon \to \varepsilon^\star \in \partial G$ if there exists a function $u^\star \in S$ such that:

$$\lim_{\varepsilon \to \varepsilon^\star} \|u_\varepsilon - u^\star\|_S = 0,$$

otherwise we say u_ε is **singular** and $(\mathcal{P}_\varepsilon)$ is **singularly perturbed**.

It is important to note that the above definition of singularly perturbed problems is norm dependent. A good discussion in this direction can be found in [5, 6]. Further, it is found that the maximum norm is the most suitable norm for studying singularly perturbed problems.

Singularly perturbed differential equations are typically characterized by a small perturbation parameter multiplied with the highest order derivative term. Unlike regularly perturbed problems, their solutions (or their derivatives) approach a discontinuous limit as perturbation parameter ε approaches zero. In general, the solutions of singularly perturbed problems exhibit multiscale phenomena, that is, solutions vary rapidly within some parts of the domain and behave smoothly away from them. The regions of the domain where this rapid change occurs are called as *layer regions* and where solutions behave smoothly are called as *regular regions*. If the layer regions appear near the boundaries of the domain, they are called as boundary layers regions, otherwise they are called as interior layer regions.

1.2 Numerical solutions of singularly perturbed differential equations

Analytical solutions of most of singular perturbed differential equations arising in physical systems is not known or very difficult to find. In this situation the mainly two approaches to find their solutions are numerical methods and asymptotic expansions. The asymptotic analysis tries to gain insight into the qualitative behavior of a family of problems. A decent literature on asymptotic analysis of singularly perturbed problems is available in the books [7–12] . Note that for most of the singularly perturbed nonlinear and partial differential equations, it is not possible to construct asymptotic expansions. So, one has to go numerical methods for such problems. Some good reference books on numerical methods for singuarly perturbed problems are [5, 6, 13, 14].

The solutions of singularly perturbed problems vary rapidly within the layer regions, which makes it unfeasible to obtain a satisfactory numerical solution with the help of traditional numerical methods, because they require a very large number of mesh points to resolve the layers. More specifically, we need the number of mesh points to be inversely proportional to some powers of the perturbation parameter, which is not practical for small perturbation parameter. Therefore, it is important to develop numerical methods that behave well for all values of the perturbation parameter, no matter how small. Such methods are called as parameter-uniform/ parameter-robust/uniformly convergent/robust convergent methods.

Definition 1.2.1. [5] Let $(\mathcal{P}_\varepsilon)$ be a problem with solution u_ε and let U_ε^N be its approximation obtained by some numerical method. The method is said to be uniformly convergent or robust with respect to the perturbation parameter ε in a given norm $\|.\|_\star$ if there exists N_0 independent of ε such that

$$\|u_\varepsilon - U_\varepsilon^N\|_\star \leq C\vartheta(N) \quad \text{for } N \geq N_0,$$

with a function ϑ that is independent of ε and $\lim\limits_{N\to\infty} \vartheta(N) = 0$; and a constant $C > 0$ that is independent of ε and N.

Mainly, there are two common strategies for constructing parameter-robust numerical methods: *fitted operator* and *fitted mesh* approaches. In *fitted operator approach* the problem is discretized on a uniform mesh by a specially designed discrete operator that captures the layer behavior of the solution. This approach was first suggested by Allen and Southwell [15] in 1955 for a fluid flow problem of a viscous fluid past a cylinder. It is possible to construct an appropriate fitted operator on a uniform mesh for singularly perturbed problems with regular boundary layers, but in [16] and [17] it is established that there exists no fitted operator on a uniform mesh for singularly perturbed problems with parabolic boundary layers. Also, for problems in higher dimensions and involving nonlinearity, construction of a fitted operator is a very difficult task. More insights about fitted operator methods can be found in [14, 15, 18–22].

