Chapter 5

Strongly convergent Algorithms to Solve Monotone Inclusion Problems

In previous chapters of the thesis, we have proposed iterative methods which are guaranteed to show weak convergence behavior under mild assumptions. Researchers assume strong conditions like strong convexity or strong monotonicity on the operators to prove strong convergence of the algorithms. This chapter is dedicated to propose and study strongly convergent algorithms to solve monotone inclusion problem without assuming strong convexity or strong monotonicity. Section 5.2 recalls some important results in nonlinear analysis. In Section 5.3, we propose a generalized Mann and normal-S iteration and study its convergence behavior. In Section 5.4,

This chapter is based on our submitted research work "**Dixit, A.**, Sahu, D. R., Gautam, P., and Som, T. (2021) Strongly convergent Algorithms to Solve Monotone Inclusion Problems. Optimization."

we propose a new forward-backward algorithm and a forward-backward type primal-dual algorithm to solve the inclusion problem and complexly structured monotone inclusion problem, respectively. In Section 5.5, we propose Douglas-Rachford type algorithms to solve monotone inclusion problems and complexly structured monotone inclusion problems of set-valued operators. In the last, we performed a numerical experiment to show the importance of proposed algorithms to solve the image deblurring problem.

5.1 Introduction

In Chapter 1, we have discussed the proximal point algorithm. Rockafellar [80] modified the proximal point agorithm and proposed an inexact proximal point algorithm as follows:

$$x_{n+1} = J_{c_nT}(x_n + v_n), \quad \forall n \in \mathbb{N}, \tag{5.1}$$

where v_n is the error term in $\mathcal{H}$. The sequence $\{x_n\}$ also converges weakly to the solution set of inclusion problem provided $\sum_{n=1}^{\infty} v_n < \infty$ and sequence $\{v_n\}$ is bounded away from zero. Guler [44] showed by an example that sequence generated by proximal point algorithm (1.7) converges weakly, but not strongly, in general. It becomes a matter of interest for the research community to modify the proximal point algorithm to obtain strong convergence. In such consequences, Tikhonov method was proposed which generates as follows,

$$x_{n+1} = J_{c_nT}(x), \tag{5.2}$$

where $x \in \mathcal{H}$ and $c_n > 0$ such that $c_n \to \infty$. Detailed study of Tikhonov regularization method can be found in [26, 93, 92, 91, 96]. Lehdili and Moudafi [55] combined the idea of proximal algorithm and Tikhonov regularization to find an algorithm converges strongly to the solution of inclusion problem 1.0.1. They solve the inclusion problem 1.0.1 by solving the inclusion problem of fixed approximation of T, which is $T_n = T + \mu_n Id$, i.e.,

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in T_n(x),$$

where μ_n is a regularization parameter. The proximal-Tikhonov algorithm is given by

$$x_{n+1} = J_{\lambda_n}^{T_n}(x_n).$$

The Tikhonov regularization term $\mu_k Id$ impelled the strong convergence to the algorithm. In the absence of Tikhonov regularization term, proximal-Tikhonov algorithm becomes the proximal algorithm which shows only weak convergence in most of the cases. Strong convergence of the algorithm can be obtained by using some other techniques also, some of them can be found in [8, 46].

The weak convergence of the algorithms reduces its applicability in infinite dimensional spaces. To achieve the strong convergence of algorithms one assumes stronger assumptions like strong monotonicity and strong convexity, which is difficult to achieve in many applications. This situation lefts a question to the research community: can we find the strongly convergent algorithms without assuming these strong assumptions? The answer to this question is replied positively by Bot et al. in [17]. They modified the Mann algorithm as follows:

$$x_{n+1} = e_n x_n + \theta_n(S(e_n x_n) - e_n x_n), \tag{5.3}$$

where e_n, θ_n are positive real numbers. The strong convergence of algorithm (5.3) for nonexpansive operator, S is studied in Bot et al. in [17] when set of fixed points of S is nonempty and parameters θ_n and e_n satisfy the following:

(i) $0 < e_n < 1$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} e_n = 1$, $\sum_{n=1}^{\infty}(1-e_n) = \infty$ and $\sum_{n=1}^{\infty} |e_n - e_{n-1}| < \infty$;

(ii) $0 < \theta_n \leq 1$ for all $n \in \mathbb{N}$, $0 < \liminf_{n\to\infty} \theta_n$, $\sum_{n=1}^{\infty} |\theta_n - \theta_{n-1}| < \infty$.

We consider the more general problem which is as follows:

Problem 5.1.1. *Consider $T, S : \mathcal{H} \to 2^{\mathcal{H}}$ are monotone operators. Find a point $x \in \mathcal{H}$ such that $0 \in Tx \cap Sx$.*

Remark 5.1. The algorithm (5.3) proposed by Bot et al. [17] can not apply to solve inclusion problem 1.0.1.

In this paper, we introduce the normal-S iteration method based fixed point algorithm to find common fixed point of nonexpansive operators $\mathrm{T}, \mathrm{S} : \mathcal{H} \to \mathcal{H}$, which converges strongly to minimal norm solutions of common fixed point problem of operators S and T. Based on the proposed fixed point algorithm, we develop a forward-backward algorithm and a Doughlas-Rachford algorithm containing Tikhonov regularization term to solve the monotone inclusion problems. In many cases, monotone inclusion problems are very complex, they contain mixtures of composite and parallel sum monotone operators. Recently, many researchers have proposed primal-dual algorithms to precisely solve the considered complex monotone inclusion system [19, 18, 35, 21, 95]. We propose a forward-backward type primal-dual algorithm and a Doughlas-Rachford type primal-dual algorithm having Tikhonov regularization term to find the common solution of the complexly structured monotone inclusion problems. The proposed algorithms have a special property that all the operators are evaluated separately.

5.2 Preliminary Results

This section devotes to some important results from nonlinear analysis and operator theory. Let (X_1, d_1) and (X_2, d_2) be metric spaces, let $T : X_1 \to X_2$, and C be a subset of X_1. Then T is Lipschitz continuous with constant $\beta \in (0, \infty)$ if

$$(\forall x \in X_1)(\forall y \in X_1)\ d_2(Tx, Ty) \leq \beta d_1(x, y).$$

Definition 5.2.1. *Let D be a nonempty subset of a Hilbert space $\mathcal{H}$ and let $T : D \to \mathcal{H}$ be a mapping. Then*

(a) T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \ \ \textit{for all} \ \ x, y \in D,$$

(b) firmly nonexpansive if

$$(\forall x \in D)(\forall y \in D)\ \|Tx - Ty\|^2 + \|(Id - T)x - (Id - T)y\|^2 \leq \|x - y\|^2,$$

(c) quasinonexpansive if

$$(\forall x \in D)(\forall y \in \mathrm{Fix}(T))\ \|Tx - y\| \leq \|x - y\|.$$

Definition 5.2.2. *Let D be a nonempty subset of $\mathcal{H}$, let $T : D \to \mathcal{H}$ and let $\beta \in (0, \infty)$. Then T is β-cocoercive (or β-inverse strongly monotone) if βT is firmly nonexpansive, i.e.*

$$(\forall x \in D)(\forall y \in D)\beta\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle.$$

Definition 5.2.3. *Let D be a nonempty subset of $\mathcal{H}$, let $T : D \to \mathcal{H}$ be nonexpansive and let $\alpha \in (0, 1)$. Then T is averaged with constant α or α-averaged, if there exists a nonexpansive operator $R : D \to \mathcal{H}$ such that $T = (1 - \alpha)Id + \alpha R$.*

Let X be a real vector space. Let C be a subset of X. C is a cone if

$$C = \mathbb{R}_{++}C,$$

where $\mathbb{R}_{++} = \{\lambda \in \mathbb{R} | \lambda > 0\}$.

Definition 5.2.4. *The intersection of all the linear subspaces of X containing C, i.e., the smallest linear subspace of X containing C is denoted by span C, its closure is the smallest closed linear subspace of X containing C and it is denoted by $\overline{\text{span}}\ C$. Let $\mathcal{C}$ be a nonempty subset of $\mathcal{H}$. Then*

interior of $\mathcal{C}$ is

$$int\ \mathcal{C} = \{x \in \mathcal{C} : (\exists \rho > 0) B(0; \rho) \subset \mathcal{C} - x\}$$

strong relative interior of $\mathcal{C}$ is

$$sri\ \mathcal{C} = \{x \in \mathcal{C} : cone(\mathcal{C} - x) = sp\bar{a}n(\mathcal{C} - x)\}$$

strong quasi-relative interior of $\mathcal{C}$ is

$$sqri\ \mathcal{C} = \{x \in \mathcal{C} : \bigcup_{\rho > 0} \rho(\mathcal{C} - x) is\ a\ closed\ linear\ subspace\ of\ space\ \mathcal{H}\}.$$

Lemma 5.2.1. *[9, Proposition 25.1(ii)] If T_1 and T_2 are monotone operators then the set of zeros of their sum* $\operatorname{zer}(T_1 + T_2) = J_{\gamma T_2}(\operatorname{Fix}(R_{\gamma T_1} R_{\gamma T_2}))\ \forall \gamma \geq 0$.

Proposition 5.2.1. *[9] Consider $T_1, T_2 : \mathcal{H} \to \mathcal{H}$ be α_1, α_2-averaged operators, respectively. Then the averaged operator $T_1 \circ T_2$ is $\alpha = \frac{\alpha_1+\alpha_2-2\alpha_1\alpha_2}{1-\alpha_1\alpha_2}$-avereaged.*

Lemma 5.2.2. [9] *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping. Let $\{u_n\}$ be a sequence in $\mathcal{H}$ and $u \in \mathcal{H}$ such that $u_n \rightharpoonup u$ and $u_n - Tu_n \to 0$ as $n \to \infty$. Then $u \in \mathrm{Fix}(T)$.*

Lemma 5.2.3. *[96] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1-\theta_n)a_n + \theta_n b_n + \epsilon_n \quad \forall n \geq 0,$$

where

(i) $0 \leq \theta_n \leq 1$ *for all* $n \geq 0$ *and* $\sum_{n\geq 0} \theta_n = \infty$;

(ii) $\limsup_{n\to\infty} b_n \leq 0$;

(iii) $\epsilon_n \geq 0$ *for all* $n \geq 0$ *and* $\sum_{n\geq 0} \epsilon_n < \infty$. *Then the sequence* $\{a_n\}$ *converges to 0.*

5.3 Strongly convergent common fixed point algorithm

This section devotes to investigate a computational theory for finding common fixed points of nonexpansive operators. We introduce a common fixed point algorithm such that sequence generated by the algorithm strongly converges to the set of common fixed points of mappings.

