Chapter 4

Convergence Analysis of Two-Step Inertial Douglas-Rachford Algorithm and Application

The last two chapters of the thesis are focused to develop the methods to solve the monotone inclusion problem of the sum of two monotone operators in which one operator is necessarily single-valued. In this chapter, we are dedicated to solve the inclusion problem for the sum of two set-valued monotone operators. We propose the normal S-iteration based inertial Douglas-Rachford splitting algorithm and study its convergence behavior in Section 4.3. In Section 4.4, we propose an inertial primal-dual algorithm based on Algorithm 4.3.1 to solve monotone inclusion problems involving composition and parallel-sum operators. In Section 4.5, we discuss the applicability of the proposed inertial primal-dual algorithm to solve the convex

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optimization problem. We also apply the proposed inertial primal-dual algorithm to solve the generalized Heron problem.

4.1 Introduction

As explained in the first chapter, the Douglas-Rachford algorithm was introduced by Lions and Mercier [60] which is given as below:

$$x_{n+1} = \frac{1}{2}(Id + R_A \circ R_B)x_n, \tag{4.1}$$

where A and B are set-valued maximal monotone operators. In [16], Bot et al. proposed an inertial Douglas-Rachford algorithm, which contains the inertial term $x_n + \theta_n(x_n - x_{n-1})$ as follows:

$$\begin{cases} y_n = J_{\lambda B}[x_n + \theta_n(x_n - x_{n-1})], \\ z_n = J_{\lambda A}[2y_n - x_n - \theta_n(x_n - x_{n-1})], \\ x_{n+1} = x_n + \theta_n(x_n - x_{n-1}) + \beta_n(z_n - y_n) \ \forall n \geq 1, \end{cases} \tag{4.2}$$

where $A, B : \mathcal{H} \to 2^{\mathcal{H}}$ are maximal monotone operators and sequences $\{\theta_n\}, \{\beta_n\}$ satisfy some suitable assumptions. Inspired by the Douglas-Rachford algorithm (4.2) and normal S-iteration method (1.21), we propose a new Douglas-Rachford algorithm to solve the monotone inclusion problem (1.8) and study its convergence behavior. Further, we consider the highly structured monotone inclusion problems containing composition with linear operators and parallel sum operators. Since the resolvent of composition and resolvent of parallel sum operators is generally not present in closed form, the classical Douglas-Rachford algorithm is unable to solve such types of problems. In order to solve the problems, we propose an inertial primal-dual algorithm based on the proposed Algorithm 4.3.1. Further, we have study the convergence behavior of the proposed primal-dual algorithm and we apply the proposed primal-dual algorithm to solve a highly structured convex optimization problem. At the end, we conduct a numerical experiment to show the performance of the proposed algorithm and compare the algorithm with already known algorithms.

4.2 Preliminary Results

In this section, we present some basic results and definitions related to the study made in this chapter.

Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be an operator. Domain of A is dom $(A) = \{x \in \mathcal{H} : Ax \neq \emptyset\}$. Range of A is denoted by ran $(A) = \cup_{x \in \mathcal{H}} Ax$. A is said to be uniformly monotone with modulus $w_A \colon [0, \infty) \to [0, \infty]$, where w_A is increasing, vanishes only at 0, and satisfies

$$\langle x - y, u - v \rangle \geq w_A(\|x - y\|), \ \forall (x, u), (y, v) \in GrA.$$

The resolvent of A is defined by $J_A = (Id + A)^{-1}$. The reflected resolvent of A is $R_A = 2J_A - Id$. Consider $f : \mathcal{H} \to [-\infty, \infty]$. The conjugate of f is defined by $f^* : \mathcal{H} \to [-\infty, \infty]$ as

$$u \mapsto \sup_{x \in \mathcal{H}} \left(\langle x, u \rangle - f(x) \right), \ \forall u \in \mathcal{H}.$$

If $f \in \Gamma(\mathcal{H})$, then ∂f is maximally monotone and resolvent of ∂f is $prox_f$.

Let $\mathcal{C}$ be a nonempty subset of $\mathcal{H}$. The indicator function $i_{\mathcal{C}} : \mathcal{H} \to [-\infty, \infty]$ is defined by

$$i_{\mathcal{C}}(x) = \begin{cases} 0, & if\ x \in \mathcal{C}, \\ \infty, & otherwise. \end{cases}$$

The projection of a point $x \in \mathcal{H}$ on $\mathcal{C}$ is denoted and defined by

$$P_C(x) = \{u \in \mathcal{C} : u \in argmin_{z\in\mathcal{C}} \|x - z\|\}.$$

If $\mathcal{C}$ is convex, then the normal cone to $\mathcal{C}$ at x is defined by

$$N_{\mathcal{C}}(x) = \begin{cases} \{u \in \mathcal{H} : \langle y - x, u\rangle \leq 0,\ \forall y \in \mathcal{C}\}, & if\ x \in \mathcal{C} \\ \emptyset, & otherwise. \end{cases}$$

Definition 4.2.1. *[9] Consider the operators $T_1, T_2 : \mathcal{H} \to 2^{\mathcal{H}}$. The parallel sum of T_1 and T_2 is denoted by $T_1 \square T_2 : \mathcal{H} \to 2^{\mathcal{H}}$ and is defined by $T_1 \square T_2 = (T_1^{-1} + T_2^{-1})^{-1}$.*

Lemma 4.2.1. *[9, Proposition 25.1(ii)] If $T_1, T_2 : \mathcal{H} \to 2^{\mathcal{H}}$ are monotone operators, then $\operatorname{zer}(T_1 + T_2) = J_{\lambda T_2}(\operatorname{Fix}(R_{\lambda T_1} \circ R_{\lambda T_2})),\ \forall \lambda > 0$.*

Lemma 4.2.2. [9, Corollary 2.14] *Let $z_1, z_2 \in \mathcal{H}$. Then the following identities hold for arbitrary $a \in \mathbb{R}$:*

(i) $\|z_1 - z_2\|^2 = \|z_1\|^2 + \|z_2\|^2 - 2\langle z_1, z_2\rangle$,

(ii) $\|az_1 + (1-a)z_2\|^2 = a\|z_1\|^2 + (1-a)\|z_2\|^2 - a(1-a)\|z_1 - z_2\|^2$.

Lemma 4.2.3. *[37] Let ρ be positive and α be nonnegative real numbers. Then, for each $z_1,\ z_2 \in \mathcal{H}$,*

$$\|z_1 \pm \alpha z_2\|^2 \geq (1 - \alpha\rho)\|z_1\|^2 + \alpha(\alpha - \frac{1}{\rho})\|z_2\|^2.$$

Lemma 4.2.4. [9, Corollary 4.18] *Let $\mathcal{C}$ be a nonempty closed convex subset of $\mathcal{H}$ and $T : \mathcal{C} \to \mathcal{H}$ be a nonexpansive mapping. Let $\{z_n\}$ be a sequence in $\mathcal{C}$ and $z \in \mathcal{H}$ be such that $z_n \rightharpoonup z$ and $z_n - Tz_n \to 0$ as $n \to \infty$. Then $z \in \operatorname{Fix}(T)$.*

Lemma 4.2.5. *[9, Corollary 25.5] Consider maximal monotone operators $T_1, T_2 \colon \mathcal{H} \to 2^{\mathcal{H}}$. Let $\{(x_n, u_n)\} \subseteq GrT_1$, $\{(y_n, v_n)\} \subseteq GrT_2$ be sequences such that $x_n \rightharpoonup x, u_n \rightharpoonup u, y_n \rightharpoonup y, v_n \rightharpoonup v$ and $u_n + v_n \to 0$ and $x_n - y_n \to 0$ as $n \to \infty$. Then $x = y \in \operatorname{zer}(T_1 + T_2)$, $(x, u) \in GrT_1$ and $(y, v) \in GrT_2$.*

Lemma 4.2.6. [5, Lemma 3] *Consider sequences $\{y_n\}, \{z_n\}$ and $\{\theta_n\}$ in $[0, \infty)$ such that*

$$y_{n+1} \leq y_n + \theta_n(y_n - y_{n-1}) + z_n \quad \textit{for all} \quad n \in \mathbb{N}, \quad \sum_{n=1}^{\infty} z_n < \infty$$

and there exists a real number θ with $0 \leq \theta_n \leq \theta < 1$ for all $n \in \mathbb{N}$. Then the following hold:

(i) $\sum_{n=1}^{\infty}[y_n - y_{n-1}]_+ < \infty$, *where* $[t]_+ = \max\{t, 0\}$,

(ii) *there exists $y^* \in [0, \infty)$ such that $y_n \to y^*$.*

Lemma 4.2.7. *[74] Consider a nonempty subset $\mathcal{C}$ of $\mathcal{H}$. Let $\{\phi_n\}$ be a sequence in $\mathcal{H}$ such that the following two conditions hold:*

(i) *for all $\phi \in \mathcal{C}$, $\lim_{n\to\infty} \|\phi_n - \phi\|$ exists,*

(ii) *every sequential weak cluster point of $\{\phi_n\}$ is in $\mathcal{C}$.*

Then the sequence $\{\phi_n\}$ converges weakly to a point in $\mathcal{C}$.

4.3 Douglas-Rachford Algorithm

In this section, we propose an inertial Douglas-Rachford algorithm based on the normal S-iteration method [82] to solve the monotone inclusion problem (1.8) and study its convergence behavior in the real Hilbert space framework.

