

*To*

*The greatest gift I ever got from God;*


***My Parents***



CERTIFICATE

It is certified that the work contained in this thesis titled “*Accelerated Iterative Techniques to Solve Inclusion Problems and Applications*” by *Avinash Dixit* has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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31/5/21

Dr. Tanmoy Som  
(Supervisor)  
Professor  
Department of Mathematical Sciences  
Indian Institute of Technology  
(Banaras Hindu University)  
Varanasi-221005

---

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(Prof. Tanmoy Som)  
Professor  
Department of Mathematical Sciences  
Indian Institute of Technology  
(Banaras Hindu University)  
Varanasi-221005



(Prof. Tanmoy Som)  
Professor and Head  
Department of Mathematical Sciences  
Indian Institute of Technology  
(Banaras Hindu University)  
Varanasi-221005

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**Avinash Dixit**



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# Abbreviations

|              |   |
|--------------|---|
| <b>PPA</b>   | Proximal Point Algorithm                        |
| <b>FISTA</b> | Fast Iterative Shrinkage Thresholding Algorithm |
| <b>DRA</b>   | Douglas-Rachford Algorithm                      |
| <b>MPG</b>   | Mann Proximal Gradient                          |
| <b>IMPG</b>  | Inertial Mann Proximal Gradient                 |
| <b>NSPG</b>  | Normal-S Proximal Gradient                      |
| <b>INSPG</b> | Inertial Normal-S Proximal Gradient             |
| <b>KKT</b>   | Karush-Kuhn-Tucker                              |
| <b>a.e.</b>  | almost everywhere                               |
| <b>AML</b>   | Acute Myeloid Leukemia                          |
| <b>ALL</b>   | Acute Lymphoblastic Leukemia                    |
| <b>RMSE</b>  | Root Mean Square Error                          |
| <b>CN</b>    | Common Neighbors                                |
| <b>AA</b>    | Adamic/Adar                                     |
| <b>RA</b>    | Resource Allocation                             |
| <b>PA</b>    | Preferential Attachment                         |





# Symbols

|                                 |   |
|---------------------------------|---|
| $\mathbb{N}$                    | The set of Natural numbers                      |
| $\mathbb{R}$                    | The set of Real numbers                         |
| $\mathbb{R}_\infty$             | $\mathbb{R} \cup \{\infty\}$                    |
| $\mathbb{N}_0$                  | $\mathbb{N} \cup \{0\}$                         |
| $Id$                            | Identity operator                               |
| $2^{\mathcal{X}}$               | Power set of a set $\mathcal{X}$                |
| $\mathcal{H}$                   | Real Hilbert space                              |
| $\langle \cdot   \cdot \rangle$ | Scalar product                                  |
| $\  \cdot \ $                   | Norm  |
| $d$                             | distance  |
| $\rightarrow$                   | Strong convergence in Hilbert spaces            |
| $\rightharpoonup$               | Weak convergence in Hilbert spaces              |
| $dom(A)$                        | Domain of an operator $A$                       |
| $ran(A)$                        | Range of an operator $A$                        |
| $int C$                         | Interior of a set $C$                           |
| $sri C$                         | Strong relative interior of a set $C$           |
| $sqri C$                        | Strong quasi relative interior of a set $C$     |
| $i_c$                           | Indicator function of a set $C$                 |
| $P_C$                           | Projector onto a nonempty closed convex set $C$ |
| $N_C$                           | Normal cone operator of a set $C$               |