Fitted mesh approach involves the construction of a fitted mesh that is adapted to the multiscale behavior of the solution of a singularly perturbed problem. On these non-uniform meshes that are condensed towards the boundary layers, only standard finite difference/element operators are enough to produce good numerical approximations and obtain parameter-robust numerical methods. The advantegeous thing about the fitted mesh methods over fitted operator methods is that the nonlinear problems and higher dimensional problems involving complicated domain structures can also be easily dealt with fitted mesh methods. For the first time in 1969 Bakhvalov [23] proposed a fitted mesh which is generated by a suitable mesh generating function which appropriately redistributes an equidistant mesh, so that the maximum number of mesh points lie inside the boundary layer region(s). Some utilizations of this mesh can be found in [24–30]. However, Bakhvalov mesh is applied to a large range of problems, its complicated construction always create difficulties to extend it to higher dimensional problems. Thereafter, two more graded meshes are introduced by Vulanović [31] and Gartland [32]. The construction of these meshes is also very complex and have the similar difficulty level in extending them to higher dimensions. Another frequently-studied and relatively simpler mesh is Shishkin mesh [33], which is actually a piecewise-uniform mesh constructed with the help of a transition point. Based on the *a priori* information about the solution transition point is chosen in such a way that half of the mesh points lie inside the boundary layer(s) and rest half lie outside. Shishkin mesh is the most favourable fitted mesh because of its simplicity and applicability to more complicated problems. However, it is observed that rate of convergence for approximate solutions on a piecewise-uniform Shishkin mesh is not optimal with respect to the discretization of the problem. Some applications of Shishkin meshes are found in [6, 18, 18, 30, 34–45].

If sufficient information about the location and width of the layers is available, appropriate layer-adapted meshes discussed above can be constructed. A more popular approach for constructing highly non-uniform layer-adaptive meshes is based on the equidistribution principle [46]. This approach involves the equidistribution of a positive monitor function and aims to cluster automatically the maximum number of mesh points in the layer regions. The monitor function automatically detects the the presence, location, and width of the layers and mesh points are accordingly distributed. It is important to note that it produces optimal convergent results. We shall discuss this approach in detail in the next section.

1.3 Mesh equidistribution

The idea of mesh equidistribution is first introduced by de Boor [47]. Starting with a uniform mesh, this approach aims to condense the maximum number of mesh points inside the layer region(s) by equidistributing a positive function of the solution over each sub-interval of the domain. This positive function is approximated from the solution of the original problem and is known as the monitor function since it defines a measurement of the numerical error. We can define the equidistributed mesh as follows:

Definition 1.3.1. The mesh $\{x_i\}_{i=0}^{N}$ is said to be defined through equidistribution of the monitor function $\mathcal{M}(y(x), x)$ if

$$\int_{x_{i-1}}^{x_i} \mathcal{M}(y(z), z)\, dz = \int_{x_i}^{x_{i+1}} \mathcal{M}(y(z), z)\, dz, \quad 1 \le i \le N-1, \tag{1.1}$$

or equivalently,

$$\int_{x_{i-1}}^{x_i} \mathcal{M}(y(z), z)\, dz = \frac{1}{N} \int_0^1 \mathcal{M}(y(z), z)\, dz, \quad 1 \leq i \leq N. \tag{1.2}$$

Equidistributing condition can also be represented by an invertible mapping from the computational uniform coordinates $\xi \in [0, 1]$ to the physical non-uniform coordinates $x \in [0, 1]$ as

$$\int_0^{x(\xi)} \mathcal{M}(y(z), z)\, dz = \xi \int_0^1 \mathcal{M}(y(z), z)\, dz. \tag{1.3}$$

