Chapter 4

Forward-backward-half forward dynamical systems for monotone inclusion problems with application to v-GNE

4.1 Introduction

The variable metric forward-backward (FB) algorithms with a symmetric positive definite operator M have been studied in [68, 69]. Moreover, Raguet et al.[70] have reviewed generalized FB algorithm using variable metric operator M and have assumed that operator M is strongly positive. Bot et al. [78] have studied primal-dual dynamical approach to solve convex minimization problems in a variable metric sense. The variable metric considered by Bot et al. [78] is a continuous linear operator $M : H \to H$, which is self-adjoint and positive semidefinite.

Briceno-Arias et al. [79] have studied forward-backward-half forward (FBHF) splitting technique to solve the monotone inclusion problems consisting of three operators of the form

$$\text{find } x \in C : 0 \in Ax + B_1x + B_2x, \tag{4.1}$$

where $C(\neq \emptyset)$ is a closed convex subset of H, the operators $A : H \to 2^H$ is maximally monotone, B_1 is β-cocoercive for $\beta > 0$,and $B_2 : H \to 2^H$ is maximally monotone such that B_2 is single-valued in $\mathcal{D}(B_2) \supset C \cup \mathcal{D}(A)$. Furthermore, suppose that B_2 is continuous on $C \cup \mathcal{D}(A)$ and $A + B_2$ is maximally monotone.

It is observed that the FBF splitting methods proposed by Tseng [20] can also solve the problem (4.1), by assuming that B_2 is L-Lipschitz continuous, as $B := B_1 + B_2$ is monotone, and $(\beta^{-1} + L)$-Lipschitz continuous. In the FBHF splitting approach, the authors [79] have avoided twice computation of B_1 by iteration and have escaped computation of B_1 in the line search to obtain adequately small step-size, which was numerically costly; however, these are considered in the FBF splitting method [20].

In this work, we have studied forward-backward-half forward dynamical system for monotone inclusion problem of the following form:

$$\text{find } x \in H : 0 \in Ax + B_1x + B_2x, \tag{4.2}$$

where H is a real Hilbert space, $A : H \to 2^H$ is maximally monotone operator, $B_1 : H \to H$ is β-cocoercive for $\beta > 0$, and $B_2 : H \to H$ is monotone and L-Lipschitz continuous and $\mathcal{D}(B_2) = H$. The FB dynamical system finds the problem (1.3) when one operator is maximally monotone; another one is cocoercive. In comparison, FBF dynamical system solves the problem (1.3) when one operator is maximally monotone; the other one is monotone, and Lipschitz continuous, these are the particular case of the problem (4.2) when $B_2 = 0$, and $B_1 = 0$, respectively.

The problem (4.1) is widely applicable to optimization problems involving variational inequalities, image processing, PDEs, saddle point problems [11, 80, 81], and the references therein.

The remaining parts of this chapter are organized as follows: some lemmas and definitions required to prove the main results are presented in section 4.2. In section 4.3, the existence and uniqueness of trajectories of the dynamical system (4.3) are proved by the use of classical Cauchy-Lipschitz-Picard Theorem. In section 4.3.2, we study the weak convergence of the trajectories generated by the dynamical system (4.3) with the help of a continuous version of the Opial lemma. In section 4.4, we have studied variable metric FBHF dynamical system. We take non-self-adjoint linear operators to compute resolvents and other operators involved. By using Lyapunov analysis and a continuous variant of the Opial lemma for the class of operators $\mathfrak{T}$, one can prove the convergence of generated trajectories. In section 4.5, we have given a numerical example to illustrate the convergence of trajectories. In section 4.6, we have applied the proposed dynamical system for solving the Nash equilibrium problem.

4.2 Preliminaries

Definition 4.2.1. [82] $\mathfrak{T} := \{T : H \to H : \mathcal{D}(T) = H \text{ and } \mathcal{F}(T) \subset h(x, Tx)\ \forall x \in H\}$, where $h(x, y) = \{u \in H : \langle u - y, x - y\rangle \leq 0\}$, for given $x, y \in H$.

Lemma 4.2.1. [83] Consider an operator $T : H \to H$ such that $T \in \mathfrak{T}$. Then we have the following:

(i) $\|Tx - x\|^2 \leq \langle y - x, Tx - x\rangle\ \forall (x, y) \in H \times \mathcal{F}(T)$.