## *Symbols*

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|                         |  |
|-------------------------|--|
| $\nabla f$              | Gradient operator of a function $f$  |
| $\partial T$            | Subdifferential of a function $T$  |
| $A^t$                   | Transpose of an operator $A$   |
| $M^*$                   | Adjoint of a bounded linear operator $M$   |
| $f^*$                   | Conjugate of a function $f$  |
| $Gr(T)$                 | Graph of an operator $T$   |
| $zer(T)$                | Set of zeros of an operator $T$  |
| $Fix(T)$                | Set of fixed points of an operator $T$   |
| $\Gamma(\mathcal{H})$   | Set of all lower semicontinuous convex functions from $\mathcal{H}$ to $[-\infty, +\infty]$        |
| $\Gamma_0(\mathcal{H})$ | Set of all proper lower semicontinuous convex functions from $\mathcal{H}$ to $(-\infty, +\infty]$ |
| $prox_f$                | proximity operator of a function $f$   |
| $J_A$                   | Resolvent of operator $A$  |
| $R_A$                   | Reflected resolvent of operator $A$  |
| $T_1 \square T_2$       | Parallel sum of operators $T_1$ and $T_2$  |

## PREFACE

---

The main goal of the work contained in the thesis is to present different methods to solve monotone inclusion problems investigated during the last few years. The basic approach to achieve the goal is to split the monotone operator into a sum of two monotone operators. The other pertinent and massive goal is to apply the proposed methods to solve real-world problems. For real-world application purposes, we are mainly concerned with computer engineering related problems. Different methods are proposed to solve monotone inclusion problems using direct as well as iterative methods. Iterative techniques have the advantage over direct methods in that they can be used to solve even the high-dimensional cases. Keeping the point in mind the author is inclined to develop iterative methods to solve the monotone inclusion problem.

Chapter 1 of the thesis is introductory. It explains the main background of the monotone inclusion problem and the different previous approaches to solve the problem. It also gives the idea about the structure of the thesis.

In Chapter 2 of the thesis, we propose a fixed point algorithm to find the fixed point of a nonexpansive operator. The fixed point algorithms are not just limited to solve the fixed point problems, these fixed point algorithms are also used to solve inclusion problems by framing the monotone inclusion problem as an equivalent fixed point problem. We use the inertial term to define the algorithm, which is motivated by the Heavy ball method proposed by Polyak. We use the proposed fixed point algorithm to solve the regression problems. We conducted numerical experiments to solve high-dimensional regression problems. We compare the performance of the proposed method to already known methods on the basis of convergence speed and accuracy.

In Chapter 3, we propose a preconditioning based inertial forward-backward algorithm and focus to solve the inclusion problem of the sum of two monotone operators. We study the convergence behavior of proposed algorithm under mild assumptions. We also propose an iterative method to solve the saddle point problem. Further, we apply the proposed methods to solve the regression and link prediction problem. A comparative study is also done for the proposed algorithm and some well-known methods to solve regression and link prediction problems.

The Chapter 4 of the thesis addresses the inclusion problem of the sum of two set-valued operators. We propose a novel two-step inertial Douglas-Rachford algorithm to solve the monotone inclusion problem of the sum of two maximally monotone operators based on the normal S-iteration method [82]. Further, we study the convergence behavior of the proposed algorithm. Based on the proposed method, we develop an inertial primal-dual algorithm to solve highly structured monotone inclusions containing the mixture of linearly composed and parallel-sum type operators. Finally, we apply the proposed inertial primal-dual algorithm to solve a highly structured minimization problem. We also perform a numerical experiment to solve the generalized Heron problem and compare the performance of the proposed inertial primal-dual algorithm with the performance of already known algorithms.

We aim to propose strongly convergent methods in Chapter 5 without assuming strong convexity or strong monotonicity. First, we propose a fixed point algorithm to find the common fixed point of nonexpansive operators. Based on proposed fixed point algorithm, we develop a new forward-backward algorithm and a Douglas-Rachford algorithm in connection with Tikhonov regularization to find the solution of splitting monotone inclusion problems. Further, we consider the complexly structured monotone inclusion problems which are quite popular these days. We also propose a strongly convergent forward-backward type primal-dual algorithm and a

Douglas-Rachford type primal-dual algorithm such that they solve the monotone inclusion problems containing the mixture of linearly composed and parallel-sum type operators. Finally, we conduct a numerical experiment to solve image deblurring problems.