The construction of equidistributed mesh is based on the optimal choice of the monitor function. Some factors affecting the optimality of the monitor function are: the type of the problem being solved, the numerical discretization being used, and the norm of the error to be controlled. In [48], three types of monitor functions are mentioned, which are of arc length type, combination of the curvature and gradient type, and based on the truncation error or the solution residual. Arc length type monitor functions have been considered by many authors, for instance [49–52]. Also, *a posteriori* error estimation corresponding to the arc length monitor function is done by Kopteva in [53]. However, in [54] it is pointed out that the arc length based monitor function is unsuitable for reaction-diffusion type problems. Beckett and Meckenzie in [55, 56] proposed a curvature based monitor function which works fine for a wider class of singularly perturbed problems (see [57–62]). The monitor function based on truncation error or solution residual is considered in [63]. Some more insights about the mesh equidistribution technique can be found in [36, 46, 49, 50, 64]. Adaptive numerical methods based on equidistributed meshes have been successfully applied to a variety of singularly perturbed problems (see [51, 54, 56, 58, 65–68]).

The success of these methods is due to the exponentially stretched mesh we obtain, which results in an improved rate of convergence as compared to the piecewise-uniform fitted meshes. The use of a strictly positive monitor function theoretically guarantees the existence and uniqueness of the equidistributed mesh. However, rarely we can find it exactly, as the integrals in (1.1) are normally approximated. A few algorithms are proposed in this regard including de Boor [47], and some of its modifications by Pryce [69] and Linß [70]. It is observed that de Boor's algorithm numerically produces better results than the other modified versions. The convergence of de Boor algorithm for singularly perturbed convection-diffusion problems is dicussed by Kopteva and Stynes in [71] and for singularly perturbed reaction diffusion problems by Chadha and Kopteva in [72]. In this thesis, for the construction of adaptive meshes using equidistribution of the proposed monitor functions de Boor's algorithm is used.

1.4 Literature review

Since the past few decades the area of singularly perturbed problems have been a centre of attraction for many researchers because of their regular occurence in the modelling of various physical phenomena in science and engineering. Many reference books like [5, 6, 18, 73] are available. Also, see the survey articles [74, 75]. We now present a brief literature review for some classes of singularly perturbed problems that are considered in this thesis.

1.4.1 Singularly perturbed degenerated convection-diffusion problems

In the recent few years singularly perturbed degenerate problems attracted the attention of various researchers due to their importance in the modeling of many physical phenomena (see [76, 77]). For the numerical solution of singularly perturbed degenerated convection-diffusion problems, in [78] a classical implicit upwind difference scheme on a piecewise-uniform Shishkin mesh in space and a uniform mesh in time is considered. In [79], the Richardson extrapolation technique is considered. In [80], a hybrid scheme on a piecewise-uniform Shishkin mesh in space and the backward Euler scheme on a uniform mesh in time is considered. In [81], a parameter-robust numerical method is given for a singularly perturbed degenerate convection–diffusion problem with discontinuous source term. In [82], the backward Euler method on a uniform mesh in time and a hybrid scheme on a generalized Shishkin mesh in space is considered. Note that the numerical methods developed in all the above papers are based on Shishkin meshes. Therefore, it is important to develop a robust numerical method based on equidistributed meshes for singularly perturbed degenerated convection-diffusion problems.

1.4.2 Singularly perturbed parabolic reaction-diffusion problems

Going through the literature, we see that the numerical solution of this class of problems has drawn a lot of attention of researchers since a long time. In [34], a standard finite difference approximation (central difference in space and backward difference in time) on a fitted piecewise-uniform Shishkin mesh is considered and the method is proved to be almost second order in space and first order in time. A discrete Green's function based approach, that avoids both the solution decomposition and use of special barrier functions, is used in [83]. On a general characterization of fitted meshes the time-independent problem is analysed for convergence in the same framework for both Shishkin and Bakhvalov meshes. In [84], the problem is discretized by a scheme which is a combination of the classical central difference and a cubic spline scheme for the spatial derivative and the backward difference scheme for the time derivative. Then, stability and error estimate on Shishkin mesh are obtained. In [85], a numerical method with convergence of almost fourth order in space and second order in time is constructed using the compact finite difference scheme on a generalized Shishkin mesh in space and Crank–Nicolson scheme on a uniform mesh in time. Some more papers constructing high order parameter-robust convergence are published by Clavero and Gracia (see [86], [87] and [88]). Then, in [59], these problems are considered with a layer-adaptive equidistributed mesh at each time level. The non-uniform spatial meshes in this paper are obtained by equidistribution of a positive monitor function. The second and first order convergence in space and time is shown. This may be noted that all the above references are with Dirichlet type boundary conditions. A very less amount of work has been done so far for parabolic reaction-diffusion problems with Robin boundary conditions (RBCs). Some papers considering only stationary reaction-diffusion problems with RBCs are [89, 90]; both the papers considered Shishkin mesh to resolve the layers. Therefore, construction of efficient higher order numerical methods based on layer-adaptive equidistributed meshes for singularly perturbed parabolic problems with Dirichlet and Robin boundary conditions are required.