Algorithm 5.3.1. *Let $S, T : \mathcal{H} \to \mathcal{H}$ be nonexpansive mappings. Select $\{e_n\}$, $\{\theta_n\} \subset (0,1)$ and compute the $(n+1)^{th}$ iteration as follows:*

$$y_{n+1} = S[(1-\theta_n)e_n y_n + \theta_n T(e_n y_n)] \quad \textit{for all } n \in \mathbb{N}. \tag{5.4}$$

We now study the convergence behavior of Algorithm 5.3.1 for finding the common fixed point of S and T.

Theorem 5.3.1. *Let $S, T : \mathcal{H} \to \mathcal{H}$ be nonexpansive mappings such that $\Omega := \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \neq \emptyset$. Let $\{y_n\}$ be a sequence in $\mathcal{H}$ defined by Algorithm 5.3.1, where $\{\theta_n\}$ and $\{e_n\}$ are real sequences satisfy the following conditions:*

(i) $0 < e_n < 1$ *for all* $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} e_n = 1$, $\sum_{n=1}^{\infty}(1-e_n) = \infty$ *and* $\sum_{n=1}^{\infty} |e_n - e_{n-1}| < \infty$;

(ii) $0 < \underline{\theta} \leq \theta_n \leq \overline{\theta} < 1$ *for all* $n \in \mathbb{N}$, *and* $\sum_{n=1}^{\infty} |\theta_n - \theta_{n-1}| < \infty$.

Then the sequence $\{y_n\}$ converges strongly to $proj_{\Omega}(0)$.

Proof. In order to prove the convergence of the sequence $\{y_n\}$, we follow the following steps:

Step 1. Sequence $\{y_n\}$ is bounded.

Let $y \in \Omega$. Since S and T are nonexpansive, we have following

$$\begin{aligned}
\|y_{n+1} - y\| &= \|S[(1-\theta_n)e_n y_n + \theta_n T(e_n y_n)] - y\| \\
&\leq \|(1-\theta_n)e_n y_n + \theta_n T(e_n y_n) - y\| \\
&\leq (1-\theta_n)\|e_n y_n - y\| + \theta_n \|T(e_n y_n) - y\| \\
&\leq \|e_n y_n - y\| \qquad (5.5) \\
&= \|e_n(y_n - y) - (1-e_n)y\| \\
&\leq e_n\|(y_n - y)\| + (1-e_n)\|y\| \\
&\leq \max\{\|y_0 - y\|, \|y\|\}.
\end{aligned}$$

Thus, $\{y_n\}$ is bounded.

Step 2. $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.

Using nonexpensitivity of S and T, we have

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|S[(1-\theta_n)e_n y_n + \theta_n T(e_n y_n)] - S[(1-\theta_{n-1})e_{n-1}y_{n-1} + \theta_{n-1}T(e_{n-1}y_{n-1})]\| \\
&\leq \|(1-\theta_n)e_n y_n + \theta_n T(e_n y_n) - (1-\theta_{n-1})e_{n-1}y_{n-1} - \theta_{n-1}T(e_{n-1}y_{n-1})\| \\
&= \|(1-\theta_n)e_n y_n - (1-\theta_{n-1})e_{n-1}y_{n-1} + \theta_n T(e_n y_n) - \theta_{n-1}T(e_{n-1}y_{n-1})\| \\
&\leq \|(1-\theta_n)(e_n y_n - e_{n-1}y_{n-1}) + (\theta_{n-1} - \theta_n)e_{n-1}y_{n-1})\| \\
&+ \|\theta_n(T(e_n y_n) - T(e_{n-1}y_{n-1})) + (\theta_n - \theta_{n-1})T(e_{n-1}y_{n-1})\| \\
&\leq \|e_n y_n - e_{n-1}y_{n-1}\| + |\theta_n - \theta_{n-1}|\mathcal{C}_1 \\
&= \|e_n(y_n - y_{n-1}) + (e_n - e_{n-1})y_{n-1}\| + |\theta_n - \theta_{n-1}|\mathcal{C}_1 \\
&\leq e_n\|y_n - y_{n-1}\| + |e_n - e_{n-1}|\mathcal{C}_2 + |\theta_n - \theta_{n-1}|\mathcal{C}_1,
\end{aligned}$$

for some $\mathcal{C}_1, \mathcal{C}_2 > 0$. By applying Lemma 5.2.3 with $a_n = \|y_n - y_{n-1}\|, b_n = 0$, $\epsilon_n = |e_n - e_{n-1}|\mathcal{C}_2 + |\theta_n - \theta_{n-1}|\mathcal{C}_1$ and $\theta_n = 1 - e_n, \forall n \in \mathbb{N}$, we obtain that $\|y_{n+1} - y_n\| \to 0$.

Step 3. $\|y_n - Ty_n\|$ and $\|y_n - Sy_n\| \to 0$ as $n \to \infty$.

Let $y \in \Omega$ Note

$$\begin{aligned}
\|y_{n+1} - y\|^2 &= \|S[(1-\theta_n)e_n y_n + \theta_n T(e_n y_n)] - y\|^2 \\
&\leq \|(1-\theta_n)e_n y_n + \theta_n T(e_n y_n) - y\|^2 \\
&= (1-\theta_n)\|e_n y_n - y\|^2 + \theta_n\|T(e_n y_n) - y\|^2 - \theta_n(1-\theta_n)\|e_n y_n - T(e_n y_n)\|^2 \\
&\leq (1-\theta_n)\|e_n y_n - y\|^2 + \theta_n\|e_n y_n - y\|^2 - \theta_n(1-\theta_n)\|e_n y_n - T(e_n y_n)\|^2 \\
&= \|e_n y_n - y\|^2 - \theta_n(1-\theta_n)\|e_n y_n - T(e_n y_n)\|^2, \qquad (5.6)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\theta_n(1-\theta_n)\|e_n y_n - T(e_n y_n)\|^2 \leq \|e_n y_n - y\|^2 - \|y_{n+1} - y\|^2. \\
\leq\ & (\|e_n y_n - y\| + \|y_{n+1} - y\|)\|e_n y_n - y_{n+1}\| \\
\leq\ & (\|e_n y_n - y\| + \|y_{n+1} - y\|)\|e_n y_n - e_n y_{n+1} + e_n y_{n+1} - y_{n+1}\| \\
\leq\ & (\|e_n y_n - y\| + \|y_{n+1} - y\|)(e_n\|y_n - y_{n+1}\| + (e_n - 1)\|y_{n+1}\|).
\end{aligned}$$

Since $\lim_{n\to\infty} e_n = 1$, by the condition (**i**), $0 < \underline{\theta} \leq \theta_n \leq \overline{\theta} < 1$ for all $n \in \mathbb{N}$ by the condition (**ii**) and $\|y_{n+1} - y\| \to 0$ by Step 2, we have $\|e_n y_n - T(e_n y_n)\| \to 0$

as $n \to \infty$. Now,

$$\begin{aligned} \|y_n - Ty_n\| &= \|y_n - e_n y_n + e_n y_n - T(e_n y_n) + T(e_n y_n) - Ty_n\| \\ &\leq \|y_n - e_n y_n\| + \|e_n y_n - T(e_n y_n)\| + \|T(e_n y_n) - Ty_n\| \\ &\leq 2(1-e_n)\|y_n\| + \|e_n y_n - T(e_n y_n)\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

and

$$\begin{aligned} \|y_n - Sy_n\| &\leq \|y_n - y_{n+1}\| + \|y_{n+1} - Sy_n\| \\ &= \|y_n - y_{n+1}\| + \|S[(1-\theta_n)e_n y_n + \theta_n T(e_n y_n)] - Sy_n\| \\ &\leq \|y_n - y_{n+1}\| + \|(1-\theta_n)e_n y_n + \theta_n T(e_n y_n) - y_n\| \\ &\leq \|y_n - y_{n+1}\| + (1-\theta_n)\|e_n y_n - y_n\| + \theta_n\|T(e_n y_n) - y_n\| \\ &\leq \|y_n - y_{n+1}\| + (1-\theta_n)(1-e_n)\|y_n\| + \theta_n\|T(e_n y_n) - Ty_n + Ty_n - y_n\| \\ &\leq \|y_n - y_{n+1}\| + (1-\theta_n)(1-e_n)\|y_n\| + \theta_n\|e_n y_n - y_n\| + \theta_n\|Ty_n - y_n\| \\ &= \|y_n - y_{n+1}\| + (1-e_n)\|y_n\| + \theta_n\|Ty_n - y_n\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Step 4. $\{y_n\}$ converges strongly to $\bar{y} = proj_\Omega(0)$.