In what follows first we introduce the Douglas-Rachford like algorithm:

Algorithm 4.3.1. *Suppose that* $A, B : \mathcal{H} \to 2^{\mathcal{H}}$ *are maximally monotone operators with nonempty set* $\operatorname{zer}(A+B)$*. Choose* $\lambda > 0$*, initial points* $x_0, x_1 \in \mathcal{H}$ *and parameters* $\{\theta_n\}, \{\beta_n\}$ *in [0,1]. Then the* $(n+1)^{th}$ *term of the algorithm is obtained as follows:*

$$\begin{cases} y_n = J_{\lambda B}[x_n + \theta_n(x_n - x_{n-1})], \\ z_n = J_{\lambda A}[2y_n - x_n - \theta_n(x_n - x_{n-1})], \\ u_n = x_n + \theta_n(x_n - x_{n-1}) + \beta_n(z_n - y_n), \\ x_{n+1} = (2J_{\lambda A} - Id)(2J_{\lambda B} - Id)(u_n), \ \forall n \in \mathbb{N}. \end{cases} \tag{4.3}$$

Remark 4.3.1. *Set* $w_n := x_n + \theta_n(x_n - x_{n-1})$*. Using the definition of the reflected resolvent, Algorithm 4.3.1 can be written as*

$$\begin{aligned} x_{n+1} &= (2J_{\lambda A} - Id)(2J_{\lambda B} - Id)\{w_n + \beta_n[J_{\lambda A}(2J_{\lambda B} - Id)w_n - J_{\lambda B}w_n]\} \\ &= (R_{\lambda A} \circ R_{\lambda B})\left\{w_n + \beta_n\left[\left(\frac{Id + R_{\lambda A}}{2} \circ R_{\lambda B}\right)w_n - \frac{Id + R_{\lambda B}}{2}w_n\right]\right\} \\ &= (R_{\lambda A} \circ R_{\lambda B})\left\{(1 - \frac{\beta_n}{2})w_n + \frac{\beta_n}{2}(R_{\lambda A} \circ R_{\lambda B})w_n\right\}. \end{aligned} \tag{4.4}$$

In view of (4.4) and (1.21), we say Algorithm 4.3.1 is normal S-iteration based inertial Douglas-Rachford splitting method (InS-DRSM).

For convergence analysis of Algorithm 4.3.1, we assume that $\{\theta_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

($\mathbf{A}1$) $\{\theta_n\} \subset [0, \theta]$ is a nondecreasing sequence, where $0 < \theta < 1$,

($\mathbf{A}2$) $\beta_n \in (0, 1)$,

($\mathbf{A}3$) the constants $\beta, \tau, \delta > 0$ satisfy

$$\delta > \tfrac{\gamma\theta(\theta(1+\theta)+\tau)}{1-\theta^2(1-\beta/2)}, \qquad 0 < \beta \leq \beta_n \leq \frac{\delta - \theta(\gamma\theta(1+\theta) + \theta\delta(1-\frac{\beta}{2}) + \gamma\tau)}{\delta[\frac{1}{2} + \gamma\theta(1+\theta) + \theta\delta(1-\frac{\beta}{2}) + \gamma\tau]},$$

where $\gamma = 1 + \frac{4}{\beta^2}$.

The convergence behavior of Algorithm 4.3.1 is summarized in the following theorem:

Theorem 4.3.1. *Consider $A, B : \mathcal{H} \to 2^{\mathcal{H}}$ are maximally monotone operators such that $\operatorname{zer}(A+B)$ is non-empty. Let $\{x_n\}$ be an orbit of InS-DRSM (4.3) with parameters $\{\theta_n\}, \{\beta_n\}$ satisfying assumption ($\mathbf{A}1$)-($\mathbf{A}3$) with $\theta_1 = 0$. Then, there exists $x \in \operatorname{Fix}(R_{\lambda A} \circ R_{\lambda B})$ such that the following are true:*

(a) $J_{\lambda B}x \in \operatorname{zer}(A+B)$,

(b) $\sum_{n=1}^{\infty} \|x_n - x_{n-1}\|^2 < \infty$,

(c) $\{x_n\}$ *converges weakly to* x,

(d) $y_n - z_n \to 0 \in \mathcal{H}$,

(e) $\{y_n\}$ *converges weakly to* $J_{\lambda B}x$,

(f) $\{z_n\}$ *converges weakly to* $J_{\lambda B}x$,

(g) If A or B is uniformly monotone, then $\{y_n\}$ and $\{z_n\}$ converge strongly to a unique point in $\operatorname{zer}(A+B)$.

Proof. (a) Note that the operators A and B are monotone. Using Remark 4.2.1, we obtain $\text{zer}(A+B) = J_{\lambda B}(\text{Fix}(R_{\lambda A} \circ R_{\lambda B}))$. Since $\text{zer}(A+B)$ is non-empty, it follows that $\text{Fix}(R_{\lambda A} \circ R_{\lambda B})$ is non-empty.

(b)-(c) Let $\bar{x} \in \text{Fix}(R_{\lambda A} \circ R_{\lambda B})$. Using Algorithm 4.3.1, Lemma 4.2.2, and nonexpansitivity of reflected resolvent, we obtain

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &= \|R_{\lambda A} \circ R_{\lambda B}[(1 - \frac{\beta_n}{2})w_n + \frac{\beta_n}{2} R_{\lambda A} \circ R_{\lambda B}(w_n)] - \bar{x}\|^2 \\
&\leq (1 - \frac{\beta_n}{2})\|w_n - \bar{x}\|^2 + \frac{\beta_n}{2}\|R_{\lambda A} \circ R_{\lambda B}(w_n) - \bar{x}\|^2 \\
&\quad -\frac{\beta_n}{2}(1 - \frac{\beta_n}{2})\|w_n - R_{\lambda A} \circ R_{\lambda B}(w_n)\|^2 \\
&\leq \|w_n - \bar{x}\|^2 - \frac{\beta_n}{2}(1 - \frac{\beta_n}{2})\|w_n - R_{\lambda A} \circ R_{\lambda B}(w_n)\|^2.
\end{aligned}$$

By Lemma 4.2.2, we have

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &\leq (1 + \theta_n)\|x_n - \bar{x}\|^2 - \theta_n\|x_{n-1} - \bar{x}\|^2 + \theta_n(1 + \theta_n)\|x_n - x_{n-1}\|^2 \\
&\quad -\frac{\beta_n}{2}(1 - \frac{\beta_n}{2})\|w_n - R_{\lambda A} \circ R_{\lambda B}(w_n)\|^2. \qquad (4.5)
\end{aligned}$$

Since $u_n = (1 - \frac{\beta_n}{2})w_n + \frac{\beta_n}{2} R_{\lambda A} \circ R_{\lambda B}(w_n)$, we get

$$\begin{aligned}
\|w_n - R_{\lambda A} \circ R_{\lambda B}(w_n)\|^2 &= \frac{4}{\beta_n^2}\|u_n - w_n\|^2 \\
&\geq \frac{4}{\beta_n^2}\|x_{n+1} - R_{\lambda A} \circ R_{\lambda B}(w_n)\|^2 \\
&= \frac{4}{\beta_n^2}\|x_{n+1} - w_n + w_n - R_{\lambda A} \circ R_{\lambda B}(w_n)\|^2.
\end{aligned}$$

Taking $\alpha = 1$ and $\rho = \frac{1}{2}$ in Lemma 4.2.3, we obtain

$$\|w_n - R_{\lambda A} \circ R_{\lambda B}(w_n)\|^2 \geq \frac{4}{\beta_n^2}\{\frac{1}{2}\|x_{n+1} - w_n\|^2 - \|w_n - R_{\lambda A} \circ R_{\lambda B}(w_n)\|^2\},$$

which implies that

$$(1 + \frac{4}{\beta_n^2})\|w_n - R_{\lambda A} \circ R_{\lambda B}(w_n)\|^2 \geq \frac{2}{\beta_n^2}\|x_{n+1} - w_n\|^2.$$

Since $\{\beta_n\}$ is bounded below by β, we have

$$\begin{aligned}(1 + \frac{4}{\beta^2})\|w_n - R_{\lambda A} \circ R_{\lambda B}(w_n)\|^2 &\geq \frac{2}{\beta_n^2}\|x_{n+1} - w_n\|^2 \\ &= \frac{2}{\beta_n^2}\|x_{n+1} - x_n - \theta_n(x_n - x_{n-1})\|^2.\end{aligned}$$

Using Lemma 4.2.3 for $\alpha = \theta_n$ and $\rho = \rho_n = \frac{1}{\theta_n + \delta\beta_n}$, we have

$$\begin{aligned}(1 + \frac{4}{\beta^2})\|w_n - R_{\lambda A} \circ R_{\lambda B}(w_n)\|^2 &\geq \frac{2(1 - \theta_n\rho_n)}{{\beta_n}^2}\|x_{n+1} - x_n\|^2 + \frac{2\theta_n}{\beta_n^2}(\theta_n - \frac{1}{\rho_n})\|x_n - x_{n-1}\|^2 \\ &= \frac{2(1 - \theta_n\rho_n)}{\beta_n^2}\|x_{n+1} - x_n\|^2 \\ &\quad - \frac{2\theta_n(1 - \theta_n\rho_n)}{\beta_n^2\rho_n}\|x_n - x_{n-1}\|^2. \qquad (4.6)\end{aligned}$$

Thus using (4.6) in (4.5), we obtain

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 \leq\; & (1 + \theta_n)\|x_n - \bar{x}\|^2 - \theta_n\|x_{n-1} - \bar{x}\|^2 + \theta_n(1 + \theta_n)\|x_n - x_{n-1}\|^2 \\ & - \frac{(1 - \frac{\beta_n}{2})(1 - \theta_n\rho_n)}{\gamma\beta_n}\|x_{n+1} - x_n\|^2 + \frac{\theta_n(1 - \frac{\beta_n}{2})(1 - \theta_n\rho_n)}{\gamma\beta_n\rho_n}\|x_n - x_{n-1}\|^2.\end{aligned}$$

Set $\phi_n = \|x_n - \bar{x}\|^2$. Then, we have

$$\begin{aligned}\phi_{n+1} &\leq (1+\theta_n)\phi_n - \theta_n\phi_{n-1} + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 - \frac{(1-\frac{\beta_n}{2})(1-\theta_n\rho_n)}{\gamma\beta_n}\|x_{n+1}-x_n\|^2 \\ &+\frac{\theta_n(1-\frac{\beta_n}{2})(1-\theta_n\rho_n)}{\gamma\beta_n\rho_n}\|x_n - x_{n-1}\|^2,\end{aligned}$$

which can be written as

$$\phi_{n+1} - (1+\theta_n)\phi_n + \theta_n\phi_{n-1} \leq -\xi_n\|x_{n+1} - x_n\|^2 + \mu_n\|x_n - x_{n-1}\|^2, \tag{4.7}$$

where $\xi_n = \frac{(1-\frac{\beta_n}{2})(1-\theta_n\rho_n)}{\gamma\beta_n}$ and $\mu_n = \theta_n(1+\theta_n) + \frac{\theta_n(1-\frac{\beta_n}{2})(1-\theta_n\rho_n)}{\gamma\beta_n\rho_n}$.