(ii) Set $T_\lambda = I + \lambda(T - I)$, where $\lambda \in [0, 2]$. Then

$$\|T_\lambda x - y\|^2 \leq \|x - y\|^2 - \lambda(2 - \lambda)\|Tx - x\|^2\ \forall (x, y) \in H \times \mathcal{F}(T).$$

Lemma 4.2.2. [79] Consider a linear, bounded and self-adjoint operator $U : H \to H$ such that $\delta\|x\|^2 \leq \langle Ux, x\rangle$ for every $x \in H$, for some $\delta > 0$ and $\|U\|^{-1} \geq \mu > 0$. Assume that operator $S : H \to H$ resides in the class $\mathfrak{T}$ in $(H, \|\cdot\|_U)$. Then, the operator $R := I - \mu U(I - S)$ resides in the class $\mathfrak{T}$ in $(H, \|\cdot\|)$ and $\mathcal{F}(S) = \mathcal{F}(R)$.

4.3 Forward-backward-half forward dynamical system

To study the Forward-backward-half forward (FBHF) dynamical systems, we need the following assumptions on the operators:

Assumption 4.3.1. (1) $A : H \to 2^H$ is maximal monotone operator;

(2) $B_1 : H \to H$ is β-cocoercive operator for $\beta > 0$;

(3) $B_2 : H \to H$ is monotone and L-Lipschitz continuous operator for $L > 0$.

4.3.1 Existence and uniqueness of trajectories

Let A, B_1 and B_2 be operators satisfying Assumption 4.3.1 with $\mathcal{D}(B_2) = H$. We introduce the following dynamical system:

$$\begin{cases} z(t) = J_{\gamma(t)A}(I - \gamma(t)(B_1 + B_2))x(t), \\ \dot{x}(t) = z(t) - x(t) + \gamma(t)(B_2x(t) - B_2z(t)), \\ x(0) = x_0, \end{cases} \tag{4.3}$$

where $x_0 \in H$, $\gamma : [0, \infty) \to (0, \chi(\beta, L))$ is a Lebesgue measurable function, and $\chi(\beta, L) = \frac{4\beta}{1+4\beta L}$.

Remark 4.3.1. (a) If $\gamma(t) = \gamma \; \forall t \in [0, \infty)$, then the dynamical system (4.3) reduces to the dynamical system

$$\begin{cases} z(t) = J_{\gamma A}(I - \gamma(B_1 + B_2))x(t), \\ \dot{x}(t) = z(t) - x(t) + \gamma(B_2x(t) - B_2z(t)), \\ x(0) = x_0, \end{cases} \tag{4.4}$$

where $x_0 \in H$ and $\gamma \in (0, \chi(\beta, L))$.

(b) If $B_1 = 0$ and $B_2 = B$, then the dynamical system (4.3) reduces to (1.12).

(c) If $B_2 = 0$ and $B_1 = B$, then the dynamical system (4.3) reduces to (1.10), where $\lambda(t) = 1 \ \forall t \in [0, \infty)$.

Define a function $F : (0, \infty) \times H \to H$ by

$$F(r, x) = ((I - rB_2) \circ J_{rA} \circ (I - r(B_1 + B_2)) - (I - rB_2))x \ \forall x \in H \ \textit{and} \ r \in (0, \infty). \tag{4.5}$$

Hence the dynamical system (4.3) can be written as follows:

$$\begin{cases} \dot{x}(t) = F(\gamma(t), x(t)), \\ x(0) = x_0 \in H. \end{cases} \tag{4.6}$$

Definition 4.3.1. A map $x : [0, \infty) \to H$ is said to be a strong global solution of the dynamical system (4.3), if the following properties hold:

(i) $x : [0, \infty) \to H$ is locally absolutely continuous;

(ii) $\dot{x}(t) = ((I - \gamma(t)B_2) \circ J_{\gamma(t)A} \circ (I - \gamma(t)(B_1 + B_2)) - (I - \gamma(t)B_2))x(t)$ for almost every $t \in [0, \infty)$;

(iii) $x(0) = x_0$.

We now discuss basic properties of function F.

Lemma 4.3.1. The following properties hold for F:

(a) For a fixed $x \in H$, $r \mapsto F(r, x)$ is continuous on $(0, \infty)$.