From (5.5), we set

$$\begin{aligned} \|y_{n+1} - \bar{y}\|^2 &\leq \|e_n y_n - \bar{y}\|^2 \\ &\leq \|e_n(y_n - \bar{y}) - (1-e_n)\bar{y}\|^2 \\ &\leq e_n^2\|y_n - \bar{y}\|^2 + 2e_n(1-e_n)\langle -\bar{y}, y_n - \bar{y}\rangle + (1-e_n)^2\|\bar{y}\|^2 \\ &\leq e_n\|y_n - \bar{y}\|^2 + 2e(1-e_n)\langle -\bar{y}, y_n - \bar{y}\rangle + (1-e_n)^2\|\bar{y}\|^2. \end{aligned} \tag{5.7}$$

Next we show that

$$\limsup_{n\to\infty}\langle -\bar{y}, y_n-\bar{y}\rangle \leq 0. \tag{5.8}$$

Contrarily assume a real number l and a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ satisfying

$$\langle -\bar{y}, y_{n_j}-\bar{y}\rangle \geq l > 0 \ \forall j \in \mathbb{N}. \tag{5.9}$$

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_j}\}$ which converges weakly to an element $y \in \mathcal{H}$. Lemma 5.2.2 alongwith Step 4 implies that $y \in \Omega$. By using variational characterization of projection, we can easily derive

$$\lim_{j\to\infty}\langle -\bar{y}, y_{n_j}-\bar{y}\rangle = \langle -\bar{y}, y-\bar{y}\rangle \leq 0, \tag{5.10}$$

which is a contradiction. Thus, (5.8) holds and

$$\limsup_{n\to\infty}\left(2e_n\langle -\bar{y}, y_n-\bar{y}\rangle + (1-e_n)\|\bar{y}\|^2\right) \leq 0. \tag{5.11}$$

Consider $a_n = \|y_n-\bar{y}\|, b_n = 2e_n\langle -\bar{y}, y_n-\bar{y}\rangle + (1-e_n)\|\bar{y}\|^2, \epsilon_n = 0$ and $\theta_n = 1-e_n$ in (5.7) and apply Lemma 5.2.3, we get the desired conclusion.

□

Corollary 5.3.1. *Let $R_1, R_2 : \mathcal{H} \to \mathcal{H}$ be α_1, α_2-averaged operators respectively, such that* Fix $(R_1) \cap$ Fix$(R_2) \neq \emptyset$. *For $y_1 \in \mathcal{H}$, let $\{y_n\}$ be sequence in $\mathcal{H}$ defined by*

$$y_{n+1} = R_2\{e_n y_n + \theta_n(R_1(e_n y_n) - e_n y_n)\} \quad \forall n \in \mathbb{N}, \tag{5.12}$$

where $\{\theta_n\}$ *and* $\{e_n\}$ *are real sequences satisfy the condition* ***(i)*** *given in Theorem 5.3.1 and the condition:*

$$0 < \underline{\Theta} \leq \alpha_1\theta_n \leq \overline{\Theta} < 1 \; for \; all \; n \in \; \mathbb{N} \; and \sum_{n=1}^{\infty} |\theta_n - \theta_{n-1}| < \infty.$$

Then the sequence $\{y_n\}$ *converges strongly to* $proj_{\mathrm{Fix}\,(R_1)\cap \mathrm{Fix}(R_2)}(0)$.

5.4 Forward-Backward type Algorithms

In this section, we propose a forward-backward algorithm based on Algorithm 5.3.1 to simultaneously solve the monotone inclusion problems of the sum of two maximally monotone operators in which one is single-valued. Further, we also propose an Algorithm 5.3.1 based forward-backward-type primal-dual algorithm to solve a complexly structured monotone inclusion problem containing composition with linear operators and parallel-sum operators.

5.4.1 Forward-Backward Algorithm

Let $A_1, A_2 : \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone operators and $B_1, B_2 : \mathcal{H} \to \mathcal{H}$ be α_1, α_2-cocoerceive operators. We consider the monotone inclusion problem

$$\text{Find } x \in \mathcal{H} \text{ such that } 0 \in (A_1 + B_1)x \cap (A_2 + B_2)x. \tag{5.13}$$

We propose a forward-backward algorithm to solve the monotone inclusion problem (5.13) such that generated sequence converges strongly to the solution set of the problem (5.13).

Theorem 5.4.1. *Suppose* $\text{zer}(A_1+B_1)\cap\text{zer}(A_2+B_2)\neq\emptyset$ *and* $\gamma_1\in(0,2\alpha_1)$ *and* $\gamma_2\in(0,2\alpha_2)$. *For* $y_1\in\mathcal{H}$, *consider the forward-backward algorithm defined as follows:*

$$y_{n+1}=J_{\gamma_2A_2}(Id-\gamma_2B_2)\left\{(1-\theta_n)e_ny_n+\theta_nJ_{\gamma_1A_1}(e_ny_n-\gamma_1B_1(e_ny_n))\right\}\forall n\in\mathbb{N}. \tag{5.14}$$

where $\{\theta_n\}$ *and* $\{e_n\}$ *are real sequences satisfy the condition (**i**) given in Theorem 5.3.1 and the condition:* $0<\underline{\Theta}\leq\frac{2\alpha_1}{4\alpha_1-\gamma_1}\theta_n\leq\overline{\Theta}<1$ *for all* $n\in\mathbb{N}$ *and* $\sum_{n=1}^{\infty}|\theta_n-\theta_{n-1}|<\infty$. *Then* $\{y_n\}$ *converges strongly to* $proj_{\text{zer}(A_1+B_1)\cap\text{zer}(A_2+B_2)}(0)$.

Proof. Set $T_1=J_{\gamma_1A_1}(Id-\gamma_1B_1)$ and $T_2=J_{\gamma_2A_2}(Id-\gamma_2B_2)$, then algorithm (5.14) can be rewritten as:

$$y_{n+1}=T_2\{(1-\theta_n)e_ny_n+\alpha_1\theta_n(T_1(e_ny_n)-e_ny_n)\}\forall n\in\mathbb{N}. \tag{5.15}$$

Since $J_{\gamma_1A_1}$ is $\frac{1}{2}$-cocoerceive and $Id-\gamma_1B_1$ is $\frac{\gamma_1}{2\alpha_1}$-averaged, T_1 is $\frac{2\alpha_1}{4\alpha_1-\gamma_1}$-averaged. Therefore, Theorem 5.4.1 follows from Corollary 5.3.1. □

Further, we consider the following minimization problem and propose a new proximal-point algorithm based on Algorithm (5.14) to solve it.

Problem 5.4.1. *Consider strictly positive real numbers* β_1,β_2. *Let* $f_1,f_2:\mathcal{H}\to\mathbb{R}\cup\{\infty\}$ *be proper convex lower semicontinuous functions and* $g_1,g_2:\mathcal{H}\to\mathbb{R}$ *be convex and Frechet-differentiable functions with* $\frac{1}{\beta_1},\frac{1}{\beta_2}$*-Lipschitz continuous gradient, respectively. The problem is to find a point* $y\in\mathcal{H}$ *such that*

$$y\in argmin(f_1+g_1)\cap argmin(f_2+g_2). \tag{5.16}$$

Corollary 5.4.1. *Consider the functions* f_1,f_2,g_1 *and* g_2 *are as in Problem 5.4.1. Let* $argmin(f_1+g_1)\cap argmin(f_2+g_2)\neq\emptyset$. *For* $\gamma_1\in(0,2\beta_1]$ *and* $\gamma_2\in(0,2\beta_2]$, *consider an algorithm with initial point* $y_1 \in \mathcal{H}$,

$$y_{n+1} = prox_{\gamma_2 f_2} o(Id - \gamma_2 \nabla g_2)\{(1-\theta_n)e_n y_n + \theta_n prox_{\gamma_1 f_1}(e_n y_n - \gamma_1 \nabla g(e_n y_n))\} \ \ \forall n \in \mathbb{N}, \tag{5.17}$$

where $\theta_n \in (0,1]$ *and* $e_n \in (0, \frac{4\beta_1 - \gamma_1}{2\beta_1})$ *are real sequences satisfy the condition **(i)** given in Theorem 5.3.1 and the condition:*

$$0 < \underline{\Theta} \leq \frac{2\beta_1}{4\beta_1 - \gamma_1}\theta_n \leq \overline{\Theta} < 1, \ \sum_{n=1}^{\infty} |\theta_n - \theta_{n-1}| < \infty$$

Then $\{y_n\}$ *converges weakly to optimization problem (5.16).*

Proof. Consider $A_1 = \partial f_1, A_2 = \partial f_2, B_1 = \nabla g_1, B_1 = \nabla g_2$. Since $\text{zer}(\partial f_i + \nabla g_i) = argmin(f_i + g_i)$, $i = 1, 2$. Note $\nabla g_1, \nabla g_1$ are β_1, β_2-cocoerceive, respectively. Thus, by Theorem 5.4.1, $\{y_n\}$ converges strongly to a point in $argmin(f_1 + g_1) \cap argmin(f_2 + g_2)$. □

5.4.2 Forward-backward type Primal-Dual algorithm with Tikhonov regularization terms

Problem 5.4.2. *Let* m *be a positive integer. Suppose* $\Omega_1, \ldots, \Omega_m$ *are real Hilbert spaces. Consider the following operators*

(a) $A, B : \mathcal{H} \to 2^{\mathcal{H}}$ *are maximally monotone operators,*

(b) $C, D : \mathcal{H} \to \mathcal{H}$ *are* μ_1, μ_2*-cocoerceive operators, respectively,* *(c)* $P_i, Q_i, R_i, S_i : \Omega_i \to 2^{\Omega_i}$ *are maximally monotone operators such that* Q_i, S_i *are* ν_i, δ_i*-cocoerceive, respectively,* $i = 1, \dots, m$*,*

(d) *nonzero continuous linear operators* $L_i : \mathcal{H} \to \Omega_i, i = 1, \dots, m$*.*

The primal inclusion problem is to find $\bar{y} \in \mathcal{H}$ *satisfying*

$$0 \in A\bar{y} + \textstyle\sum_{i=1}^{m} L_i^*(P_i \Box Q_i)(L_i\bar{y}) + C\bar{y}$$

$$\text{and}$$

$$0 \in B\bar{y} + \textstyle\sum_{i=1}^{k} L_i^*(R_i \Box S_i)(L_i\bar{y}) + D\bar{y}$$

together with dual inclusion problem

$$\textit{find} \ \ \bar{v}_1 \in \mathit{\Omega}_1, \dots, \bar{v}_m \in \mathit{\Omega}_m \ \textit{such that} \begin{cases} -\sum_{i=1}^{m} L_i^*\bar{v}_i \in Ay + Cy \\ \bar{v}_i \in (P_i \Box Q_i)(L_i x) \\ \ \textit{and} \\ -\sum_{i=1}^{m} L_i^*\bar{v}_i \in By + Dy \\ \bar{v}_i \in (R_i \Box S_i)(L_i y) \end{cases} \tag{5.18}$$