Denote $\psi_n \equiv \|x_n - \bar{x}\|^2 - \theta_n\|x_{n-1} - \bar{x}\|^2 + \mu_n\|x_n - x_{n-1}\|^2$. Since $\{\theta_n\}$ is non-decreasing, from (4.7), we obtain

$$\begin{aligned}\psi_{n+1} - \psi_n &= \phi_{n+1} - (1+\theta_{n+1})\phi_n + \theta_n\phi_{n-1} + \mu_{n+1}\|x_{n+1} - x_n\|^2 - \mu_n\|x_n - x_{n-1}\|^2 \\ &\leq \phi_{n+1} - (1+\theta_n)\phi_n + \theta_n\phi_{n-1} + \mu_{n+1}\|x_{n+1} - x_n\|^2 - \mu_n\|x_n - x_{n-1}\|^2 \\ &\leq (-\xi_n + \mu_{n+1})\|x_{n+1} - x_n\|^2. \end{aligned}\tag{4.8}$$

Note that $\mu_n = \theta_n(1+\theta_n) + \frac{\theta_n(1-\frac{\beta_n}{2})(1-\theta_n\rho_n)}{\gamma\beta_n\rho_n} > 0$, since $\theta_n\rho_n < 1$ and $\beta_n \in (0,1)$. Again, taking into account the choice of ρ_n, we have

$$\delta = \frac{1-\theta_n\rho_n}{\rho_n\beta_n}.$$

Note

$$\mu_n = \theta_n(1+\theta_n) + \frac{\theta_n(1-\frac{\beta_n}{2})\delta}{\gamma} \leq \theta(1+\theta) + \frac{\theta\delta(1-\frac{\beta}{2})}{\gamma} \text{ for all } n \in \mathbb{N}. \tag{4.9}$$

Suppose that

$$-\xi_n + \mu_{n+1} \leq -\tau, \ \forall n \in \mathbb{N}, \tag{4.10}$$

where $\tau > 0$ is a real number. Inequality (4.8) along with (4.10) implies that $\{\psi_n\}$ is nonincreasing. Since $\{\theta_n\}$ is bounded above by θ, we get

$$-\theta\phi_{n-1} \leq \phi_n - \theta\phi_{n-1} \leq \psi_n \leq \psi_1.$$

Thus, we obtain

$$\begin{aligned}
\phi_n &\leq \theta\phi_{n-1} + \psi_1 \\
&\leq \theta(\theta\phi_{n-2} + \psi_1) + \psi_1 \\
&\vdots \\
&\leq \theta^n\phi_0 + \psi_1 \sum_{k=0}^{n-1} \theta^k \leq \theta^n\phi_0 + \frac{\psi_1}{1-\theta}.
\end{aligned}$$

From (4.8) and (4.10), we have

$$\begin{aligned}
\tau \sum_{k=1}^{n} \|x_{k+1} - x_k\|^2 &\leq \sum_{k=1}^{n} \psi_k - \psi_{k+1} \\
&= \psi_1 - \psi_{n+1} \\
&\leq \psi_1 + \theta\phi_n \\
&\leq \psi_1 + \theta(\theta^n\phi_0 + \frac{\psi_1}{1-\theta}) \\
&= \theta^{n+1}\phi_0 + \frac{\psi_1}{1-\theta}.
\end{aligned}$$

Since $\theta^{n+1} \to 0$ as $n \to \infty$, we obtain that

$$\sum_{n=0}^{\infty} \|x_{n+1} - x_n\|^2 < \infty. \tag{4.11}$$

The proof will be complete if we show that $-\xi_n + \mu_{n+1} \le -\tau,\ \forall n \in \mathbb{N}$. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
-\xi_n + \mu_{n+1} \le -\tau \quad &\Leftrightarrow \quad \frac{(1-\frac{\beta_n}{2})(\theta_n\rho_n - 1)}{\gamma\beta_n} + (\mu_{n+1}+\tau) \le 0 \\
&\Leftrightarrow \quad (1-\frac{\beta_n}{2})(\theta_n\rho_n - 1) + \gamma\beta_n(\mu_{n+1}+\tau) \le 0 \\
&\Leftrightarrow \quad -(1-\frac{\beta_n}{2})\delta\rho_n\beta_n + \gamma\beta_n(\mu_{n+1}+\tau) \le 0 \\
&\Leftrightarrow \quad -\frac{(1-\frac{\beta_n}{2})\delta}{\theta_n+\delta\beta_n} + \gamma(\mu_{n+1}+\tau) \le 0 \\
&\Leftrightarrow \quad -(1-\frac{\beta_n}{2})\delta + \gamma(\mu_{n+1}+\tau)(\theta_n+\delta\beta_n) \le 0 \\
&\Leftrightarrow \quad \gamma(\mu_{n+1}+\tau)(\theta_n+\delta\beta_n) + \frac{\beta_n}{2}\delta \le \delta.
\end{aligned}$$

By using (4.9), we have

$$\begin{aligned}
\gamma(\mu_{n+1}+\tau)(\theta_n+\delta\beta_n) + \frac{\beta_n}{2}\delta \quad &\le \gamma\left(\theta(1+\theta) + \frac{\theta\delta(1-\frac{\beta}{2})}{\gamma} + \tau\right)(\theta+\delta\beta_n) + \frac{\beta_n}{2}\delta \\
&\le \delta, \qquad (4.12)
\end{aligned}$$

which follows from the upper bound on β_n as in assumption (**A**3). Hence

$$-\xi_n + \mu_{n+1} \le -\tau, \forall\ n \in \mathbb{N}.$$

Applying inequality (4.7), (4.9) and claim (b) to Lemma 4.2.6, we get that $\lim_{n\to\infty}\|x_n - \bar{x}\|$ exists.

Suppose that the sequence $\{x_n\}$ has a sequential weak cluster point $x^*\in \mathcal{H}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in \mathcal{H}$.

From Algorithm 4.3.1, we have

$$\begin{aligned}\|w_n - x_{n+1}\| &\leq \|x_n - x_{n+1}\| + \theta_n\|x_n - x_{n-1}\| \\ &\leq \|x_n - x_{n+1}\| + \theta\|x_n - x_{n-1}\|,\end{aligned}$$

which implies that

$$\|w_n - x_{n+1}\| \to 0 \text{ as } n \ \to \infty. \tag{4.13}$$

Thus, the sequential weak cluster points of $\{x_n\}$ and $\{w_n\}$ are the same.

Now,

$$\begin{aligned}\|R_{\lambda A} \circ R_{\lambda B} w_n - w_n\| &\leq \|R_{\lambda A} \circ R_{\lambda B} w_n - x_{n+1}\| + \|x_{n+1} - w_n\| \\ &= \|R_{\lambda A} \circ R_{\lambda B} w_n - R_{\lambda A} \circ R_{\lambda B} u_n\| + \|x_{n+1} - w_n\| \\ &\leq \|w_n - u_n\| + \|x_{n+1} - w_n\| \\ &= \|w_n - (1 - \frac{\beta_n}{2})w_n - \frac{\beta_n}{2} R_{\lambda A} \circ R_{\lambda B} w_n\| + \|x_{n+1} - w_n\| \\ &= \frac{\beta_n}{2}\|w_n - R_{\lambda A} \circ R_{\lambda B} w_n\| + \|x_{n+1} - w_n\| \\ &\leq \frac{1}{2}\|w_n - R_{\lambda A} \circ R_{\lambda B} w_n\| + \|x_{n+1} - w_n\|,\end{aligned}$$

thus, we obtain

$$\frac{1}{2}\|R_{\lambda A} \circ R_{\lambda B} w_n - w_n\| \ \leq \ \|x_{n+1} - w_n\|. \tag{4.14}$$

Using (4.13) and (4.14), we get

$$R_{\lambda A} \circ R_{\lambda B} w_{n_k} - w_{n_k} \to 0 \text{ as } k \to \infty. \tag{4.15}$$

Applying Lemma 4.2.4 for subsequence $\{x_{n_k}\}$, we conclude that $x^* \in \mathrm{Fix}(R_{\lambda A} \circ R_{\lambda B})$.

Since $\lim_{n\to\infty} \|x_n - \bar{x}\|$ exists for each $\bar{x} \in \text{Fix}(R_{\lambda A} \circ R_{\lambda B})$ and each sequential weak cluster point of $\{x_n\}$ is in $\text{Fix}(R_{\lambda A} \circ R_{\lambda B})$, Lemma 4.2.7 implies that there exists $x \in \text{Fix}(R_{\lambda A} \circ R_{\lambda B})$ such that $\{x_n\}$ converges weakly to x.

(d) From Algorithm 4.3.1, we have that $z_n - y_n = \frac{1}{2}(R_{\lambda A} \circ R_{\lambda B} w_n - w_n)$, for all $n \in \mathbb{N}$. From (4.14), we have that $\{(R_{\lambda A} \circ R_{\lambda B} w_n - w_n)\}$ converges strongly to 0 as $n \to \infty$. Thus, the conclusion follows.

(e) Using nonexpansiveness of $J_{\lambda B}$, we have

$$\begin{aligned} \|y_n - y_1\| &= \|J_{\lambda B} w_n - J_{\lambda B} w_1\| \\ &\leq \|w_n - w_1\| \\ &= \|x_n - x_1 + \theta_n (x_n - x_{n-1})\|. \end{aligned}$$

This shows that $\{y_n\}$ is bounded as $\{x_n\}$ is bounded. Set

$$a_n := 2y_n - w_n - z_n \text{ and } b_n := w_n - y_n.$$

Using the definition of resolvent, $(z_n, a_n) \in Gr(\lambda A)$, $(y_n, b_n) \in Gr(\lambda B)$ and $a_n + b_n = y_n - z_n$. Let y be a sequential weak cluster point of $\{y_n\}$. Then, there exists a subsequence $\{y_{n_k}\} \subseteq \{y_n\}$ such that $y_{n_k} \rightharpoonup y$. From (c) and (d), we have $z_{n_k} \rightharpoonup y$, $w_{n_k} \rightharpoonup x$, $a_{n_k} \rightharpoonup y - x$ and $b_{n_k} \rightharpoonup x - y$ as $k \to \infty$. Using Lemma 4.2.5, we conclude that $y \in \text{zer}(\lambda A + \lambda B) = \text{zer}(A + B)$, $(y, y - x) \in Gr(\lambda A)$ and $(y, x - y) \in Gr(\lambda B)$. Thus, in turn, we have $y = J_{\lambda B} x$.

(f) From (d) and (e), $\{y_n\}$ converges weakly to $J_{\lambda B} x$ and $\{y_n - z_n\}$ converges to 0, thus $\{z_n\}$ converges weakly to $J_{\lambda B} x$.