(b) For $x \in \mathcal{D}(A)$,

$$\lim_{r\downarrow 0} F(r,x) = 0.$$

Proof. (a) The first implication is obvious.

(b) Let $x \in \mathcal{D}(A)$ and $r \in (0,\infty)$. Since J_{rA} is nonexpansive, so we have

$$\|J_{rA} \circ (I - r(B_1 + B_2))x - J_{rA}x\| \leq r\|(B_1 + B_2)x\| \ \forall r > 0.$$

For $x \in \mathcal{D}(A)$, $J_{rA}x = P_{\overline{\mathcal{D}(A)}}x = x$ as $r \to 0$, where $P_{\overline{\mathcal{D}(A)}}$ denotes the projection operator on closer of the set $\mathcal{D}(A)$ [11, Theorem 23.47]. Thus $J_{rA} \circ (I - r(B_1 + B_2))x \to x$ as $r \to 0$, and assertion follows from the fact that B_2 is Lipschitz continuous on $\mathcal{D}(B_2) \supset \mathcal{D}(A)$.

□

Lemma 4.3.2. For $r \in (0, \chi(\beta, L))$ with

$$\chi(\beta, L) = \frac{4\beta}{1 + 4\beta L} \leq \min\{4\beta, L^{-1}\}. \tag{4.7}$$

There exists a positive real number $\mathcal{L}$ such that

$$\|F(r,x) - F(r,y)\| \leq \mathcal{L}\|x - y\| \ \forall x, y \in H.$$

Proof. To avoid complexity, we write $C_1 := I - r(B_1 + B_2)$, $C_2 := (I - rB_2)$ and $J := J_{rA}$.

$$\begin{aligned}
&\|F(r,x) - F(r,y)\|^2 \\
&= \|C_2 \circ J \circ C_1 x - C_2 x - C_2 \circ J \circ C_1 y + C_2 y\|^2 \\
&= \|C_2 \circ J \circ C_1 x - C_2 \circ J \circ C_1 y\|^2 + \|C_2 x - C_2 y\|^2
\end{aligned}$$

$$\begin{aligned}
&- 2\langle C_2 \circ J \circ C_1 x - C_2 \circ J \circ C_1 y, C_2 x - C_2 y\rangle \\
&= \|J \circ C_1 x - J \circ C_1 y\|^2 + r^2\|B_2 \circ J \circ C_1 x - B_2 \circ J \circ C_1 y\|^2 \\
&- 2r\langle J \circ C_1 x - J \circ C_1 y, B_2 \circ J \circ C_1 x - B_2 \circ J \circ C_1 y\rangle \\
&+ \|C_2 x - C_2 y\|^2 - 2\langle C_2 \circ J \circ C_1 x - C_2 \circ J \circ C_1 y, C_2 x - C_2 y\rangle.
\end{aligned}$$

Since B_2 is L-Lipschitz continuous and monotone, and J is firmly nonexpansive, so we deduce

$$\begin{aligned}
&\|F(r, x) - F(r, y)\|^2 \\
&\leq (1 + r^2L^2)\langle C_1 x - C_1 y, J \circ C_1 x - J \circ C_1 y\rangle - 2\langle C_2 \circ J \circ C_1 x - C_2 \circ J \circ C_1 y, C_2 x - C_2 y\rangle \\
&\quad + \|C_2 x - C_2 y\|^2 - 2r\langle J \circ C_1 x - J \circ C_1 y, B_2 \circ J \circ C_1 x - B_2 \circ J \circ C_1 y\rangle \\
&\leq (1 + r^2L^2)\langle C_1 x - C_1 y, J \circ C_1 x - J \circ C_1 y\rangle + \|C_2 x - C_2 y\|^2 \\
&\quad - 2\langle J \circ C_1 x - J \circ C_1 y, C_2 x - C_2 y\rangle + 2r\langle B_2 \circ J \circ C_1 x - B_2 \circ J \circ C_1 y, C_2 x - C_2 y\rangle \\
&= (1 + r^2L^2)\langle C_1 x - C_1 y, J \circ C_1 x - J \circ C_1 y\rangle + \|C_2 x - C_2 y\|^2 \\
&\quad - 2\langle J \circ C_1 x - J \circ C_1 y, C_1 x - C_1 y + rB_1 x - rB_1 y\rangle \\
&\quad + 2r\langle B_2 \circ J \circ C_1 x - B_2 \circ J \circ C_1 y, C_2 x - C_2 y\rangle \\
&= (1 + r^2L^2)\langle C_1 x - C_1 y, J \circ C_1 x - J \circ C_1 y\rangle + \|C_2 x - C_2 y\|^2 \\
&\quad - 2r\langle B_1 x - B_1 y, J \circ C_1 x - J \circ C_1 y\rangle - 2\langle J \circ C_1 x - J \circ C_1 y, C_1 x - C_1 y\rangle \\
&\quad + 2r\langle B_2 \circ J \circ C_1 x - B_2 \circ J \circ C_1 y, C_2 x - C_2 y\rangle \\
&= (1 + r^2L^2 - 2)\langle C_1 x - C_1 y, J \circ C_1 x - J \circ C_1 y\rangle - 2r\langle B_1 x - B_1 y, J \circ C_1 x - J \circ C_1 y\rangle \\
&\quad + \|C_2 x - C_2 y\|^2 + 2r\langle B_2 \circ J \circ C_1 x - B_2 \circ J \circ C_1 y, C_2 x - C_2 y\rangle.
\end{aligned}$$