$i = 1, 2 \dots m.$

A point $(\bar{y}, \bar{v}_1, \dots, \bar{v}_m) \in \mathcal{H} \times \mathit{\Omega}_1 \times \cdots \times \mathit{\Omega}_m$ be a primal-dual solution of Problem 5.4.2 if it satisfies the following:

$$\begin{cases} -\sum_{i=1}^{m} L_i^*\bar{v}_i \in A\bar{y} + C\bar{y}, \\ -\sum_{i=1}^{m} L_i^*\bar{v}_i \in B\bar{y} + D\bar{y}, \\ \bar{v}_i \in (P_i \Box Q_i)(L_i\bar{y}), \\ \bar{v}_i \in (R_i \Box S_i)(L_i\bar{y}) \end{cases} \tag{5.19}$$

$i = 1, 2, \ldots, m.$

Theorem 5.4.2. *Consider the operators as in Problem 5.4.2. Assume*

$$0 \in ran \left(A + \sum_{i=1}^{m} L_i^* \circ (P_i \Box Q_i) \circ L_i + C \right) \bigcap ran \left(B + \sum_{i=1}^{m} L_i^* \circ (R_i \Box S_i) \circ L_i + D \right). \tag{5.20}$$

Let $\tau, \sigma_1, \ldots, \sigma_m > 0$ *such that*

$$2\rho \min\{\beta_1, \beta_2\} \geq 1,$$

where $\rho = \min\{\frac{1}{\tau}, \frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_m}\} \left(1 - \sqrt{\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2}\right)$, $\beta_1 = \min\{\mu_1, \nu_1, \ldots, \nu_m\}$ *and* $\beta_2 = \min\{\mu_2, \delta_1, \ldots, \delta_m\}$. *Consider the algorithm with intial point* $(y_1, v_{1,1}, \ldots, v_{m,1}) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ *and defined by*

Algorithm 5.4.1: To optimize the complexly structured Problem 5.4.2

Input:

1. *initial points* $(y_1, v_{1,1}, \ldots, v_{m,1}) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$

2. *Real numbers* $\tau, \sigma_i > 0,\ i = 1, 2, ..., m$ *be such that* $\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2 < 4$.

3. $\theta_n \in (0, \frac{4\beta_1\rho - 1}{2\beta_1\rho}], e_n \in (0, 1)$.

For $k = 1, \ldots, n$;

$p_n = J_{\tau A} \left[e_n y_n - \tau \left(e_n \sum_{i=1}^{m} L_i^* v_{i,n} + C(e_n y_n)\right)\right]$

$r_n = e_n y_n + \theta_n (p_n - e_n y_n)$

For $i = 1, \ldots, m$;

$q_{i,n} = J_{\sigma_i P_i^{-1}} \left[e_n v_{i,n} + \sigma_i (L_i(2p_n - e_n y_n) - Q_i^{-1}(e_n v_{i,n}))\right]$

$u_{i,n} = e_n v_{i,n} + \theta_n (q_{i,n} - e_n v_{i,n})$

$y_{n+1} = J_{\tau B} \left[r_n - \tau \left(\sum_{i=1}^{m} L_i^* u_{i,n} + D(u_n)\right)\right]$

$v_{i,n+1} = J_{\sigma_i R_i^{-1}} \left[u_{i,n} + \sigma_i (L_i(2y_{n+1} - r_n) - S_i^{-1}(u_{i,n}))\right]$

Output: $(y_{n+1}, \zeta_{1,n+1}, \ldots, \zeta_{m,n+1})$

---

where, sequences $\{\theta_n\}$ *and* $\{e_n\}$ *are real sequences satisfy the condition (**i**) given in Theorem 5.3.1 and the condition:*

$$0 < \underline{\Theta} \leq \frac{2\beta_1\rho}{4\beta_1\rho - 1}\theta_n < \overline{\Theta} < 1, \ \sum_{n=1}^{\infty}|\theta_n - \theta_{n-1}| \leq \infty$$

Then there exists $(\bar{y}, \bar{v}_1, \ldots, \bar{v}_m) \in \mathcal{H}\times\Omega_1\times\cdots\times\Omega_m$ *such that sequence* $\{(y_n, v_{1,n}, \ldots, v_{m,n})\}$ *converges strongly to* $(\bar{y}, \bar{v}_1, \ldots, \bar{v}_m)$ *and satisfies the Problem 5.4.2.*

Proof. Consider the real Hilbert space $\mathcal{K} \equiv \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ endowed with innner product

$$\langle (x, u_1, \ldots, u_m), (y, v_1, \ldots, v_m)\rangle_{\mathcal{K}} = \langle x, y\rangle_{\mathcal{H}} + \sum_{i=1}^{m}\langle u_i, v_i\rangle_{\Omega_i}$$

and corresponding norm

$$\|(x, u_1, \ldots, u_m)\|_{\mathcal{K}} = \sqrt{\|x\|^2_{\mathcal{H}} + \sum_{i=1}^{m}\|u_i\|^2_{\Omega_i}}, \quad \forall (x, u_1, \ldots u_m), (y, v_1, \ldots, v_m) \in \mathcal{K}.$$

Further we consider following operators on real Hilbert space $\mathcal{K}$

1. $\phi_1 : \mathcal{K} \to 2^{\mathcal{K}}$, defined by $(x, u_1, \ldots, u_m) \to (Ax, P_1^{-1}u_1, \ldots, P_m^{-1}u_m)$,
2. $\phi_2 : \mathcal{K} \to 2^{\mathcal{K}}$, defined by $(x, u_1, \ldots, u_m) \to (Bx, R_1^{-1}u_1, \ldots, R_m^{-1}u_m)$,
3. $\xi : \mathcal{K} \to \mathcal{K}$, defined by $(x, u_1, \ldots, u_m) \to \left(\sum_{i=1}^{m} L_i^* u_i, -L_1x, \ldots, -L_mx\right)$,
4. $\psi_1 : \mathcal{K} \to \mathcal{K}$, defined by $(x, u_1, \ldots, u_m) \to \left(Cx, Q_1^{-1}u_1, \ldots, Q_m^{-1}u_m\right)$,
5. $\psi_2 : \mathcal{K} \to \mathcal{K}$, defined by $(x, u_1, \ldots, u_m) \to \left(Dx, S_1^{-1}u_1, \ldots, S_m^{-1}u_m\right)$.

These operators are maximally monotone as $A, B, P_i, R_i, Q_i, S_i,\ i = 1, 2, \ldots, m$ are maximally monotone and ξ is skew-symmetric, i.e., $\xi_i^* = \xi_i$. Now, define the continuous linear operator $\mathbf{V} : \mathcal{K} \to \mathcal{K}$ by,

$$(x, u_1, \ldots, u_m) \to \left(\frac{x}{\tau} - \sum_{i=1}^{m} L_i^* u_i, \frac{u_1}{\sigma_1} - L_1 x, \ldots, \frac{u_m}{\sigma_m} - L_m x \right),$$

which is selfadjoint and ρ-strongly positive, i.e., $\langle \mathbf{x}, \mathbf{V}\mathbf{x} \rangle_{\mathcal{K}} \geq \rho \|\mathbf{x}\|_{\mathcal{K}}^2 \ \ \forall \mathbf{x} \in \mathcal{K}$. Therefore inverse of operator $\mathbf{V}$ exists and satisfy $\|\mathbf{V}^{-1}\| \leq \frac{1}{\rho}$.

Using the definition of resolvent operator, the Algorithm 5.4.1 can be rewritten as

$$\begin{cases} e_n \left(\tau^{-1} x_n - \sum_{i=1}^m L_i^* v_{i,n}\right) - \tau^{-1} p_n + \sum_{i=1}^m L_i^* q_{i,n} - C(e_n x_n) \in A p_n + \sum_{i=1}^m L_i^* q_{i,n} \\ r_n = e_n x_n + \theta_n (p_n - e_n x_n) \\ For \quad i = 1, \ldots, m \\ \begin{cases} e_n(\sigma_i^{-1} v_{i,n} - L_i x_n) - \sigma_i^{-1} q_{i,n} + L_i p_n - Q_i^{-1}(e_n v_{i,n}) \in P_i^{-1}(q_{i,n}) - L_i p_n \\ u_{i,n} = e_n v_{i,n} + \theta_n (q_{i,n} - e_n v_{i,n}) \end{cases} \\ \tau^{-1} r_n - \sum_{i=1}^m L_i^* u_{i,n} - \tau^{-1} x_{n+1} + \sum_{i=1}^m L_i^* v_{i,n+1} - D(x_n) \in B x_{n+1} + \sum_{i=1}^m L_i^* v_{i,n+1} \\ For\ i = 1, 2 \ldots, m \\ \sigma_i^{-1} u_{i,n} - L_i r_n - \sigma_i^{-1} x_{n+1} + L_i v_{i,n+1} - S_i^{-1} v_{i,n} \in R_i^{-1} v_{i,n+1} - L_i x_{n+1}. \end{cases} \tag{5.21}$$

Now, consider the sequences $\mathbf{x}_n$=$(x_n, v_{1,n}, \ldots, v_{m,n})$, $\mathbf{u}_n$=$(u_n, u_{1,n}, \ldots, u_{m,n})$ and $\mathbf{y}_n = (p_n, q_{1,n}, \ldots, q_{m,n})\ \forall n \in \mathbb{N}$. By taking into account the sequences $\{\mathbf{x}_n\}$,$\{\mathbf{y}_n\}$ and $\{\mathbf{u}_n\}$ and operator $\mathbf{V}$, Algorithm 5.4.1 can be rewritten as

$$\begin{cases} e_n \mathbf{V}(\mathbf{x}_n) - \mathbf{V}(\mathbf{y}_n) - \psi_1(e_n \mathbf{x}_n) \in (\phi_1 + \xi)(\mathbf{y}_n) \\ \mathbf{u}_n = e_n \mathbf{x}_n + \theta_n (\mathbf{y}_n - e_n \mathbf{x}_n) \\ \mathbf{V} u_n\text{-}\mathbf{V}\mathbf{x}_n - \psi_2 u_n \in (\phi_2 + \xi)\mathbf{x}_{n+1}. \end{cases} \tag{5.22}$$