(g) Without any loss of generality, we assume that A is uniformly monotone (in case B is uniformly monotone, the proof follows in a similar pattern). Thus an increasing function $w_A : [0,\infty) \to [0,\infty]$ exists which vanishes only at 0 and satisfies

$$\lambda w_A(\|z_n - y\|) \leq \langle z_n - y, a_n - y + x\rangle, \ \forall n \in \mathbb{N}. \tag{4.16}$$

Using monotonicity of B, we have

$$0 \leq \langle y_n - y, b_n - x + y\rangle = \langle y_n - y, y_n - z_n - a_n - x + y\rangle, \ \forall n \in \mathbb{N}. \tag{4.17}$$

Adding (4.16) and (4.17), we have

$$\begin{aligned} \lambda w_A(\|z_n - y\|) &\leq \langle z_n - y_n, a_n - y_n + x\rangle \\ &= \langle z_n - y_n, y_n - z_n - w_n + x\rangle, \ \forall n \in \mathbb{N}. \end{aligned}$$

From (c), (d) and (4.13) we get $\lim_{n\to\infty} w_A(\|z_n - y_n\|) = 0$, hence $z_n \to y$, which in turn implies that $y_n \to y$ as $n \to \infty$. □

Remark 4.3.2. *In the proof of Theorem 4.3.1, we required the nonnegativity of μ_1 which is assured by assuming $\theta_1 = 0$. The condition can also be satisfied by choosing the same initial points x_0 and x_1, i.e., $x_0 = x_1$.*

Example 4.3.1. *Let $\mathcal{H} = \mathbb{R}^2$ with Euclidean norm. Consider a circular disk $C := \{(h,k) \in \mathbb{R}^2 : (h-5)^2 + k^2 \leq 2\}$ and a box $D = \{(h,k) \in \mathbb{R}^2 : 2 \leq h \leq 4, 0.5 \leq k \leq 2.5\}$. Consider the minimization problem:*

$$\min_{x\in\mathbb{R}^2} F(x) = i_C(x) + i_D(x). \tag{4.18}$$

Note that minimization problem (4.18) is equivalent to the following inclusion problem:

$$\textit{find } x \in \mathbb{R}^2 \textit{ such that } 0 \in N_C x + N_D x.$$

Set $A = N_C$, $B = N_D$ in Algorithm 4.3.1 and apply Theorem 4.3.1 to solve the minimization problem (4.18), we obtain that sequence $\{x_n\}$ generated by Algorithm 4.3.1 converges to a solution of minimization problem (4.18). Here in the numerical example, we choose $\theta_n = \frac{n-1}{14n+2.5}$, which is similar to the inertial parameter in [29], $\beta_n = 0.5 + \frac{1}{200n}$ with initial points $x_0 = x_1 = (10, -20)$ and $x_0 = x_1 = (20, -53)$. We take $\|P_C(x_n) - x_n\|^2 + \|P_D(x_n) - x_n\|^2 < 10^{-5}$ as a stopping criterion. We plot the semilog graph between $\|P_C(x_n) - x_n\|^2 + \|P_D(x_n) - x_n\|^2$ and the number of iterations for two different initial points.

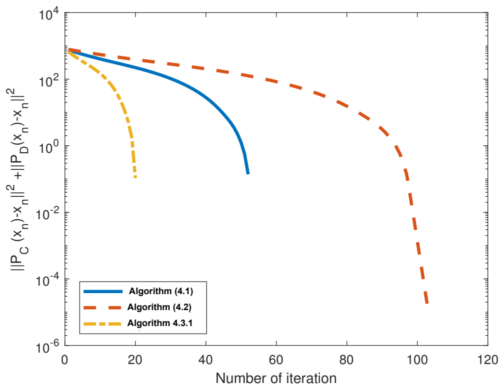

FIGURE 4.1: Initial points $x_0 = x_1 = (10, -20)$.

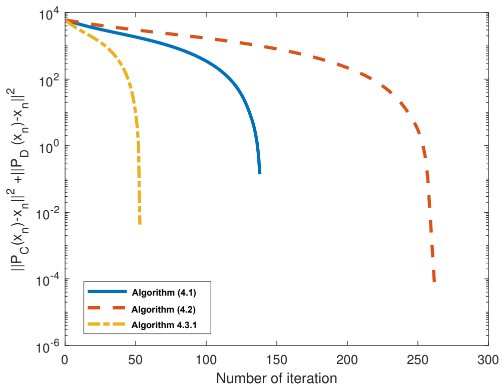

FIGURE 4.2: Initial points $x_0 = x_1 = (20, -53)$.

FIGURE 4.3: Semilog graph between number of iterations and sum of distance of iterative points to sets C and D for different initial points.

From Fig 4.3, we can observe that the sum of the distances of iterative values to the sets C and D by our Algorithm 4.3.1 are less than that obtained by the classical Douglas-Rachford algorithm (4.1), which is also less than that by Algorithm (4.2). Thus, we can conclude that Algorithm 4.3.1 has a better convergence speed in particular for optimization problem (4.18).

4.4 Accelerated normal-S primal-dual algorithm

This section is devoted to obtain a solution to the highly structured monotone inclusion problem containing set-valued operators, composition with linear operator and parallel-sum operators. Since in general resolvent of the composition of operators and resolvent of parallel sum operators are not present in closed form, classical Douglas-Rachford algorithm is not applicable to solve the structured monotone inclusion problem.

Let m be a strictly positive integer and $\mathcal{I}$ denote the set $\{1, 2, \dots, m\}$. For $i \in \mathcal{I}$, let Ω_i be a real Hilbert space. For $i \in \mathcal{I}$, let $P : \mathcal{H} \to 2^{\mathcal{H}}$, $Q_i : \Omega_i \to 2^{\Omega_i}$, $R_i : \Omega_i \to 2^{\Omega_i}$ be maximally monotone operators and $T_i : \mathcal{H} \to \Omega_i$ be a non-zero linear continuous operator. Construct the Hilbert space $\mathcal{X} = \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ with inner product and induced norm defined by,

$$\langle(\xi, \xi_1, \cdots, \xi_m), (\zeta, \zeta_1, \cdots, \zeta_m)\rangle_{\mathcal{X}} = \langle \xi, \zeta\rangle_{\mathcal{H}} + \sum_{i=1}^{m} \langle \xi_i, \zeta_i\rangle_{\Omega_i}$$

and

$$\|(\xi, \xi_1, \cdots, \xi_m)\|_{\mathcal{X}} = \left(\|\xi\|^2_{\mathcal{H}} + \sum_{i=1}^{m} \|\xi_i\|^2_{\Omega_i}\right)^{\frac{1}{2}},$$

respectively, for $(\xi, \xi_1, \cdots, \xi_m), (\zeta, \zeta_1, \cdots, \zeta_m) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$.

Suppose $(\xi, \xi_1, \dots, \xi_m)$ and $(w, h_1, h_2, \dots, h_m) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$. Define operators

(a) $\mathcal{M} : \mathcal{X} \to 2^{\mathcal{X}} : (\xi, \xi_1, \dots, \xi_m) \mapsto (-w + P\xi, h_1 + Q_1^{-1}\xi_1, \dots, h_m + Q_m^{-1}\xi_m)$.

(b) $\mathcal{S} : \mathcal{X} \to \mathcal{X} : (\xi, \xi_1, \dots, \xi_m) \mapsto (\sum_{i=1}^{m} T_i^* \xi_i, -T_1\xi, \dots, -T_m\xi)$.

(c) $\mathcal{Q} : \mathcal{X} \to \mathcal{X} : (\xi, \xi_1, \dots, \xi_m) \mapsto (0, R_1^{-1}\xi_1, \dots, R_m^{-1}\xi_m)$.

(e) $F : \mathcal{X} \to \mathcal{X} : F = \frac{1}{2}\mathcal{S} + \mathcal{Q}$.

(f) $G : \mathcal{X} \to 2^{\mathcal{X}} : G = \frac{1}{2}\mathcal{S} + \mathcal{M}$.

Remark 4.4.1. *(a) The operators $\mathcal{M}$ and $\mathcal{Q}$ are maximally monotone as P, Q and R are maximally monotone ([9, Proposition 20.22, and Proposition 20.23]).*

(b) $\mathcal{S}$ is maximally monotone as $\mathcal{S}$ is skew-symmetric ([9, Proposition 20.30]).

(c) The operators F and G are maximally monotone as $\operatorname{dom}(\mathcal{S}) = \mathcal{X}$ ([9, Corollary 24.4(i)]) .

We now consider the following problem:

Problem 4.4.1. *For $i \in \mathcal{I}$, let $w \in \mathcal{H}$ and $h_i \in \Omega_i$. The monotone inclusion problem is to find $\bar{\xi} \in \mathcal{H}$ such that*

$$w \in P\bar{\xi} + \sum_{i=1}^{m} T_i^*(Q_i \Box R_i)(T_i\bar{\xi} - h_i) \tag{4.19}$$

and the dual-inclusion problem is to find $\bar{\zeta}_i \in \Omega_i$, $\forall i \in \mathcal{I}$ such that $\exists\, \xi \in \mathcal{H}$ satisfying

$$\forall n \in \mathbb{N} \begin{cases} w - \sum_{i=1}^{m} T_i^*\bar{\zeta}_i \in P\xi \\ \bar{\zeta}_i \in (Q_i \Box R_i)(T_i\xi - h_i), \ i \in \mathcal{I}. \end{cases} \tag{4.20}$$

Remark 4.4.2. *[16]*

(a) $(\bar{\xi}, \bar{\zeta}_1, \cdots, \bar{\zeta}_m) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ is a primal-dual solution to Problem 4.4.1 then $\bar{\xi}$ solves (4.19) and $(\bar{\zeta}_1, \cdots, \bar{\zeta}_m)$ solves (4.20).

(b) $\bar{\xi} \in \mathcal{H}$ solves (4.19) if and only if there exists $(\bar{\zeta}_1, \cdots, \bar{\zeta}_m) \in \Omega_1 \times \cdots \times \Omega_m$ such that $(\bar{\zeta}_1, \cdots, \bar{\zeta}_m)$ solves (4.20) and $(\bar{\xi}, \bar{\zeta}_1, \cdots, \bar{\zeta}_m)$ solves Problem 4.4.1.

