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It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy and SOTA for the award of Ph.D. degree.

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DECLARATION BY THE CANDIDATE

I, ***Sumit***, certify that the work embodied in this thesis is my own bona fide work and carried out by me under the supervision of ***Dr. Sunil Kumar*** from ***July 2016*** to ***April 2021*** at the ***Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi***. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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Varanasi-221005 **Sumit**

Dedicated

to

My Parents

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Symbols

$\mathbb{R}$	set of real numbers
$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
ε	perturbation parameter
$N,\ M,\ M_\tau$	discretization parameters
$\mathcal{O}$	Landau symbol
$\mathcal{L},\ \mathcal{L}_\varepsilon$	differential operators
$L^N,\ L^{N,M},\ L_\varepsilon^N,\ L_\varepsilon^{N,M}$	difference operators
G	given problem domain
$G^{N,M}$	discrete problem domain
G_x	open interval $(0,1)$
$\bar{G}_x$	closed interval $[0,1]$
$G_x^N,\ \bar{G}_x^N$	discrete spatial domain
$g_{i,j}$	$g(x_i,t_j)$
$\lVert g \rVert_Q$	$\max_{(x,t)\in Q} \lvert g(x,t) \rvert$
$\lVert g \rVert_{Q^{N,M}}$	$\max_{(x_i,t_j)\in Q^{N,M}} \lvert g(x_i,t_j) \rvert$

C generic positive constant, independent of ε, N, and M

PREFACE

This thesis is concerned with the construction and analysis of robust adaptive numerical methods for singularly perturbed problems in integro and partial differential equations. The main feature of these problems is that their solutions possess boundary layers, due to which classical numerical methods on uniform meshes fail to provide good numerical numerical approximations to the solutions of these problems. So, special techniques are required to resolve the boundary layers and obtain numerical approximation that converges to the exact solution independent of the perturbation parameter.

Firstly, a class of singularly perturbed degenerate parabolic convection-diffusion problems is considered on a rectangular domain. The finite difference discretization consists of an upwind finite difference scheme on a layer-adaptive non-uniform mesh in the spatial direction and an implicit Euler scheme on a uniform mesh in the time direction. The adaptive mesh in spatial direction is generated via equidistribution of a suitably chosen monitor function. The error analysis is performed for the proposed method using truncation error and barrier function approach and the method is proved to be robust convergent of first order in both time and space.

Next, a robust convergent adaptive numerical method is developed for solving a class of singularly perturbed parabolic reaction-diffusion problems with Robin boundary conditions. The discretization consists of a modified Euler scheme in time, a central difference scheme in space, and a special finite difference scheme for the Robin boundary conditions. The adaptive mesh in spatial direction is generated via equidistribution of a suitably chosen monitor function. We discuss error analysis and prove that the method is robust convergent of order two in space and order one in time. Then, this adaptive numerical method is extended for a singularly perturbed time delayed parabolic reaction-diffusion problem with Robin boundary conditions. The method is proved to be robust convergent of order two in space and order one in time.

Thereafter, a high order numerical method is constructed for a class of singularly perturbed time dependent reaction-diffusion boundary value problem on a layer-adaptive equidistribution mesh. The numerical scheme comprises of the implicit Euler scheme to discretize in time and a high order non-monotone finite difference scheme to discretize in space. The analysis of the method is done in two steps, splitting the contribution to the error from the time and space discretizations. It is shown that the method is robust convergent having order one in time and order four in space. Further, we use the Richardson extrapolation technique to improve the order of convergence from one to two in time.

At the end, a class of nonlinear singularly perturbed Volterra integro-differential equation is considered. The problem is discretized by an implicit finite difference scheme on arbitrary non-uniform meshes. The scheme comprises of an implicit difference operator for the derivative term and an appropriate quadrature rule for the integral term. The numerical scheme is proved to be uniformly stable on an arbitrary non-uniform mesh. We derive a posteriori error estimate for the scheme that holds true uniformly in the small perturbation parameter.

Extensive numerical experiments are performed to validate the obtained theoretical error estimates.