On further analysing Algorithm 5.4.1, we get

$$\begin{cases} \mathbf{y}_n = J_{\mathbf{A}_1}(e_n\mathbf{x}_n - \mathbf{B}_1(e_n\mathbf{x}_n)) \\ \mathbf{u}_n = e_n\mathbf{x}_n + \theta_n(\mathbf{y}_n - e_n\mathbf{x}_n) \\ \mathbf{x}_{n+1} = J_{\mathbf{A}_2}(\mathbf{u}_n - \mathbf{B}_2\mathbf{u}_n), \end{cases} \tag{5.23}$$

where $\mathbf{A}_1 = \mathbf{V}^{-1}(\phi_1 + \xi)$, $\mathbf{B}_1 = \mathbf{V}^{-1}\psi_1$, $\mathbf{A}_2 = \mathbf{V}^{-1}(\phi_2 + \xi)$ and $\mathbf{B}_2 = \mathbf{V}^{-1}\psi_2$. Now, we define the real Hilbert space $\mathcal{K}_\mathbf{V} \equiv \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ endowed with inner product $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{K}_\mathbf{V}} = \langle \mathbf{x}, \mathbf{V}\mathbf{y} \rangle_{\mathcal{K}}$ and corresponding norm is given by, $\|\mathbf{x}\|_{\mathcal{K}_\mathbf{V}} = \sqrt{\langle \mathbf{x}, \mathbf{V}\mathbf{x} \rangle_{\mathcal{K}}}$ $\forall \mathbf{x}, \mathbf{y} \in \mathcal{K}_\mathbf{V}$.

In view of real Hilbert space $\mathcal{K}_\mathbf{V}$ and Algorithm 5.4.1, we observe the following:

1. $\mathbf{A}_i$ and $\mathbf{B}_i$ are maximally monotone on $\mathcal{K}_\mathbf{V}$ as $\phi_i + \xi$ and ψ_i are maximally monotone on $\mathcal{K}$, for $i = 1, 2$.

2. $\mathbf{B}_i$ are $\beta_i\rho$-cocoerceive on $\mathcal{K}_\mathbf{V}$ as ψ_i are β_i-cocoerceive in $\mathcal{K}$, for $i = 1, 2$.

3. zer$(\mathbf{A}_i+\mathbf{B}_i)$=zer$(\mathbf{V}^{-1}(\phi_i+\xi+\psi_i))$=zer$(\phi_i+\xi+\psi_i)$, $i = 1, 2$ and from condition (5.20), we can easily obtain that zer$(\mathbf{A}_1+\mathbf{B}_1)\cap$zer$(\mathbf{A}_2+\mathbf{B}_2) \neq \emptyset$. Assume that $A_i = \mathbf{A}_i$ and $B_i = \mathbf{B}_i$, $i = 1, 2$ thus $\mathbf{A}_i, \mathbf{B}_i$, $i = 1, 2$ and sequences $\{\theta_n\}, \{e_n\}$ satisfy the assumptions in Theorem 5.4.1. Thus, according to Theorem 5.4.1, $\{\mathbf{x}_n\}$ converges strongly to $(\bar{y}_n, \bar{v}_1, \ldots, \bar{v}_m) \in proj_{\text{zer}(\mathbf{A}_1+\mathbf{B}_1)\cap\text{zer}(\mathbf{A}_2+\mathbf{B}_2)}(0, \ldots, 0)$ in the space $\mathcal{K}_\mathbf{V}$ as $n \to \infty$. Thus, we obtain the conclusion as $(\bar{y}_n, \bar{v}_1, \ldots, \bar{v}_m) \in$ zer$(\phi_1 + \xi + \psi_1) \cap$ zer$(\phi_2 + \xi + \psi_2)$, will also satisfy primal-dual problem 5.4.2.

□

Next, we define a complexly structured convex optimization problem and their Fenchel duals. Further, we propose an algorithm to solve the considered problem and study the convergence property of the algorithm to find simultaneously the common solutions of optimiztion problems and common solutions of their Fenchel duals. The considered problem is as follows:

Problem 5.4.3. *Let $f_1, f_2 \in \Gamma(\mathcal{H})$ and h_1, h_2 be convex differentiable function with μ_1^{-1}, μ_2^{-1}- Lipschitz continuous gradient, for some $\mu_1, \mu_2 > 0$. Let Ω_i be real Hilbert spaces and $g_i, l_i, s_i, t_i \in \Gamma(\Omega_i)$ such that l_i, t_i are $\nu_i, \delta_i(> 0)$-strongly convex, respectively, and $L_i : \mathcal{H} \to \Omega_i$ be non-zero linear continuous operator $\forall i = 1, 2, \dots, m$, where $m > 0$ is an integer. The opmization problem under consideration is*

$$\inf_{x\in\mathcal{H}} \left\{ f_1(x) + \sum_{i=1}^{m} (g_i \Box l_i)(L_i x) + h_1(x) \right\} \bigcap \inf_{x\in\mathcal{H}} \left\{ f_2(x) + \sum_{i=1}^{m} (s_i \Box t_i)(L_i x) + h_2(x) \right\} \tag{5.24}$$

with its Fenchel-dual problem

$$\begin{aligned} &\sup_{v_i\in\Omega, i\in 1,\dots,m} \left\{ -(f_1^* \Box h_1^*)(-\sum_{i=1}^{m} L_i^* v_i) - \sum_{i=1}^{m} (g_i^*(v_i) + l_i^*(v_i)) \right\} \\ \cap &\sup_{v_i\in\Omega, i\in 1,\dots,m} \left\{ -(f_2^* \Box h_2^*)(-\sum_{i=1}^{m} L_i^* v_i) - \sum_{i=1}^{m} (s_i^*(v_i) + t_i^*(v_i)) \right\}. \end{aligned} \tag{5.25}$$

In following corollary, we propose an algorithm and study its convergence behavior. The point of convergence will be the solution of Problem 5.4.3.

Corollary 5.4.2. *Assume in Problem 5.4.3*

$$0 \in ran \left(\partial f_1 + \sum_{i=1}^{m} L_i^* \circ (\partial g_i \Box \partial l_i) \circ L_i + \nabla h_1 \right) \bigcap ran \left(\partial f_2 + \sum_{i=1}^{m} L_i^* \circ (\partial s_i \Box \partial t_i) \circ L_i + \nabla h_2 \right). \tag{5.26}$$

Consider $\tau > 0$, $\sigma_i > 0 \;\; i = 1, 2, \dots, m$ such that

$$2\rho \min\{\beta_1, \beta_2\} \geq 1,$$

where $\rho = \min\{\tau^{-1}, \sigma_1^{-1}, \dots, \sigma_m^{-1}\}\left(1 - \sqrt{\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2}\right)$, $\beta_1 = \min\{\mu_1, \nu_1, \dots, \nu_m\}$ *and* $\beta_2 = \min\{\mu_2, \delta_1, \dots, \delta_m\}$. *Consider the iterative scheme with intial point* $(x_1, v_{1,1}, \dots, v_{m,1}) \in \mathcal{H} \times \varOmega_1 \times \dots \times \varOmega_m$ *and defined by*

$$\begin{cases} p_n = prox_{\tau f_1}\left[e_n x_n - \tau\left(e_n \sum_{i=1}^{m} L_i^* v_{i,n} + \nabla h_1(e_n x_n)\right)\right] \\ r_n = e_n x_n + \theta_n(p_n - e_n x_n) \\ For\ i = 1, 2, \dots, m \\ q_{i,n} = prox_{\sigma_i g_i^*}\left[e_n v_{i,n} + \sigma_i(L_i(2p_n - e_n x_n) - \nabla l_i^*(e_n x_n))\right] \\ u_{i,n} = e_n x_n + \theta_n(q_{i,n} - e_n x_n) \\ x_{n+1} = prox_{\tau f_2}\left[r_n - \tau\left(\sum_{i=1}^{m} L_i^* u_{i,n} + \nabla h_2(u_n)\right)\right] \\ v_{i,n+1} = prox_{\sigma_i s_i^*}\left[u_{i,n} + \sigma_i(L_i(2x_{n+1} - r_n) - \nabla t_i^*(u_{i,n}))\right] \end{cases} \tag{5.27}$$

where sequences $\{\theta_n\}$ *and* $\{e_n\}$ *are real sequences satisfy the condition* ***(i)*** *given in Theorem 5.3.1 and the condition:*

$$0 < \underline{\Theta} \leq \frac{2\beta_1\rho}{4\beta_1\rho - 1}\theta_n \overline{\Theta} < 1,\ \sum_{n=1}^{\infty} |\theta_n - \theta_{n-1}| < \infty.$$

Then, there exists $(\bar{x}_n, \bar{v}_1, \dots, \bar{v}_m) \in \mathcal{H} \times \varOmega_1 \times \dots \times \varOmega_m$ *such that sequence* $(x_n, v_{1,n}, \dots, v_{m,n})$ *converges strongly to* $(\bar{x}_n, \bar{v}_1, \dots, \bar{v}_m)$ *as* $n \to \infty$ *and* $(\bar{x}_n, \bar{v}_1, \dots, \bar{v}_m)$ *satisfies Problem 5.4.3.*

5.5 Douglas-Rachford type Algorithms

In this section, using Algorithm 5.3.1 we propose a new Douglas-Rachford algorithm to solve monotone inclusion problem of sum of two maximally monotone operators.

Further, using Algorithm 5.3.1 we propose a Douglas-Rachford type primal-dual algorithm to solve complexly structured monotone inclusion problem containing composite and parallel-sum operators.