(c) $(\bar{\zeta}_1, \cdots, \bar{\zeta}_m) \in \Omega_1 \times \cdots \times \Omega_m$ solves (4.20) if and only if there exists $\bar{\xi} \in \mathcal{H}$ such that $\bar{\xi}$ solves (4.19) and $(\bar{\xi}, \bar{\zeta}_1, \cdots, \bar{\zeta}_m)$ solves Problem 4.4.1.

(d) *If* $(\bar{\xi}, \bar{\zeta}_1, \cdots, \bar{\zeta}_m) \in \operatorname{zer}(\mathcal{M}+\mathcal{S}+\mathcal{Q})$ *then* $(\bar{\xi}, \bar{\zeta}_1, \cdots, \bar{\zeta}_m)$ *is a primal-dual solution of Problem 4.4.1.*

In order to solve Problem 4.4.1, we propose the following inertial primal-dual algorithm based on Algorithm 4.3.1, as follows:

Algorithm 4.4.1:

Input:

1. initial points $(\xi_0, \zeta_{1,0}, \dots, \zeta_{m,0}), (\xi_1, \zeta_{1,1}, \dots, \zeta_{m,1}) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$.

2. $0 \le \tau, \sigma_i \in \mathbb{R}$, $i \in \mathcal{I}$ such that $\tau \sum_{i=1}^m \sigma_i \|T_i\|^2 < 4$.

Procedure:

$x_{1,n} = J_{\tau P}(\xi_n + \theta_n(\xi_n - \xi_{n-1}) - \frac{\tau}{2} \sum_{i=1}^m T_i^*(\zeta_{i,n} + \theta_n(\zeta_{i,n} - \zeta_{i,n-1})) + \tau w)$

$y_{1,n} = 2x_{1,n} - \xi_{1,n} - \theta_n(\xi_n - \xi_{n-1})$

For $i = 1, \dots, m$;

$x_{2,i,n} = J_{\sigma_i Q_i^{-1}}(\zeta_{i,n} + \theta_n(\zeta_{i,n} - \zeta_{i,n-1}) + \frac{\sigma_i}{2} T_i y_{1,n} - \sigma_i h_i)$

$y_{2,i,n} = 2x_{2,i,n} - \zeta_{i,n} - \theta_n(\zeta_{i,n} - \zeta_{i,n-1})$

$u_{1,n} = y_{1,n} - \frac{\tau}{2} \sum_{i=1}^m T_i^* y_{2,i,n}$

$v_{1,n} = \xi_n + \theta_n(\xi_n - \xi_{n-1}) + \beta_n(u_{1,n} - x_{1,n})$

For $i = 1, \dots, m$;

$u_{2,i,n} = J_{\sigma_i R_i^{-1}}(y_{2,i,n} + \frac{\sigma_i}{2} T_i(2u_{1,n} - y_{1,n}))$

$v_{2,i,n} = \zeta_{i,n} + \theta_n(\zeta_{i,n} - \zeta_{i,n-1}) + \beta_n(u_{2,i,n} - x_{2,i,n})$

$s_{1,n} = J_{\tau P}(v_{1,n} - \frac{\tau}{2} \sum_{i=1}^m T_i^*(v_{2,i,n}) + \tau w)$

$t_{1,n} = 2s_{1,n} - v_{1,n}$

For $i = 1, \dots, m$;

$s_{2,i,n} = J_{\sigma_i Q_i^{-1}}(v_{2,i,n} + \frac{\sigma_i}{2} T_i t_{1,n} - \sigma_i h_i))$

$t_{2,i,n} = 2s_{2,i,n} - v_{2,i,n}$

$q_{1,n} = t_{1,n} - \frac{\tau}{2} \sum_{i=1}^m T_i^*(t_{2,i,n})$

$\xi_{n+1} = 2q_{1,n} - t_{1,n}$

For $i = 1, \dots, m$;

$q_{2,i,n} = J_{\sigma_i R_i^{-1}}(t_{2,i,n} + \frac{\sigma_i}{2} T_i(\xi_{n+1}))$

$\zeta_{i,n+1} = 2q_{2,i,n} - t_{2,i,n}$

Output: $(\xi_{n+1}, \zeta_{1,n+1}, \dots, \zeta_{m,n+1})$.

We say Algorithm 4.4.1 is normal S-iteration based inertial primal-dual (InS-PD) algorithm. We now establish the convergence theory of Algorithm 4.4.1.

Theorem 4.4.1. *Consider the Problem 4.4.1 with the point*

$$w \in ran\left(P + \sum_{i=1}^{m} T_i^*(Q_i \square R_i)(T_i(\cdot) - h_i)\right). \tag{4.21}$$

Let $\{\theta_n\}$ *and* $\{\beta_n\}$ *be sequences satisfying the assumptions as in Theorem 4.3.1. Then* $\exists\, \bar{a} = (\bar{\xi}, \bar{\zeta}_1, \ldots, \bar{\zeta}_m) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ *and the sequence* $\{(\xi_n, \zeta_{1,n}, \ldots, \zeta_{m,n})\}$ *generated by Algorithm 4.4.1 satisfies the following:*

(a) If $\bar{x}_1 = J_{\tau P}(\bar{\xi} - \frac{\tau}{2}\sum_{i=1}^{m} T_i^* \bar{\zeta}_i + \tau w)$ *and*
$\bar{x}_{2,i} = J_{\sigma_i Q_i^{-1}}(\bar{\zeta}_i + \frac{\sigma_i}{2} T_i(2\bar{x}_1 - \bar{\xi}) - \sigma_i h_i)$, $i \in \mathcal{I}$,
then $(\bar{x}_1, \bar{x}_{2,1}, \cdots, \bar{x}_{2,m}) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ *solves Problem 4.4.1.*

(b) $\sum_{n=1}^{\infty} \|\xi_{n+1} - \xi_n\|^2 < \infty$ *and* $\sum_{n=1}^{\infty} \|\zeta_{i,n+1} - \zeta_{i,n}\|^2 < \infty$, $i \in \mathcal{I}$.

(c) $\{(\xi_n, \zeta_{1,n}, \cdots, \zeta_{m,n})\} \rightharpoonup (\bar{\xi}, \bar{\zeta}_1, \cdots, \bar{\zeta}_m)$.

(d) $\{(x_{1,n} - u_{1,n}, x_{2,1,n} - u_{2,1,n}, \cdots, x_{2,m,n} - u_{2,m,n})\} \to 0$.

(e) $\{(x_{1,n}, x_{2,1,n}, \cdots, x_{2,m,n})\} \rightharpoonup (\bar{x}_1, \bar{x}_{2,1}, \cdots, \bar{x}_{2,m})$.

(f) $\{(u_{1,n}, u_{2,1,n}, \cdots, u_{2,m,n})\} \rightharpoonup (\bar{x}_1, \bar{x}_{2,1}, \cdots, \bar{x}_{2,m})$.

(g) $\{(x_{1,n}, x_{2,1,n}, \cdots, x_{2,m,n})\}$ *and* $\{(u_{1,n}, u_{2,1,n}, \cdots, u_{2,m,n})\}$ *converge strongly to unique primal-dual solution* $(\bar{x}_1, \bar{x}_{2,1}, \cdots, \bar{x}_{2,m})$ *to Problem 4.4.1, when* P *and* Q_i^{-1} *are uniformly monotone, for* $i \in \mathcal{I}$.

Proof. The idea of the proof is inspired by [16]. From (4.21), we have zer$(\mathcal{M}+\mathcal{S}+\mathcal{Q})$ is nonempty. Define a linear continuous operator $V : \mathcal{X} \to \mathcal{X}$ by

$$(\xi, \zeta_1, \cdots, \zeta_m) \mapsto \left(\frac{\xi}{\tau} - \frac{1}{2}\sum_{i=1}^{m} T_i^* \zeta_i, \frac{\zeta_1}{\sigma_1} - \frac{1}{2}T_1\xi, \cdots, \frac{\zeta_m}{\sigma_m} - \frac{1}{2}T_m\xi\right).$$

Then V is selfadjoint and ρ-strongly (see [18]) positive for

$$\rho = \left(1 - \frac{1}{2}\sqrt{\tau \sum_{i=1}^{m} \sigma_i \|T_i\|^2}\right) \min\left\{\frac{1}{\tau}, \frac{1}{\sigma_1}, \cdots, \frac{1}{\sigma_m}\right\} > 0.$$

Observe that the following inequality holds:

$$\langle \xi, V\xi \rangle_{\mathcal{X}} \geq \rho \|\xi\|_{\mathcal{X}}^2, \ \forall \xi \in \mathcal{X}.$$

This ensures the existence of inverse operator, V^{-1} such that $\|V^{-1}\| \leq \frac{1}{\rho}$.

Consider the sequences

$$\begin{cases} \mathbf{a}_n = (\xi_n, \zeta_{1,n}, \dots, \zeta_{m,n}) \\ \mathbf{b}_n = (x_{1,n}, x_{2,1,n}, \dots, x_{2,m,n}) \\ \mathbf{c}_n = (u_{1,n}, u_{2,1,n}, \dots, u_{2,m,n}) \\ \mathbf{d}_n = (v_{1,n}, v_{2,1,n}, \dots, v_{2,m,n}) \\ \mathbf{p}_n = (s_{1,n}, s_{2,1,n}, \dots, s_{2,m,n}) \\ \mathbf{q}_n = (q_{1,n}, q_{2,1,n}, \dots, q_{2,m,n}). \end{cases}$$