Since $rL \leq 1$ and the fact that J is monotone, we have

$$\begin{aligned}
&\|F(r,x) - F(r,y)\|^2 \\
&\leq 2r\langle B_1y - B_1x, J \circ C_1x - J \circ C_1y\rangle + \|C_2x - C_2y\|^2 \\
&\quad + 2r\langle B_2 \circ J \circ C_1x - B_2 \circ J \circ C_1y, C_2x - C_2y\rangle.
\end{aligned}$$

Now, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\|F(r,x) - F(r,y)\|^2 \\
&\leq 2r\|B_1y - B_1x\|\|C_1x - C_1y\| + \|C_2x - C_2y\|^2 + 2rL\|C_1x - C_1y\|\|C_2x - C_2y\| \\
&= 2\gamma\|B_1y - B_1x\|\|x - rB_1x - rB_2x - y + rB_1y + rB_2y\| \\
&\quad + \|x - rB_2x - y + rB_2y\|^2 \\
&\quad + 2rL\|x - rB_1x - rB_2x - y + rB_1y + rB_2y\|\|x - rB_2x - y + rB_2y\| \\
&\leq 2r\|B_1y - B_1x\|(\|x - y\| + r\|B_1x - B_1y\| + r\|B_2x - B_2y\|) \\
&\quad + \|x - y\|^2 + r^2\|B_2x - B_2y\|^2 - 2r\langle x - y, B_2x - B_2y\rangle \\
&\quad + 2rL(\|x - y\| + r\|B_1x - B_1y\| + r\|B_2x - B_2y\|)(\|x - y\| + r\|B_2x - B_2y\|) \\
&\leq 2r(\|B_1y - B_1x\|\|x - y\| + r\|B_1x - B_1y\|^2 \\
&\quad + r\|B_1x - B_1y\|\|B_2x - B_2y\|) \\
&\quad + (1 + r^2L^2)\|x - y\|^2 + 2rL(\|x - y\|^2 + r\|x - y\|\|B_2x - B_2y\| \\
&\quad + r\|x - y\|\|B_1x - B_1y\| + r^2\|B_1x - B_1y\|\|B_2x - B_2y\| \\
&\quad + r\|x - y\|\|B_2x - B_2y\| + r^2\|B_2x - B_2y\|^2) \\
&\leq \left(\frac{2r}{\beta} + \frac{2r^2}{\beta^2} + \frac{2r^2L}{\beta} + 1 + r^2L^2 + 2rL + 2r^2L^2 + \frac{2r^2L}{\beta} + 2\frac{r^3L^2}{\beta} + 2r^2L^2 + 2r^3L^3\right)\|x - y\|^2 \\
&\leq \mathfrak{K}\|x - y\|^2,
\end{aligned}$$

for some $\mathfrak{K} > 0$. Thus

$$\|F(r,x) - F(r,y)\| \leq \mathcal{L}\|x - y\|, \text{ where } \mathcal{L} = \sqrt{\mathfrak{K}}\ .$$

$\square$

Lemma 4.3.3. Let $F(r,x)$ be a function defined by (4.5). Then

$$\|F(r,x)\| \leq \mathcal{K}(1 + \|x\|)\ \forall x \in H \text{ and } \forall r \in (0, \chi(\beta, L)),$$

where $\mathcal{K}$ is a positive real number.