5.5.1 Douglas-Rachford Algorithm

Let $A, B : \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone operators. In this section, we consider the following monotone inclusion problem:

$$\text{Find } x \in \mathcal{H} \text{ such that } 0 \in (A+B)x. \tag{5.28}$$

We propose a Douglas-Rachford algorithm based on Algorithm 5.3.1 such that the generated sequence converges strongly to a point in the solution set.

Theorem 5.5.1. *Consider $x_1 \in \mathcal{H}$ and $\gamma > 0$, then algorithm is given by:*

$$n \in \mathbb{N} \begin{cases} y_n = J_{\gamma B}(e_n x_n) \\ z_n = J_{\gamma A}(2y_n - e_n x_n) \\ u_n = e_n x_n + \theta_n (z_n - y_n) \\ x_{n+1} = (2J_{\gamma A} - Id)(2J_{\gamma B} - Id)u_n. \end{cases} \tag{5.29}$$

*Let $\operatorname{zer}(A+B) \neq \emptyset$ and sequences $e_n \in (0,1)$ and $\theta_n \in (0,2]$ are real sequences satisfy the condition **(i)** given in Theorem 5.3.1 and the condition:*

$$0 < \underline{\Theta} \leq \theta_n \leq \overline{\Theta} < 2, \ \sum_{n=1}^{\infty} |\theta_n - \theta_{n-1}| < \infty.$$

Then the following statements are true:

(a) *$\{x_n\}$ converges strongly to $\bar{x} = proj_{\operatorname{Fix} R_{\gamma A} R_{\gamma B}}(0)$ as $n \to \infty$.*

(b) *$\{y_n\}$ and $\{z_n\}$ converges strongly to $J_{\gamma B}(\bar{x}) \in \operatorname{zer}(A+B)$ as $n \to \infty$.*

Proof. Consider the operator $T \equiv R_{\gamma A} \circ R_{\gamma B} : \mathcal{H} \to 2^{\mathcal{H}}$. From the definition of reflected resolvent, and definitions of operator T, algorithm (5.29) can be rewritten as

$$\begin{aligned} x_{n+1} &= R_{\gamma A} R_{\gamma B}\{e_n x_n + \frac{\theta_n}{2}(R_{\gamma A} R_{\gamma B})(e_n x_n) - e_n x_n\} \\ &= T\{e_n x_n + \frac{\theta_n}{2}(T(e_n x_n) - e_n x_n)\}. \end{aligned} \tag{5.30}$$

Since resolvent operator is nonexpansive [9], T is nonexpansive. Suppose $x^* \in \operatorname{zer}(A+B)$ and results from [9], we have $\operatorname{zer}(A+B) = J_{\gamma B}\operatorname{Fix}(T)$, which collectively implies that $\operatorname{Fix}(T) \neq \emptyset$. Applying Theorem 5.3.1 with $A_1 = A_2 = A, B_1 = B_2 = B$, we conclude that $\{x_n\}$ conveges strongly to $\bar{x} = proj_{\operatorname{Fix}(T)}(0)$ as $n \to \infty$.

The continuity of resolvent operator forces the sequence $\{y_n\}$ to converge strongly to $J_{\gamma B}\bar{x} \in \operatorname{zer}(A+B)$. Finally, since $z_n - y_n = \frac{1}{2}(T(e_n x_n) - e_n x_n)$, which converges strongly to 0, concludes **(b)** of Theorem 5.5.1.

□

Problem 5.5.1. *Let $f, g : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be convex proper and lower semicontinuous functions. Consider the minimization problem*

$$\min_{x \in \mathcal{H}} f(x) + g(x). \tag{5.31}$$

Using Karush-Kuhn-Tucker condition, (5.31) is equivalent to solve the inclusion problem

$$\text{find } x \in \mathcal{H} \; 0 \in \partial f(x) + \partial g(x). \tag{5.32}$$

In order to solve such type of problem, we propose an iterative scheme and study its convergence behavior which can be summarized in the following corollary.

Corollary 5.5.1. *Let f, g be as in Problem 5.5.1 with $argmin_{x\in\mathcal{H}}\{f(x)+g(x)\} \neq \emptyset$ and $0 \in sqri(dom\ f - dom\ g)$. Consider the following iterative scheme with $x_1 \in \mathcal{H}$:*

$$\begin{cases} y_n = prox_{\gamma g}(e_n x_n) \\ z_n = J_{\gamma f}(2y_n - e_n x_n) \\ u_n = e_n x_n + \theta_n(z_n - y_n) \\ x_{n+1} = (2prox_{\gamma f} - Id)(2prox_{\gamma g} - Id)u_n, \quad n \in \mathbb{N}, \end{cases} \tag{5.33}$$

*where $\gamma > 0$ and sequences $\{\theta_n\} \subseteq (0,2]$ and $\{e_n\}$ are real sequences satisfy the condition **(i)** given in Theorem 5.3.1 and the condition:*

$$0 < \underline{\Theta} \leq \theta_n \leq \overline{\Theta} < 2, \ \sum_{n=1}^{\infty} |\theta_n - \theta_{n-1}| < \infty.$$

Then we have the following:

(a) *converges strongly to $\bar{x} = proj_{\mathrm{Fix}(T)}$ where $T = (2prox_{\gamma f} - Id)(2prox_{\gamma g} - Id)$.*

(b) *$\{y_n\}$ and $\{z_n\}$ converge strongly to $prox_{\gamma g}(\bar{x}) \in argmin_{x\in\mathcal{H}}\{f(x) + g(x)\}$ as $n \to \infty$.*

Proof. Since $argmin_{x\in\mathcal{H}}\{f(x) + g(x)\} \neq \emptyset$ and $0 \in sqri(dom\ f - dom\ g)$ ensures that $\mathrm{zer}(A+B) = argmin_{x\in\mathcal{H}}\{f(x)+g(x)\}$. The results can be obtained by choosing $A = \partial f, B = \partial g$ in Theorem 5.5.1. □

5.5.2 Douglas-Rachford type Primal-Dual algorithm with Tikhonov regularization terms

In this section, we propose a Douglas-Rachford type primal-dual algorithm to solve the comple structured monotone inclusion problem having mixture of composite and parallel-sum operators. We consider the monotone inclusion problem is as follows:

Problem 5.5.2. *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximally monotone operator. Consider for each $i = 1, \ldots, m$, Ω_i is a real Hilbert space, $P_i, Q_i : \Omega_i \to 2_i^{\Omega}$ are maximally monotone operators and $L_i : \mathcal{H} \to \Omega_i$ are nonzero linear continuous operator. The problem is to find $\bar{x} \in \mathcal{H}$ satisfying the primal inclusion problem*

$$0 \in A\bar{x} + \sum_{i=1}^{m} L_i^*(P_i \Box Q_i)(L_i\bar{x})$$

together with dual inclusion problem

$$find \;\; \bar{v}_1 \in \Omega_1, \ldots, \bar{v}_m \in \Omega_m \;\; such \; that \; (\exists x \in \mathcal{H}) \begin{cases} -\sum_{i=1}^{m} L_i^*\bar{v}_i \in Ax \\ \bar{v}_i \in (P_i \Box Q_i)(L_i x) \;\; i = 1, \ldots, m. \end{cases} \tag{5.34}$$

Here, operators $P_i, Q_i, i = 1, \ldots, m$ are not cocoerceive, thus to solve the Problem 5.5.2, we have to evaluate the resolvent of each operator, which makes the Douglas-Rachford algorithm based primal-dual algorithm is more appropriate to solve the problem.

Theorem 5.5.2. *In addition to assumption in Problem 5.5.2, we assume that*

$$0 \in ran\left(A + \sum_{i=1}^{m} L_i^* \circ (P_i \Box Q_i) \circ L_i\right). \tag{5.35}$$

Consider the strictly positive integers $\tau, \sigma_i, i = 1, \ldots, m$ *satisfying*

$$\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2 < 4. \tag{5.36}$$

Consider the initial point $(x_1, v_{1,1}, \ldots, v_{m,1}) \in \mathcal{H} \times \Omega_i \times \cdots \times \Omega_m$. *The primal- dual algorithm to solve Problem 5.5.2 is given by*

Algorithm 5.5.1: To optimize the complexly structured monotone inclusion problem 5.5.2

Input:

1. *initial points* $(x_1, v_{1,1}, \dots, v_{m,1}) \in \mathcal{H} \times \Omega_i \times \cdots \times \Omega_m$.
2. *Positive real numbers* $\tau, \sigma_i,\ i = 1, 2, ..., m$ *be such that* $\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2 < 4$.
3. *The sequences* $e_n \in (0,1), \theta_n \in (0,2]$

For $k = 1, \dots, n$;

$p_{1,n} = J_{\tau A}(e_n x_n - \frac{\tau}{2} e_n \sum_{i=1}^{m} L_i^* v_{i,n})$

$w_{1,n} = 2p_{1,n} - e_n x_n$

For $i = 1, \dots, m$;

$p_{2,i,n} = J_{\sigma_i P_i^{-1}}(e_n v_{i,n} + \frac{\sigma_i}{2} L_i w_{1,n})$

$w_{2,i,n} = 2p_{2,i,n} - e_n v_{i,n}$

$z_{1,n} = w_{1,n} - \frac{\tau}{2} \sum_{i=1}^{m} L_i^* w_{2,i,n}$

$\xi_{1,n} = e_n x_n + \theta_n (z_{1,n} - p_{1,n})$

For $i = 1, \dots, m$;

$z_{2,i,n} = J_{\sigma_i Q_i^{-1}}(w_{2,i,n} + \frac{\sigma_i}{2} L_i (2z_{1,n} - w_{1,n}))$

$\xi_{2,i,n} = e_n v_{i,n} + \theta_n (z_{2,i,n} - p_{2,i,n})$

$q_{1,n} = J_{\tau A}(\xi_{1,n} - \frac{\tau}{2} \sum_{i=1}^{m} L_i^* (\xi_{2,i,n}))$

$s_{1,n} = 2q_{1,n} - \xi_{1,n}$

For $i = 1, \dots, m$;