Then Algorithm 4.4.1 reduces to

$$\begin{cases} V(\mathbf{a}_n - \mathbf{b}_n + \theta_n(\mathbf{a}_n - \mathbf{a}_{n-1})) = (\frac{1}{2}\mathcal{S} + \mathcal{M})\mathbf{b}_n \\ V(2\mathbf{b}_n - \mathbf{a}_n - \mathbf{c}_n - \theta_n(\mathbf{a}_n - \mathbf{a}_{n-1})) = (\frac{1}{2}\mathcal{S} + \mathcal{Q})\mathbf{c}_n \\ \mathbf{d}_n = \mathbf{a}_n + \theta_n(\mathbf{a}_n - \mathbf{a}_{n-1}) + \beta_n(\mathbf{c}_n - \mathbf{b}_n) \\ V(\mathbf{d}_n - \mathbf{p}_n) \in (\frac{1}{2}\mathcal{S} + \mathcal{M})\mathbf{p}_n \\ V(2\mathbf{p}_n - \mathbf{d}_n - \mathbf{q}_n) \in (\frac{1}{2}\mathcal{S} + \mathcal{Q})\mathbf{q}_n \\ \mathbf{a}_{n+1} = 2\mathbf{q}_n - \mathbf{p}_n, \ \forall n \in \mathbb{N}, \end{cases}$$

which is equivalent to

$$\begin{cases} \mathbf{b}_n = \big(Id + V^{-1}(\frac{1}{2}\mathcal{S} + \mathcal{M})\big)^{-1}(\mathbf{a}_n + \theta_n(\mathbf{a}_n - \mathbf{a}_{n-1})) \\ \mathbf{c}_n = (Id + V^{-1}(\frac{1}{2}\mathcal{S} + \mathcal{Q}))^{-1}(2\mathbf{b}_n - \mathbf{a}_n - \theta_n(\mathbf{a}_n - \mathbf{a}_{n-1})) \\ \mathbf{d}_n = \mathbf{a}_n + \theta_n(\mathbf{a}_n - \mathbf{a}_{n-1}) + \beta_n(\mathbf{c}_n - \mathbf{b}_n) \\ \mathbf{p}_n \in (Id + V^{-1}(\frac{1}{2}\mathcal{S} + \mathcal{M}))^{-1}\mathbf{d}_n \\ \mathbf{q}_n \in (Id + V^{-1}(\frac{1}{2}\mathcal{S} + \mathcal{Q}))^{-1}(2\mathbf{p}_n - \mathbf{d}_n) \\ \mathbf{a}_{n+1} = 2\mathbf{q}_n - \mathbf{p}_n, \ \forall n \in \mathbb{N}. \end{cases} \tag{4.22}$$

Consider the Hilbert space $\mathcal{X}_V = \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ equipped with inner product and norm $\langle \xi, \zeta \rangle_{\mathcal{X}_V} = \langle \xi, V\zeta \rangle_{\mathcal{X}}$ and $\|\xi\|_{\mathcal{X}_V} = \sqrt{\langle \xi, V\xi \rangle_{\mathcal{X}}}$, $\forall \xi, \zeta \in \mathcal{X}_V$, respectively. Define the operators $\mathbf{A} = V^{-1}F$ and $\mathbf{B} = V^{-1}G$. Then $\mathbf{A}$ and $\mathbf{B}$ are maximally monotone on $\mathcal{X}_V$ as F and G are maximally monotone on $\mathcal{X}$. Since V is selfadjoint and ρ-strongly positive, this implies that weak and strong convergence is equivalent in both the Hilbert spaces $\mathcal{X}$ and $\mathcal{X}_V$.

Using the definition of resolvent of $\mathbf{A}$ and $\mathbf{B}$ in (4.22), Algorithm 4.4.1 becomes,

$$\begin{cases} \mathbf{b}_n = J_{\mathbf{B}}(\mathbf{a}_n + \theta_n(\mathbf{a}_n - \mathbf{a}_{n-1})) \\ \mathbf{c}_n = J_{\mathbf{A}}(2\mathbf{b}_n - \mathbf{a}_n - \theta_n(\mathbf{a}_n - \mathbf{a}_{n-1})) \\ \mathbf{d}_n = \mathbf{a}_n + \theta_n(\mathbf{a}_n - \mathbf{a}_{n-1}) + \beta_n(\mathbf{c}_n - \mathbf{b}_n) \\ \mathbf{a}_{n+1} = (2J_{\mathbf{A}} - Id)(2J_{\mathbf{B}} - Id)\mathbf{d}_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.23}$$

which is in the form of InS-DRSM (4.3) in the space $\mathcal{X}_V$ for $\lambda = 1$. Note that $\text{zer}(\mathbf{A} + \mathbf{B}) = \text{zer}(V^{-1}(\mathcal{M} + \mathcal{S} + \mathcal{Q})) = \text{zer}(\mathcal{M} + \mathcal{S} + \mathcal{Q})$. Thus, (4.21) implies that $\text{zer}(\mathbf{A} + \mathbf{B})$ is non-empty.

(a) By applying Theorem 4.3.1(a) to Algorithm 4.4.1, we get a point $\bar{a}=(\bar{\xi}, \bar{\zeta}_1, \cdots, \bar{\zeta}_m) \in \text{Fix}(R_{\mathbf{A}} \circ R_{\mathbf{B}})$ satisfying $J_{\mathbf{B}}\bar{a} \in \text{zer}(\mathbf{A} + \mathbf{B}) = \text{zer}(\mathcal{M} + \mathcal{S} + \mathcal{Q})$. From Remark

4.4.2 (d), $J_{\mathbf{B}}\bar{a}$ solves inclusion Problem 4.4.1 and the claim follows by identifying $J_{\mathbf{B}}\bar{a}$.

(b) From Theorem 4.3.1(b), $\sum_{n=1}^{\infty} \|\mathbf{a}_{n+1} - \mathbf{a}_n\|^2_{\mathcal{X}_V} < \infty$. Since V is ρ-strongly positive,

$$\rho \sum_{n=1}^{\infty} \|\mathbf{a}_{n+1} - \mathbf{a}_n\|^2_{\mathcal{X}} \leq \sum_{n=1}^{\infty} \|\mathbf{a}_{n+1} - \mathbf{a}_n\|^2_{\mathcal{X}_V}.$$

Thus, $\sum_{n=1}^{\infty} \|\mathbf{a}_{n+1} - \mathbf{a}_n\|^2_{\mathcal{X}}$ is finite. Using the definition of $\|\cdot\|_{\mathcal{X}}$, $\sum_{n=1}^{\infty} \|\xi_{n+1} - \xi_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|\zeta_{i,n+1} - \zeta_{i,n}\|^2 < \infty$, $i \in \mathcal{I}$.

(c) From Theorem 4.3.1(c), the sequence $\{\mathbf{a}_n\} = \{(\xi_n, \zeta_{1,n}, \cdots, \zeta_{m,n})\}$ converges weakly to $\bar{a} = (\bar{\xi}, \bar{\zeta}_1, \ldots, \bar{\zeta}_m)$.

(d) From Theorem 4.3.1 (d), the sequence $\{\mathbf{b}_n - \mathbf{c}_n\} = \{(x_{1,n} - u_{1,n}, x_{2,1,n} - u_{2,1,n}, \cdots, x_{2,m,n} - u_{2,m,n})\}$ converges strongly to 0.

(e) From Theorem 4.3.1 (e), the sequence $\{\mathbf{b}_n\} = \{(x_{1,n}, x_{2,1,n}, \cdots, x_{2,m,n})\}$ converges weakly to $J_{\mathbf{B}}\bar{a}$ and the result follows by identifying $J_{\mathbf{B}}\bar{a}$.

(f) From (d) and (e), we have sequence $\{\mathbf{c}_n\}$ converges weakly to $(\bar{x}_1, \bar{x}_{2,1}, \cdots, \bar{x}_{2,m})$.

(g) Since P and Q_i^{-1} are uniformly monotone, $\mathcal{M}$ is uniformly monotone on $\mathcal{X}$ ([18, Theorem 2.1]). Since V is ρ-strongly positive, $\mathbf{B}$ is uniformly monotone on $\mathcal{X}$. From Theorem 4.3.1 (g), sequences $\{\mathbf{b}_n\} = \{(x_{1,n}, x_{2,1,n}, \cdots, x_{2,m,n})\}$ and $\{\mathbf{c}_n\}$ $= (u_{1,n}, u_{2,1,n}, \cdots, u_{2,m,n})$ converges strongly to unique $J_{\mathbf{B}}\bar{a}= (\bar{x}_1, \bar{x}_{2,1}, \cdots, \bar{x}_{2,m})$.

□

4.5 Applications to solve convex optimization problem

In this section, we aim to solve a highly structured convex optimization problem with the help of InS-PD Algorithm 4.4.1. We have also conducted a numerical experiment to solve the generalized Heron problem and compare the performance of InS-PD Algorithm 4.4.1 with already known algorithms. Let m be a strictly positive integer and $\mathcal{I}$ denote the set $\{1, 2, \ldots, m\}$. The optimization problem we consider is as follows:

Problem 4.5.1. *Consider $f \in \Gamma(\mathcal{H})$ and $g_i, l_i \in \Gamma(\Omega_i)$, where $\Omega_i, i \in \mathcal{I}$ are real Hilbert spaces. Let $T_i : \mathcal{H} \to \Omega_i$ be nonzero bounded linear operators and $h_i \in \Omega_i$, for each $i \in \mathcal{I}$. For $w \in \mathcal{H}$, we define the convex optimization problem.*

$$\inf_{\xi \in \mathcal{H}} \left\{ f(\xi) + \sum_{i=1}^{m} (g_i \Box l_i)(T_i \xi - h_i) - \langle \xi, w \rangle \right\} \tag{4.24}$$

and its conjugate dual problem

$$\sup_{(\zeta_1, \cdots, \zeta_m) \in \Omega_1 \times \cdots \times \Omega_m} \left\{ -f^* \left(w - \sum_{i=1}^{m} T_i^* \zeta_i \right) - \sum_{i=1}^{m} \left(g_i^*(\zeta_i) + l_i^*(\zeta_i) + \langle \zeta_i, h_i \rangle \right) \right\}. \tag{4.25}$$

Suppose $P = \partial f$, $Q_i = \partial g_i$ and $R_i = \partial l_i$, $\forall i \in \mathcal{I}$ in Problem 4.4.1, then primal inclusion problem (4.19) is to find $\bar{\xi} \in \mathcal{H}$ such that

$$w \in \partial f(\bar{\xi}) + \sum_{i=1}^{m} T_i^* (\partial g_i \Box \partial l_i)(T_i \bar{\xi} - h_i) \tag{4.26}$$

and corresponding dual inclusion problem (4.20) becomes find $(\bar{\zeta}_1, \cdots, \bar{\zeta}_m) \in \Omega_1 \times \cdots \times \Omega_m$ such that $(\exists \xi \in \mathcal{H})$ satisfying

$$\begin{cases} w - \sum_{i=1}^{m} T_i^* \bar{\zeta}_i \in \partial f(\xi) \\ \bar{\zeta}_i \in (\partial g_i \Box \partial l_i)(T_i \xi - h_i), \ \forall i \in \mathcal{I}. \end{cases} \tag{4.27}$$

A point $\bar{x} \in \mathcal{H}$ solves the primal inclusion problem (4.26) and $(\bar{\zeta}_1, \cdots, \bar{\zeta}_m) \in \Omega_1 \times \cdots \times \Omega_m$ solves its dual problem (4.27) then $\bar{x}$ optimizes the primal optimization problem (4.24) and $(\bar{\zeta}_1, \cdots, \bar{\zeta}_m)$ optimizes dual optimization problem (4.25), i.e., $(\bar{\xi}, \bar{\zeta}_1, \cdots, \bar{\zeta}_m)$ is the solution of primal-dual optimization Problem 4.5.1 .