1.4.3 Singularly perturbed time delayed parabolic reaction-diffusion problems

Singularly perturbed delay differential equations often arise in the modelling of various physical, biological and chemical systems, such as in population dynamics, variational problems in control theory, epidemiology, circadian rhythms, respiratory system, chemostat models, tumor growth and neural networks. The delay term in these models enable us to include some past behaviour to get more practical models for the phenomena. A wide range of examples of delay models can be found in [91]. In [92], a numerical method comprising a standard finite difference operator (central differencing in space and the backward difference in time) on a rectangular piecewise-uniform fitted mesh is developed and is proved to be almost second order convergent in space and first order convergent in time. A high order (almost fourth order in space and second order in time) parameter-robust method for these problems is presented in [93]. In this paper, a hybrid scheme on a generalized Shishkin mesh is considered in spatial direction and the implicit Euler scheme on a uniform mesh is considered in time direction. Then, the order of convergence in time direction is increased by Richardson extrapolation scheme. Some more references in this sequence are [94–96]. Then, in [97] a numerical method for the delay problem consisting of the implicit Euler scheme for the time derivative and the classical central difference scheme for the spatial derivative together with the domain discretized by a uniform mesh in the time and a non-uniform equidistributed mesh in space. Note that in all the above papers the time delay problems with Dirichlet boundary conditions are considered. However, the study of time delay problems with Robin boundary conditions is still at an infant stage. Recently, in [98] the authors considered a singularly perturbed parabolic reaction-diffusion problems with time delay and constructed a finite difference method that is almost second order convergent in space and first order convergent in time. But, best to our knowledge, no work has been done to develop a parameter-robust numerical method based on automatically generated adaptive meshes for the partial differential equations of singularly reaction-diffusion type with Robin boundary conditions. This motivated us to construct an efficient higher order numerical method based on layer-adaptive equidistributed meshes for the same.

1.4.4 Nonlinear singularly perturbed Volterra integro-differential equation

These problems are an important class of problems, because of their regular appearance in many applications in various physical and biological systems, such as diffusion-dissipation processes, filament stretching problems, epidemic dynamics, and synchronous control systems [99–102]. Numerical methods for a linear singularly perturbed Volterra integro-differential equation (VIDE) are developed in [103–107]. More specifically, an exponential type difference scheme is developed in [104]. In [106] the problem is solved using a fitted operator technique on a piecewise-uniform Shishkin mesh. In [105] a tension spline collocation method is presented. In [103] a backward difference formula is used for the derivative and a repeated quadrature rule is used for the integral term. Further, a Bakhvalov type mesh is used to resolve the layer. In [107] the integrand is considered to be $(x-s)^{-\alpha}y(s)$ with $0<\alpha<1$. Assuming the source term $g(x)$ such that $|g'(x)| \leq (1+x^{-\alpha})$, a posteriori error estimate for a linear VIDE is derived. More precisely, it is proved that $||\widetilde{Y}-y||_\infty \leq C \max_{1\leq i\leq N}(h_i^{1-\alpha}+h_i|D^-Y_i|)$, where $\widetilde{Y}$ is the piecewise linear interpolant of the computed solution Y and D^- is the backward difference operator. But surprisingly this a posteriori error estimate is not used for the adaptive mesh generation, instead an arc-length based monitor function is used. Numerical methods for a nonlinear singularly perturbed VIDE are developed in [108, 109]. In [108] the nonlinear problem with a special Kernel is solved by asymptotic expansions and an implicit Runge-Kutta method. In [109] a first order parameter-robust finite difference scheme is constructed on a Bakhvalov type mesh. To the best of our knowledge, no published paper developed a numerical method based on equidistributed meshes for nonlinear singularly perturbed VIDEs. This gap in the literature is the motivation of our work for this problem.

