To

The greatest gift I ever got from God;

My Parents

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It is certified that the work contained in this thesis titled "***Accelerated Iterative Techniques to Solve Inclusion Problems and Applications***" by ***Avinash Dixit*** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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DECLARATION BY THE CANDIDATE

I, ***Avinash Dixit***, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of ***Dr. Tanmoy Som*** from ***July, 2016*** to ***May, 2021*** at the ***Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi***. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports, dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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Avinash Dixit

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Abbreviations

PPA	Proximal Point Algorithm
FISTA	Fast Iterative Shrinkage Thresholding Algorithm
DRA	Douglas-Rachford Algorithm
MPG	Mann Proximal Gradient
IMPG	Inertial Mann Proximal Gradient
NSPG	Normal-S Proximal Gradient
INSPG	Inertial Normal-S Proximal Gradient
KKT	Karush-Kuhn-Tucker
a.e.	almost everywhere
AML	Acute Myeloid Leukemia
ALL	Acute Lymphoblastic Leukemia
RMSE	Root Mean Square Error
CN	Common Neighbors
AA	Adamic/Adar
RA	Resource Allocation
PA	Preferential Attachment

Symbols

$\mathbb{N}$	The set of Natural numbers
$\mathbb{R}$	The set of Real numbers
$\mathbb{R}_\infty$	$\mathbb{R} \cup \{\infty\}$
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
Id	Identity operator
$2^{\mathcal{X}}$	Power set of a set $\mathcal{X}$
$\mathcal{H}$	Real Hilbert space
$\langle \cdot \mid \cdot \rangle$	Scalar product
$\| \cdot \|$	Norm
d	distance
$\rightarrow$	Strong cpnvergence in Hilbert spaces
$\rightharpoonup$	Weak convergence in Hilbert spaces
$dom(A)$	Domain of an operator A
$ran(A)$	Range of an operator A
$int\ C$	Interior of a set C
$sri\ C$	Strong relative interior of a set C
$sqri\ C$	Strong quasi relative interior of a set C
i_c	Indicator function of a set C
P_C	Projector onto a nonempty closed convex set C
N_C	Normal cone operator of a set C

∇f	Gradient operator of a function f
∂T	Subdifferential of a function T
A^t	Transpose of an operator A
M^*	Adjoint of a bounded linear operator M
f^*	Conjugate of a function f
$Gr(T)$	Graph of an operator T
$\text{zer}(T)$	Set of zeros of an operator T
$\text{Fix}(T)$	Set of fixed points of an operator T
$\Gamma(\mathcal{H})$	Set of all lower semicontinuous convex functions from $\mathcal{H}$ to $[-\infty, +\infty]$
$\Gamma_0(\mathcal{H})$	Set of all proper lower semicontinuous convex functions from $\mathcal{H}$ to $(-\infty, +\infty]$
$prox_f$	proximity operator of a function f
J_A	Resolvent of operator A
R_A	Reflected resolvent of operator A
$T_1 \Box T_2$	Parallel sum of operators T_1 and T_2

PREFACE

The main goal of the work contained in the thesis is to present different methods to solve monotone inclusion problems investigated during the last few years. The basic approach to achieve the goal is to split the monotone operator into a sum of two monotone operators. The other pertinent and massive goal is to apply the proposed methods to solve real-world problems. For real-world application purposes, we are mainly concerned with computer engineering related problems. Different methods are proposed to solve monotone inclusion problems using direct as well as iterative methods. Iterative techniques have the advantage over direct methods in that they can be used to solve even the high-dimensional cases. Keeping the point in mind the author is inclined to develop iterative methods to solve the monotone inclusion problem.

Chapter 1 of the thesis is introductory. It explains the main background of the monotone inclusion problem and the different previous approaches to solve the problem. It also gives the idea about the structure of the thesis.

In Chapter 2 of the thesis, we propose a fixed point algorithm to find the fixed point of a nonexpansive operator. The fixed point algorithms are not just limited to solve the fixed point problems, these fixed point algorithms are also used to solve inclusion problems by framing the monotone inclusion problem as an equivalent fixed point problem. We use the inertial term to define the algorithm, which is motivated by the Heavy ball method proposed by Polyak. We use the proposed fixed point algorithm to solve the regression problems. We conducted numerical experiments to solve high-dimensional regression problems. We compare the performance of the proposed method to already known methods on the basis of convergence speed and accuracy.

In Chapter 3, we propose a preconditioning based inertial forward-backward algorithm and focus to solve the inclusion problem of the sum of two monotone operators. We study the convergence behavior of proposed algorithm under mild assumptions. We also propose an iterative method to solve the saddle point problem. Further, we apply the proposed methods to solve the regression and link prediction problem. A comparative study is also done for the proposed algorithm and some well-known methods to solve regression and link prediction problems.

The Chapter 4 of the thesis addresses the inclusion problem of the sum of two set-valued operators. We propose a novel two-step inertial Douglas-Rachford algorithm to solve the monotone inclusion problem of the sum of two maximally monotone operators based on the normal S-iteration method [82]. Further, we study the convergence behavior of the proposed algorithm. Based on the proposed method, we develop an inertial primal-dual algorithm to solve highly structured monotone inclusions containing the mixture of linearly composed and parallel-sum type operators. Finally, we apply the proposed inertial primal-dual algorithm to solve a highly structured minimization problem. We also perform a numerical experiment to solve the generalized Heron problem and compare the performance of the proposed inertial primal-dual algorithm with the performance of already known algorithms.

We aim to propose strongly convergent methods in Chapter 5 without assuming strong convexity or strong monotonicity. First, we propose a fixed point algorithm to find the common fixed point of nonexpansive operators. Based on proposed fixed point algorithm, we develop a new forward-backward algorithm and a Douglas-Rachford algorithm in connection with Tikhonov regularization to find the solution of splitting monotone inclusion problems. Further, we consider the complexly structured monotone inclusion problems which are quite popular these days. We also propose a strongly convergent forward-backward type primal-dual algorithm and a

Douglas-Rachford type primal-dual algorithm such that they solve the monotone inclusion problems containing the mixture of linearly composed and parallel-sum type operators. Finally, we conduct a numerical experiment to solve image deblurring problems.