We apply Algorithm 4.4.1 to solve Problem 4.5.1. For the choice of $P = \partial f$, $Q_i = \partial g_i$ and $R_i = \partial l_i$, $\forall i \in \mathcal{I}$, Algorithm 4.4.1 can be reformulated as the following algorithm:

Algorithm 4.5.1:

Input:

1. Initial points $(\xi_0, \zeta_{1,0}, \dots, \zeta_{m,0}), (\xi_1, \zeta_{1,1}, \dots, \zeta_{m,1}) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$.

2. $0 \leq \tau, \sigma_i \in \mathbb{R}$, $i \in \mathcal{I}$, such that $\tau \sum_{i=1}^m \sigma_i \|T_i\|^2 < 4$.

Procedure:

$x_{1,n} = prox_{\tau f}(\xi_n + \theta_n(\xi_n - \xi_{n-1}) - \frac{\tau}{2} \sum_{i=1}^m T_i^*(\zeta_{i,n} + \theta_n(\zeta_{i,n} - \zeta_{i,n-1})) + \tau w)$

$y_{1,n} = 2x_{1,n} - \xi_{1,n} - \theta_n(\xi_n - \xi_{n-1})$

For $i = 1, \dots, m;$

$x_{2,i,n} = prox_{\sigma_i g_i^*}(\zeta_{i,n} + \theta_n(\zeta_{i,n} - \zeta_{i,n-1}) + \frac{\sigma_i}{2} T_i y_{1,n} - \sigma_i h_i)$

$y_{2,i,n} = 2x_{2,i,n} - \zeta_{i,n} - \theta_n(\zeta_{i,n} - \zeta_{i,n-1})$

$u_{1,n} = y_{1,n} - \frac{\tau}{2} \sum_{i=1}^m T_i^* y_{2,i,n}$

$v_{1,n} = \xi_n + \theta_n(\xi_n - \xi_{n-1}) + \beta_n(u_{1,n} - x_{1,n})$

For $i = 1, \dots, m;$

$u_{2,i,n} = prox_{\sigma_i l_i^*}(y_{2,i,n} + \frac{\sigma_i}{2} T_i(2u_{1,n} - y_{1,n}))$

$v_{2,i,n} = \zeta_{i,n} + \theta_n(\zeta_{i,n} - \zeta_{i,n-1}) + \beta_n(u_{2,i,n} - x_{2,i,n})$

$s_{1,n} = prox_{\tau f}(v_{1,n} - \frac{\tau}{2} \sum_{i=1}^m T_i^*(v_{2,i,n}) + \tau w)$

$t_{1,n} = 2s_{1,n} - v_{1,n}$

For $i = 1, \dots, m;$

$s_{2,i,n} = prox_{\sigma_i g_i^*}(v_{2,i,n} + \frac{\sigma_i}{2} T_i t_{1,n} - \sigma_i h_i))$

$t_{2,i,n} = 2s_{2,i,n} - v_{2,i,n}$

$\xi_{n+1} = 2\{t_{1,n} - \frac{\tau}{2} \sum_{i=1}^m T_i^*(t_{2,i,n})\} - t_{1,n}$

For $i = 1, \dots, m;$

$\zeta_{i,n+1} = 2prox_{\sigma_i l_i^*}(t_{2,i,n} + \frac{\sigma_i}{2} T_i(\xi_{n+1})) - t_{2,i,n}$

Output: $(\xi_{n+1}, \zeta_{1,n+1}, \cdots, \zeta_{m,n+1})$

We establish the convergence behavior of Algorithm 4.5.1.

Theorem 4.5.1. *In Problem 4.5.1, consider*

$$w \in ran \left(\partial f + \sum_{i=1}^m T_i^*(\partial g_i \Box \partial l_i)(T_i(\cdot) - h_i) \right)$$

and $\{(\xi_n, \zeta_{1,n}, \ldots, \zeta_{m,n})\}$ *be the sequence generated by Algorithm 4.5.1. Let sequences* $\{\theta_n\}, \{\beta_n\}$ *satisfy the assumption as in Theorem 4.4.1, Then there exists* $\bar{\boldsymbol{a}} = (\bar{\xi}, \bar{\zeta}_1, \ldots, \bar{\zeta}_m) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ *such that following holds:*

(a) $(\bar{x}_1, \bar{x}_{2,1}, \cdots, \bar{x}_{2,m}) \in \mathcal{H} \times \Omega_1 \times \cdots \times \Omega_m$ *solves Problem 4.5.1, where*

$$\bar{x}_1 = prox_{\tau f}\left(\bar{\xi} - \frac{\tau}{2}\sum_{i=1}^{m} T_i^* \bar{\zeta}_i + \tau w\right) \text{ and}$$
$$\bar{x}_{2,i} = prox_{\sigma_i g_i^*}\left(\bar{\zeta}_i + \frac{\sigma_i}{2} T_i(2\bar{x}_1 - \bar{\xi}) - \sigma_i h_i\right), i \in \mathcal{I}.$$

(b) $\sum_{n=1}^{\infty} \|\xi_{n+1} - \xi_n\|^2 < \infty$ *and* $\sum_{n=1}^{\infty} \|\zeta_{i,n+1} - \zeta_{i,n}\|^2 < \infty$*,* $i \in \mathcal{I}$*.*

(c) $\{(\xi_n, \zeta_{1,n}, \cdots, \zeta_{m,n})\} \rightharpoonup (\bar{\xi}, \bar{\zeta}_1, \cdots, \bar{\zeta}_m)$*.*

(d) $\{(x_{1,n} - u_{1,n}, x_{2,1,n} - u_{2,1,n}, \cdots, x_{2,m,n} - u_{2,m,n})\} \to 0$ *as* $n \to \infty$*.*

(e) $\{(x_{1,n}, x_{2,1,n}, \cdots, x_{2,m,n})\} \rightharpoonup (\bar{x}_1, \bar{x}_{2,1}, \cdots, \bar{x}_{2,m})$*.*

(f) $\{(u_{1,n}, u_{2,1,n}, \cdots, u_{2,m,n})\} \rightharpoonup (\bar{x}_1, \bar{x}_{2,1}, \cdots, \bar{x}_{2,m})$*.*

(g) $\{(x_{1,n}, x_{2,1,n}, \cdots, x_{2,m,n})\}$ *and* $\{(u_{1,n}, u_{2,1,n}, \cdots, u_{2,m,n})\}$ *converge strongly to a unique solution* $(\bar{x}_1, \bar{x}_{2,1}, \cdots, \bar{x}_{2,m})$ *of Problem 4.5.1, when* f *and* g_i^* *are uniformly convex,* $i \in \mathcal{I}$*.*

Numerical Experiment 4.5.1. In this section, we perform a numerical experiment to solve the generalized Heron problem using Algorithm 4.5.1. In the classical Heron problem, we search for a point on a given straight line such that the sum of its distances to two given points is minimum. The generalized Heron problem was examined in [69, 70] by replacing the points and straight line with non-empty closed convex sets. Thus, in the generalized Heron problem, we search a point on a given nonempty closed convex set $\Omega \subset \mathbb{R}^n$ such that sum of its distances to given non-empty closed convex sets $\Omega_i \subset \mathbb{R}^n$ is minimal, $i \in \mathcal{I}$. Mathematically, generalized

Heron problem can be written as

$$\inf_{x\in\Omega}\sum_{i=1}^{m} d(x,\Omega_i),$$

where the distance of a point x to a non-empty set Ω is defined as $\inf\{\|x-y\| : y \in \Omega\}$. The generalized Heron problem can be reduced into the framework of optimization Problem 4.5.1 by setting $f = \delta_\Omega$, $g_i = \|\cdot\|$ and $l_i = \delta_{\Omega_i}, i \in \mathcal{I}$.

For numerical experiment in $\mathbb{R}^2$, we take a ball of radius 1 centered at $(-2, 4)$ as Ω and choose constraints from balls having radius 1 with centers $C_1 = (-10, 0)$, $C_2 = (-1, 8)$, $C_3 = (2, -4)$, $C_4 = (7, 6)$, $C_5 = (7, 1)$, and $C_6 = (8, -3)$ (Fig 4.4). For numerical experiment in $\mathbb{R}^3$, we take ball with radii 1 and center $(0, 2, 0)$ as Ω and choose constraints from balls having radii 1 with centers $C_1 = (0, -4, 0)$, $C_2 = (-4, 2, -3)$, $C_3 = (-3, -4, 2)$, $C_4 = (-5, 4, 4)$, and $C_5 = (-1, 8, 1)$ (Fig 4.5).

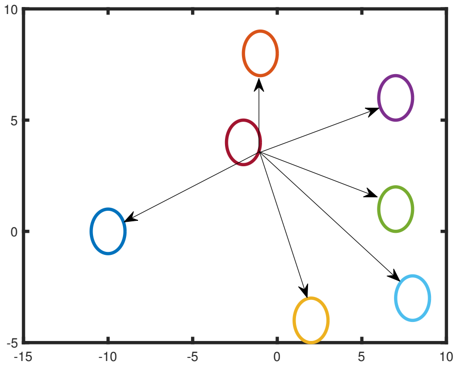

FIGURE 4.4: Circle with circle constraints.

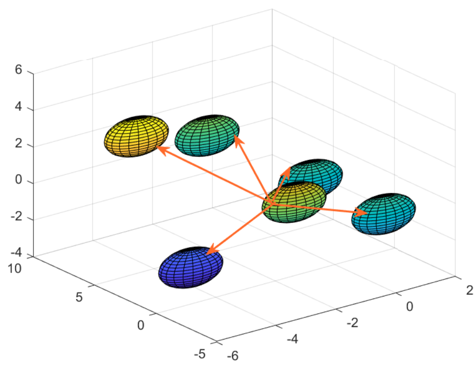

FIGURE 4.5: Sphere with sphere constraints.

FIGURE 4.6: Generalized Heron problem for different convex set and contraints.