1.5 Outline of the thesis

This thesis is concerned with the construction and analysis of robust adaptive numerical methods for singularly perturbed problems in integro and partial differential equations. After our literature survey on adaptive mesh generation (by equidistributing a monitor function over the domain) for singularly perturbed problems, we found that there are many interesting and challenging problems still open. The main hindrance that is faced by the researchers are: a proper choice of the monitor function and convergence analysis of the designed methods. We can see from literature survey that there are some advances in this direction, but the subject is very far from being developed. The work embodied in the thesis is divided into six chapters.

Chapter 1 consists of an introduction to singularly perturbed problems, parameter-robust numerical methods, and adaptive mesh generation based on the equidistribution principle. It also provides a brief literature review, thesis objectives, and outline of the thesis.

In Chapter 2, we propose an adaptive numerical method for a class of singularly perturbed degenerate parabolic convection-diffusion problems posed on a rectangular domain. The problem is discretized using the implicit Euler scheme on a uniform mesh in time and upwind finite difference scheme on a layer-adaptive non-uniform spatial mesh generated through the equidistribution of a suitably chosen monitor function. The error analysis of the proposed method is given based on the truncation error and barrier function approach. It is proved that the proposed method is robust convergent of order one in both space and time. Numerical results are provided in support of theoretical findings.

In Chapter 3, we develop a parameter-robust numerical method on equidistributed meshes for solving a class of singularly perturbed parabolic reaction-diffusion problems with Robin boundary conditions. The discretization consists of a modified Euler scheme in time, a central difference scheme in space, and a special finite difference scheme for the Robin boundary conditions. On the adaptively generated equidistributed mesh we discuss error analysis and prove that the method is parameter-robust convergent of order two in space and order one in time. To support the theoretical result, numerical results on some test examples are provided.

In Chapter 4, a robust finite difference method is presented on the adaptive mesh for a singularly perturbed parabolic reaction-diffusion problem with time delay and Robin type boundary conditions. The adaptive mesh generation through the equidistribution of a positive monitor function and the finite difference discretization is analogous to that in Chapter 3. For the proposed method, parameter-robust convergence of second order in space and first order in time is shown through rigorous error analysis. Some numerical experiments are conducted in support of the theory.

In Chapter 5, we propose a high order parameter-robust numerical method for singularly perturbed time dependent reaction-diffusion boundary value problems.

The numerical scheme comprises of the implicit Euler scheme to discretize in time and a high order non-monotone finite difference scheme to discretize in space. The analysis of the method is done in two steps, splitting the contribution to the error from the time and space discretizations. It is shown that the method is robust convergent having order one in time and order four in space. Further, we use the Richardson extrapolation technique to improve the order of convergence from one to two in time. Numerical experiments are presented to confirm the theoretically proven convergence result.

In Chapter 6, we consider a nonlinear singularly perturbed Volterra integro-differential equation. The problem is discretized by an implicit finite difference scheme on arbitrary non-uniform meshes. The scheme comprises of an implicit difference operator for the derivative term and an appropriate quadrature rule for the integral term. The numerical scheme is proved to be uniformly stable on an arbitrary non-uniform mesh. We establish a posteriori error estimate for the scheme that holds true uniformly in the small perturbation parameter. Numerical experiments are performed and results are reported for validation of the theoretical error estimate.

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