$q_{2,i,n} = J_{\sigma_i P_i^{-1}}(\xi_{2,i,n} + \frac{\sigma_i}{2} L_i s_{1,n})$

$s_{2,i,n} = 2q_{2,i,n} - \xi_{2,i,n}$

$t_{1,n} = s_{1,n} - \frac{\tau}{2} \sum_{i=1}^{m} L_i^* (s_{2,i,n})$

$x_{n+1} = 2t_{1,n} - s_{1,n}$

For $i = 1, \dots, m$;

$t_{2,i,n} = J_{\sigma_i Q_i^{-1}}(s_{2,i,n} + \frac{\sigma_i}{2} L_i (x_{n+1}))$

$v_{2,i,n} = 2t_{2,i,n} - s_{2,i,n}$

Output: $(\xi_{n+1}, \zeta_{1,n+1}, \dots, \zeta_{m,n+1})$

*where sequences $\{\theta_n\}$ and $\{e_n\}$ are real sequences satisfy the condition **(i)** given in Theorem 5.3.1 and the condition:*

$$0 < \underline{\Theta} \leq \theta_n \leq \overline{\Theta} < 2, \ \sum_{n=1}^{\infty} |\theta_n - \theta_{n-1}| < \infty.$$

Then there exists an element $(\bar{x}, \bar{v}_1, \ldots, \bar{x}_m) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ such that following statements are true:

1. *Denote*
   $\bar{p}_1 = J_{\tau A}\left(\bar{x} - \frac{\tau}{2}\sum_{i=1}^m L_i^* \bar{v}_i\right)$
   $\bar{p}_{2,i} = J_{\sigma_i P_i^{-1}}\left(\bar{v}_i + \frac{\sigma_i}{2} L_i(2\bar{p}_1 - \bar{x})\right), i = 1, \ldots, m$. *Then the point* $(\bar{p}_1, \bar{p}_{2,1}, \ldots, \bar{p}_{2,m}) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ *is a primal-dual solution to Problem 5.5.2.*
2. $(x_n, v_{1,n}, \ldots, v_{m,n})$ *converges strongly to* $(\bar{x}, \bar{v}_1, \ldots, \bar{v}_m)$.
3. $(p_{1,n}, p_{2,1,n}, \ldots, p_{2,m,n})$ *and* $(z_{1,n}, z_{2,1,n}, \ldots, z_{2,m,n})$ *converges strongly to* $(\bar{p}_1, \bar{p}_{2,1}, \ldots, \bar{p}_{2,m})$.

Proof. Consider the real Hilbert space $\mathcal{K}$ and operators ϕ, ξ as in the Theorem 5.4.2. Now define the operator $\psi : \mathcal{K} \to \mathcal{K}$, defined by

$\psi(x, u_1, \ldots, u_m) = \left(0, Q_1^{-1}u_1, \ldots, Q_m^{-1}u_m\right)$. We can observe the following

1. operator $\frac{1}{2}\xi + \psi$ and $\frac{1}{2}\xi + \phi$ are maximally monotone as dom $\xi = \mathcal{K}$,
2. condition (5.35) implies $\text{zer}(\phi + \xi + \psi) \neq \emptyset$,
3. every point in $\text{zer}(\phi + \xi + \psi)$ will solve Problem 5.5.2.

Define the linear continuous operator $\mathbf{W} : \mathcal{K} \to \mathcal{K}$, defined by

$$\mathbf{W}(x, u_1, \ldots, u_m) = \left(\frac{x}{\tau} - \frac{1}{2}\sum_{i=1}^m L_i^* u_i, \frac{u_1}{\sigma_1} - \frac{1}{2}L_1 x, \ldots, \frac{u_m}{\sigma_m} - \frac{1}{2}L_m x\right)$$

which is selfadjoint. Consider

$$\rho = \left(1 - \frac{1}{2}\sqrt{\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2}\right) \min\left\{\frac{1}{\tau}, \frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_m}\right\} > 0$$

. The operator $\mathbf{V}$ is ρ- strongly positive and satisfies the following inequality

$$\langle x, \mathbf{W}x\rangle_{\mathcal{K}} \geq \|x\|^2_{\mathcal{K}} \quad \forall x \in \mathcal{K}.$$

Thus the inverse of $\mathbf{W}$ exits and satisfies $\|\mathbf{W}^{-1}\| \leq \frac{1}{\rho}$. Consider the sequences

$$\forall n \in \mathbb{N} \begin{cases} \mathbf{x}_n = (x_n, v_{1,n}, \ldots, v_{m,n}) \\ \mathbf{y}_n = (p_{1,n}, p_{2,1,n}, \ldots, p_{2,m,n}) \\ \mathbf{z}_n = (z_{1,n}, z_{2,1,n}, \ldots, z_{2,m,n}) \\ \mathbf{u}_n = (u_{1,n}, u_{2,1,n}, \ldots, u_{2,m,n}) \\ \mathbf{c}_n = (c_{1,n}, c_{2,1,n}, \ldots, c_{2,m,n}) \\ \mathbf{d}_n = (d_{1,n}, d_{2,1,n}, \ldots, d_{2,m,n}). \end{cases} \tag{5.37}$$

Using the definition of operators ϕ, ξ, ψ and $\mathbf{W}$, Algorithm 5.5.1 can be written equivalently as

$$\forall n \in \mathbb{N} \begin{cases} \mathbf{W}(x_n - y_n) \in (\frac{1}{2}\xi + \phi)y_n \\ \mathbf{W}(2y_n - x_n - z_n) \in (\frac{1}{2}\xi + \psi)z_n \\ u_n = x_n + \theta_n(z_n - y_n) \\ \mathbf{W}(u_n - c_n) \in (\frac{1}{2}\xi + \phi)z_n \\ \mathbf{W}(2c_n - u_n - d_n) \in (\frac{1}{2}\xi + \psi)(2c_n - u_n) \\ x_{n+1} = 2d_n - c_n, \end{cases} \tag{5.38}$$

which is further equivalent to

$$\forall n \in \mathbb{N} \begin{cases} y_n = (Id + \mathbf{W}^{-1}(\frac{1}{2}\xi + \phi))^{-1}(x_n) \\ z_n = (Id + \mathbf{W}^{-1}(\frac{1}{2}\xi + \psi))^{-1}(2y_n - x_n) \\ u_n = x_n + \theta_n(z_n - y_n) \\ c_n = (Id + \mathbf{W}^{-1}(\frac{1}{2}\xi + \phi))^{-1}(u_n) \\ d_n = (Id + \mathbf{W}^{-1}(\frac{1}{2}\xi + \psi))^{-1}(2c_n - u_n) \\ x_{n+1} = 2d_n - c_n. \end{cases} \tag{5.39}$$

Now, consider the real Hilbert space $\mathcal{K}_{\mathbf{W}} = \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ with inner product and norm defined as

$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{K}_{\mathbf{W}}} = \langle \mathbf{x}, \mathbf{W}\mathbf{y} \rangle$ and $\|\mathbf{x}\|_{\mathcal{K}_{\mathbf{W}}} = \sqrt{\langle \mathbf{x}, \mathbf{W}\mathbf{x} \rangle_{\mathcal{K}}}$ respectively.

Now, define the operators $\mathbf{A} \equiv \mathbf{W}^{-1}(\frac{1}{2}\xi + \psi)$ and $\mathbf{B} \equiv \mathbf{W}^{-1}(\frac{1}{2}\xi + \phi)$, which are maximally monotone on $\mathcal{K}_{\mathbf{W}}$ as $\frac{1}{2}\xi + \phi$ and $\frac{1}{2}\xi + \psi$ are maximally monotone on $\mathcal{K}$. The Algorithm 5.5.1 can be written in the form of Douglas-Rachford algorithm as

$$\forall n \in \mathbb{N} \begin{cases} \mathbf{y}_n = \mathbf{J_B}(e_n \mathbf{x}_n) \\ \mathbf{z}_n = \mathbf{J_A}(2\mathbf{y}_n - e_n \mathbf{x}_n) \\ \mathbf{x}_{n+1} = (2\mathbf{J_A} - Id)(2\mathbf{J_B} - Id)\mathbf{z}_n, \end{cases} \tag{5.40}$$

which is of the form Algorithm (5.29) for $\gamma = 1$. From assumption (5.35), we have

$$\text{zer}(\mathbf{A} + \mathbf{B}) = \text{zer}(\mathbf{W}^{-1}(\mathbf{M} + \mathbf{S} + \mathbf{Q})) = \text{zer}(\mathbf{M} + \mathbf{S} + \mathbf{Q}).$$

Applying Theorem 5.5.1, we can find $\bar{\mathbf{x}} \in \text{Fix}(R_{\mathbf{A}} R_{\mathbf{B}})$ such that $\mathbf{J_B}\bar{\mathbf{x}} \in \text{zer}(\mathbf{A} + \mathbf{B})$. □

At the end of this section, we study iterative technique to solve the following convex optimization problem

Problem 5.5.3. *Let $f \in \Gamma(\mathcal{H})$ and $m \in \mathbb{N}$. Consider for each $i = 1, \dots, m$, Ω_i are real Hilbert spaces, $g_i, l_i \in \Gamma(\Omega_i)$ and $L_i : \mathcal{H} \to \Omega_i$ are linear continuous operators. The optimization problem is given by*

$$\inf_{x \in H} \left[f(x) + \sum_{i=1}^{m} (g_i \Box l_i)(L_i x) \right] \tag{5.41}$$

with conjugate-dual problem is given by

$$\sup_{v_i \in \Omega, i=1,2,\dots,m} \left\{ -f_1^*(-\sum_{i=1}^{m} L_i^* v_i) - \sum_{i=1}^{m} (g_i^*(v_i) + l_i^*(v_i)) \right\}. \tag{5.42}$$

Consider stricly positive integers $\tau, \sigma_i, i = 1, \dots, m$ and initial point $(x_1, v_{1,1}, \dots, v_{m,1}) \in \mathcal{H} \times \Omega_i \times \dots \times \Omega_m$. The primal-dual algorithm to solve Problem 5.5.3 is given by

Algorithm 5.5.2: To optimize the complexly structured monotone inclusion Problem 5.5.3

Input:

1. initial points $(x_1, v_{1,1}, \dots, v_{m,1}) \in \mathcal{H} \times \Omega_i \times \cdots \times \Omega_m$.
2. Positive real numbers $\tau, \sigma_i,\ i = 1, 2, ..., m$ be such that $\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2 < 4$.
3. The sequences $\{\theta_n\}, \{e_n\}$ satisfying the assumptions in Theorem 5.5.2.