We solve the generalized Heron problem using Douglas-Rachford algorithm ([18, Algorithm 3.1]), inertial Douglas-Rachford algorithm ([16, Algorithm 15]), and proposed Algorithm 4.5.1 and we compare on the basis of the number of iterations required to achieve the root mean square error (RMSE) less than 0.001 and 0.00001. The performance of algorithms is depicted in Table 4.1. We have also plotted the graph between the number of iterations and RMSE to have a clear visualization of the experiment (Fig 4.15). In $\mathbb{R}^2$, we initialize with points $x_0 = x_1 = (-1, 4)$ and with $x_0 = x_1 = (0, 2, 0)$ in $\mathbb{R}^3$. We set $\theta_n = \frac{n-1}{14n+2.5}$ which is similar to the inertial parameter in [29], $\beta_n = 0.5 + \frac{1}{200n}$, $\sigma_i = 0.15$ and $\tau = 5/3$.

	RMSE $<$ 0.001			RMSE $<$ 0.00001		
Dimension	Algo 4.5.1	[16, Algo 15]	[18, Algo 3.1]	Algo 4.5.1	[16, Algo 15]	[18, Algo 3.1]
m=3,n=2	**11**	28	30	**24**	38	41
m=5,n=2	**12**	26	28	**29**	47	51
m=6,n=2	**21**	28	30	**32**	48	52
m=3,n=3	**16**	21	23	**26**	40	43
m=5,n=3	**12**	26	28	**19**	47	50

TABLE 4.1: Number of iterations required to have different accuracy for different algorithms. The best results are presented in bold letters.

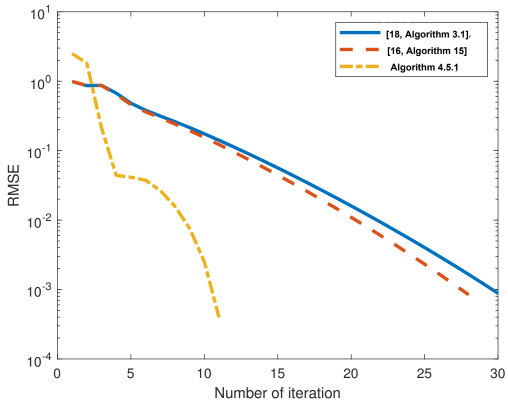

FIGURE 4.7: $m = 3, n = 2$.

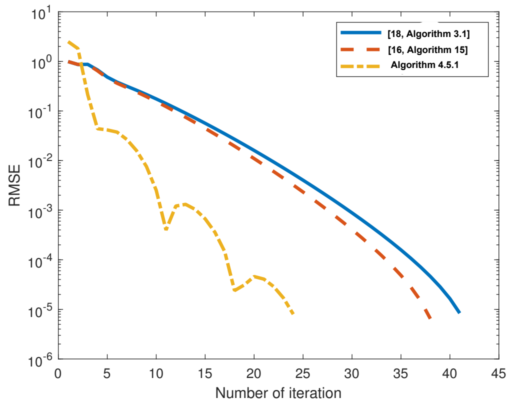

FIGURE 4.8: $m = 3, n = 2$

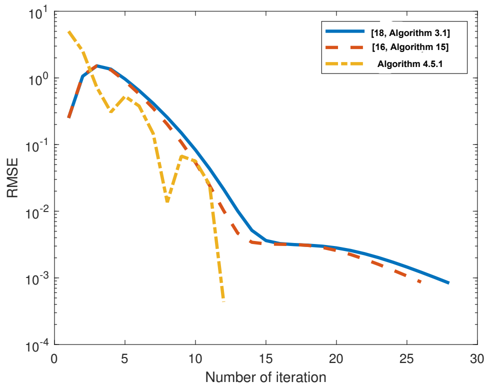

FIGURE 4.9: $m = 5, n = 2$

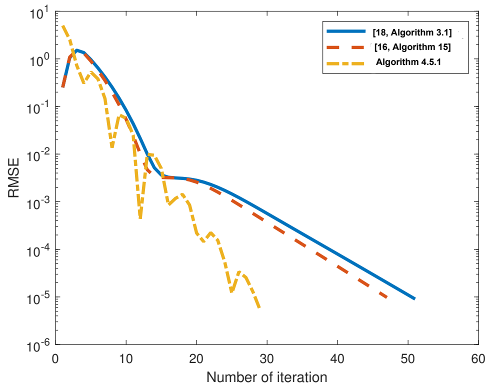

FIGURE 4.10: $m = 5, n = 2$

From Table 4.1, we can observe that proposed Algorithm 4.5.1 takes the least number of iterations to achieve the RMSE less than 0.001 as well as 0.00001 while inertial Douglas-Rachford algorithm ([16, Algorithm 15]) remains the second fastest algorithm in terms of the number of iterations to achieve the RMSE less than 0.001 as well as 0.00001. Number of iterations taken by inertial Douglas-Rachford algorithm ([16, Algorithm 15]) and Douglas-Rachford algorithm ([18, Algorithm 15]) are very close.

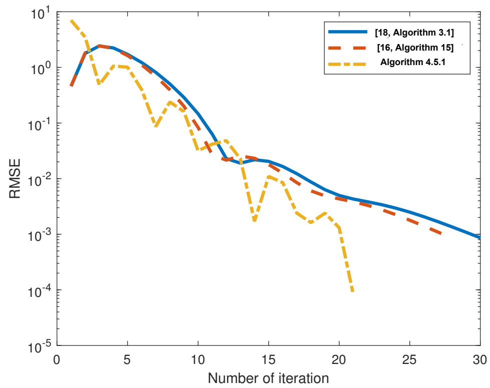

FIGURE 4.10: $m = 6, n = 2$

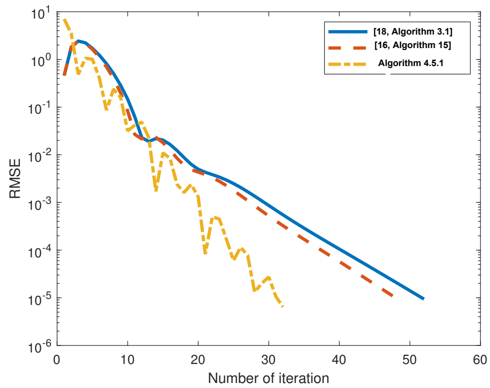

FIGURE 4.11: $m = 6, n = 2$

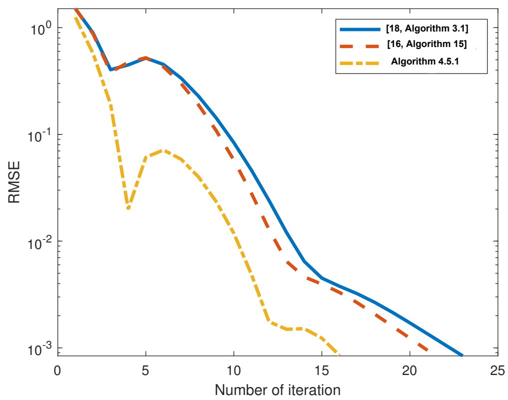

FIGURE 4.12: $m = 3, n = 3$

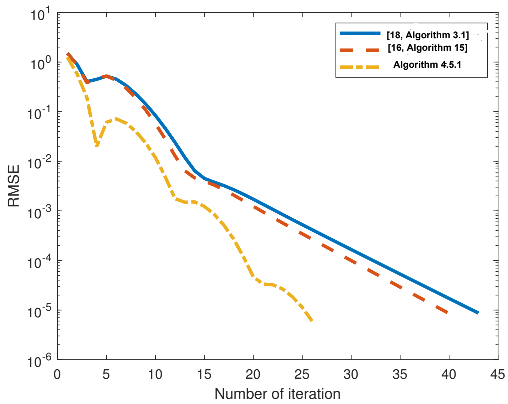

FIGURE 4.13: $m = 3, n = 3$

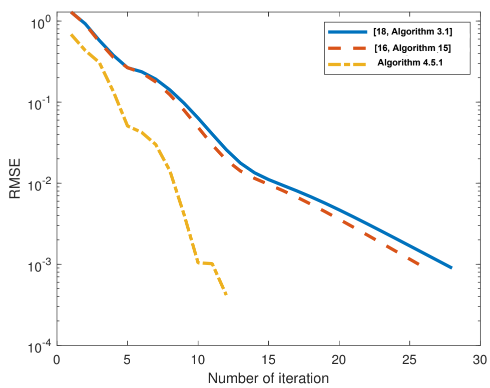

FIGURE 4.13: $m = 5, n = 3$

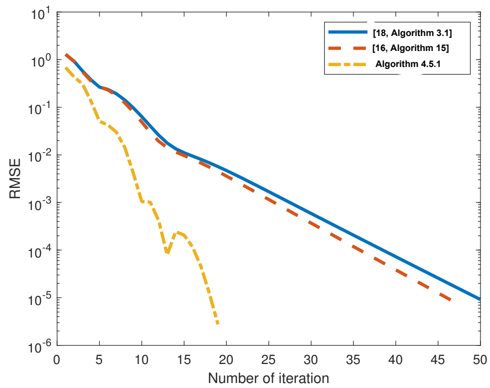

FIGURE 4.14: $m = 5, n = 3$

FIGURE 4.15: The semilog graph between number of iterations and RMSE for different choices of m and n as in Table 4.1. Figure 4.7, 4.9, 4.11, 4.13 are plotted for RMSE < 0.001 and Figure 4.8, 4.10, 4.12, 4.14 are plotted for RMSE < 0.00001.

4.6 Conclusion

In this chapter, we have introduced an InS-DRSM (4.3) to solve the inclusion problem of the sum of set-valued operators. The sequence generated by Algorithm 4.3.1 converges weakly to the solution set of inclusion problem. We have presented an example in support of Theorem 4.3.1. We have also introduced normal-S based inertial primal-dual (InS-PD) algorithm to solve a highly structured monotone inclusion problem having linearly composed and parallel-sum type operators and studied its convergence behavior. Further, we have applied InS-PD algorithm to solve a highly structured minimization problem. Numerical experiment shows that proposed algorithm takes fewer iterations than inertial Douglas-Rachford algorithm [16, Algorithm 15] and Douglas-Rachford algorithm [18, Algorithm 3.1]. In future work, we will study the theoretical convergence rate analysis of Algorithm 4.3.1 and evaluate the performance on some large scale real datasets.

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