For $k = 1, \dots, n$;

$p_{1,n} = prox_{\tau f}(e_n x_n - \frac{\tau}{2} e_n \sum_{i=1}^{m} L_i^* v_{i,n})$

$w_{1,n} = 2p_{1,n} - e_n x_n$

For $i = 1, \dots, m$;

$p_{2,i,n} = prox_{\sigma_i g_i^*}(e_n v_{i,n} + \frac{\sigma_i}{2} L_i w_{1,n})$

$w_{2,i,n} = 2p_{2,i,n} - e_n v_{i,n}$

$z_{1,n} = w_{1,n} - \frac{\tau}{2} \sum_{i=1}^{m} L_i^* w_{2,i,n}$

$\xi_{1,n} = e_n x_n + \theta_n (z_{1,n} - p_{1,n})$

For $i = 1, \dots, m$;

$z_{2,i,n} = prox_{\sigma_i l_i^*}(w_{2,i,n} + \frac{\sigma_i}{2} L_i (2z_{1,n} - w_{1,n}))$

$\xi_{2,i,n} = e_n v_{i,n} + \theta_n (z_{2,i,n} - p_{2,i,n})$

$q_{1,n} = prox_{\tau f_2}(\xi_{1,n} - \frac{\tau}{2} \sum_{i=1}^{m} L_i^*(\xi_{2,i,n}))$

$s_{1,n} = 2q_{1,n} - \xi_{1,n}$

For $i = 1, \dots, m$;

$q_{2,i,n} = prox_{\sigma_i g_i^*}(\xi_{2,i,n} + \frac{\sigma_i}{2} L_i s_{1,n})$

$s_{2,i,n} = 2q_{2,i,n} - \xi_{2,i,n}$

$t_{1,n} = s_{1,n} - \frac{\tau}{2} \sum_{i=1}^{m} L_i^*(s_{2,i,n})$

$x_{n+1} = 2t_{1,n} - s_{1,n}$

For $i = 1, \dots, m$;

$t_{2,i,n} = prox_{\sigma_i l_i^*}(s_{2,i,n} + \frac{\sigma_i}{2} L_i (x_{n+1}))$

$v_{2,i,n} = 2t_{2,i,n} - s_{2,i,n}$

Output: $(\xi_{n+1}, \zeta_{1,n+1}, \dots, \zeta_{m,n+1})$

where $\{\theta_n\}$ and $\{e_n\}$ are real sequences.

Corollary 5.5.2. *In addition to assumptions in Problem 5.5.3, consider*

$$0 \in ran(\partial f + \sum_{i=1}^{m} L_i^* \circ (\partial g_i \Box \partial l_i) \circ L_i). \tag{5.43}$$

Then, there exists an element $(\bar{x}, \bar{v}_1, \ldots, \bar{x}_m) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ *such that sequence* $\{(\xi_n, \zeta_{1,n}, \ldots, \zeta_{m,n}\}$ *generated by Algorithm 5.5.2 satisfy the following:*

1. *Denote*
   $\bar{p}_1 = prox_{\tau f}\left(\bar{x} - \frac{\tau}{2}\sum_{i=1}^{m} L_i^* \bar{v}_i\right)$
   $\bar{p}_{2,i} = prox_{\sigma_i g_i^*}\left(\bar{v}_i + \frac{\sigma_i}{2} L_i(2\bar{p}_i - \bar{x})\right), i = 1, \ldots, m$. *Then the point* $(\bar{p}_1, \bar{p}_{2,1}, \ldots, \bar{p}_{2,m}) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ *is a primal-dual solution to Problem 5.5.3.*

2. $(x_n, v_{1,n}, \ldots, v_{m,n})$ *converges strongly to* $(\bar{x}, \bar{v}_1, \ldots, \bar{v}_m)$.

3. $(p_{1,n}, p_{2,1,n}, \ldots, p_{2,m,n})$ *and* $(z_{1,n}, z_{2,1,n}, \ldots, z_{2,m,n})$ *converges strongly to* $(\bar{p}_1, \bar{p}_{2,1}, \ldots, \bar{p}_{2,m})$.

5.6 Numerical Experiment

In this section, we make an experimental setup to solve the wavelet based image deblurring problem. In image deblurring, we develop mathematical methods to recover the original, sharp image from the blurred image. The mathematical formulation of the blurring process can be written as a linear inverse problem,

$$\text{find } x \in \mathbb{R}^d \text{ such that } Ax = b + w \tag{5.44}$$

where $A \in \mathbb{R}^{m \times d}$ is blurring operator, $b \in \mathbb{R}^m$ is blurred image and w is an unknown noise. A classical approach to solve problem (5.44) is to minimize the least-square term $\|Ax - b\|^2$. In the deblurring case, the problem is ill-conditioned as the solution has huge norm. To remove the difficulty, the ill-conditioned problem is replaced by a nearly well-conditioned problem. In the wavelet domain, most images are sparse in nature, thats why we choose l_1 regularization. For l_1 regularization, the image processing problem becomes

$$\min_{x \in \mathbb{R}^2} F(x) = \|Ax - b\|^2 + \lambda \|x\|_1 \tag{5.45}$$

where λ is a sparsity controlling parameter and provides a tradeoff between fidelity to the measurements and noise sensitivity. The l_1 regularization produces sparse images having sharp edges since it is less sensitive to outliers. Using subdifferential characterization of the minimum of a convex function, a point x^* minimizes $F(x)$ if and only if

$$0 \in A^T(Ax^* - b) + \partial \lambda \|x^*\|_1$$

Thus we can apply the forward-backward Algorithm (5.14) to solve the deblurring problem (5.45).

For Numerical experiment purposes, we have chosen images from publically available domain and assumed reflexive (Neumann) boundary conditions. We blurred the images using gaussian blur of size 9×9 and standard deviation 4. We have compared the algorithm (5.14) with [17, Algorithm 8]. The operator $A = RW$, where W is the three stage Haar wavelet transform and R is the blur operator. The original and corresponding blurred images were shown in Figure 5.5. The regularization parameter was chosen to be $\lambda = 2 \times 10^{-5}$, and the initial image was the blurred image. The objective function value is denoted by $F(x^*)$ and function value at n^{th} iteration is denoted by $F(x_n)$. Sequences $\{\lambda_n\}$ and $\{\beta_n\}$ are chosen as $\{1 - \frac{1}{n+1}\}$ and $\{0.9\}$ respectively. The images recovered by the algorithms for 1000 iterations are shown in figure. The graphical representation of convergence of $F(x_n) - F(x^*)$

FIGURE 5.1: Original.

FIGURE 5.2: Blurred

FIGURE 5.3: Original

FIGURE 5.4: Blurred

FIGURE 5.5: The original and blurred images of Lenna and crowd.

is depicted in Figure 5.8. For deblurring methods, lower the value of $F(x_n) - F(x^*)$ higher the quality of recovered images.

It can be observed from Figure 5.8 and 5.13 that the proposed Algorithm (5.14) outperforms [17, Algorithm 8].

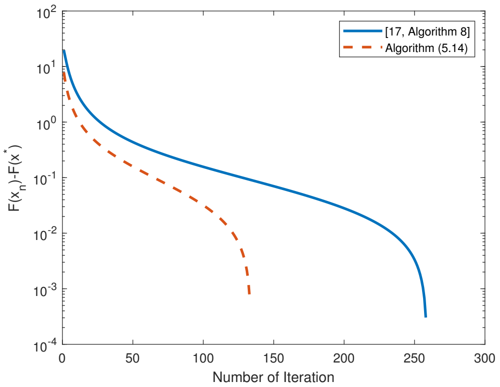

FIGURE 5.6: Lenna.

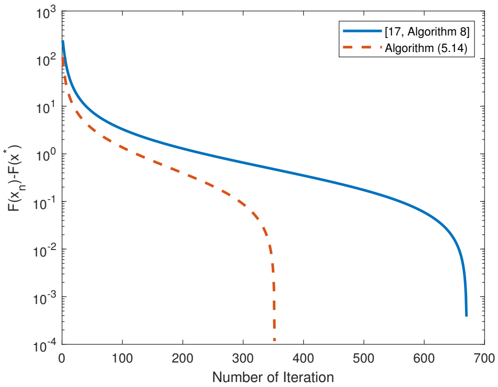

FIGURE 5.7: Crowd

FIGURE 5.8: The variation of $F(x_n) - F(x^*)$ with respect to number of iteration for different images.

5.7 Conclusion

In this chapter, we have proposed the normal-S iteration method based fixed point algorithm to find common fixed point of nonexpansive operators which converges strongly to minimal norm solutions of common fixed point problem of the considered operators. Based on the proposed fixed point algorithm, we develop a new forward-backward algorithm and a Doughlas-Rachford algorithm containing Tikhonov regularization term to solve the monotone inclusion problems. We have also proposed a forward-backward type primal-dual algorithm and a Doughlas-Rachford type primal-dual algorithm having Tikhonov regularization term to find the common solution of the complexly structured monotone inclusion problems containing mixtures of composite and parallel sum monotone operators. In the last, we have conducted a numerical experiment to solve the image deblurring problem using proposed methods. The numerical experiment shows that the proposed Algorithm (5.14) outperforms [17, Algorithm 8].

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FIGURE 5.9: Algorithm (5.14).

FIGURE 5.10: [17, Algorithm 8]

FIGURE 5.11: Algorithm (5.14).

FIGURE 5.12: [17, Algorithm 8].

FIGURE 5.13: The recovered images using different algorithms for 1000 iterations.