Proof. The proof is similar to [38, Lemma 3]. $\square$

Proposition 4.3.1. Suppose that $\gamma : [0,\infty) \to (0, \chi(\beta, L))$ is a Lebesgue measurable function, and $x_0 \in H$. Then there exists a strong global solution of the dynamical system (4.3).

Proof. Let $x, y \in H$. For every $t \in [0,\infty)$, by Lemma 4.3.2 we can see

$$\|F(\gamma(t), x) - F(\gamma(t), y)\| \leq \mathcal{L}\|x - y\|.$$

Also, the map $r \mapsto F(r,x)$ is continuous on $(0,\infty)$ for each $x \in H$, hence the map $t \mapsto F(\gamma(t), x)$ is measurable, and bounded by Lemma 4.3.3, so locally integrable. Therefore, by using Theorem 1.2.1 to the map $(t,x) \mapsto F(\gamma(t), x)$, we have the result. $\square$

4.3.2 Convergence of trajectories generated by dynamical system (4.3)

To study the asymptotic behavior of trajectories of (4.3), we need the following lemmas:

Lemma 4.3.4. Let $\theta \in (0, 1]$ and let $\alpha : (0, \infty) \to [0, \infty)$, $\beta : (0, \infty) \to [0, \infty)$ and $\bar{\epsilon} : (0, \infty) \to [0, \infty)$ be functions such that $\int_0^\infty \bar{\epsilon}(t)dt < \infty$ satisfying the following relation:

$$\dot{\alpha}(t) \leq (\theta - 1)\alpha(t) - \beta(t) + \bar{\epsilon}(t), \ t > 0.$$

Then, we have the following:

(a) $\alpha(t)$ is bounded.

(b) $\lim_{t\to\infty} \alpha(t)$ exists.

(c) $\int_0^\infty \beta(t)dt < \infty$.

Proof. The proof is obvious. □

Lemma 4.3.5. Let $\mathrm{Zer}(A + B_1 + B_2) \neq \emptyset$ and $x^* \in \mathrm{Zer}(A + B_1 + B_2)$. Suppose that $\gamma : [0, \infty) \to (0, \chi(\beta, L))$ is a Lebesgue measurable function, where $\chi(\beta, L) = \frac{4\beta}{1+4\beta L}$. Then $\lim_{t\to\infty} \|x(t) - x^*\|$ exists and $\int_0^\infty (\chi(\beta, L) - \gamma(t)) \|x(t) - z(t)\|^2 < \infty$.

Proof. For $t \in [0, \infty)$, we obtain

$$\begin{aligned}\frac{d}{dt}\|x(t) - x^*\|^2 &= 2\langle x(t) - x^*, \dot{x}(t)\rangle \\ &= 2\langle x(t) - x^*, z(t) - x(t) + \gamma(t)(B_2x(t) - B_2z(t))\rangle\end{aligned}$$

$$= \|z(t) - x^*\|^2 - \|x(t) - z(t)\|^2 - \|x(t) - x^*\|^2$$
$$+ 2\gamma(t)\langle x(t) - x^*, B_2x(t) - B_2z(t)\rangle. \tag{4.8}$$

Since $x^* \in \mathrm{Zer}(A + B_1 + B_2)$, we have $-(B_1 + B_2)x^* \in Ax^*$. From the dynamical system (4.3), we get

$$\frac{x(t) - z(t)}{\gamma(t)} - (B_1 + B_2)x(t) \in Az(t).$$

Now, by the monotonicity of A, we have

$$\begin{aligned} 0 &\leq 2\langle x(t) - \gamma(t)(B_1 + B_2)x(t) - z(t) + \gamma(t)(B_1 + B_2)x^*, z(t) - x^*\rangle \\ &= \|x(t) - x^*\|^2 - \|x(t) - z(t)\|^2 - \|z(t) - x^*\|^2 + 2\gamma(t)\langle B_1x^* - B_1x(t), z(t) - x^*\rangle \\ &\quad + 2\gamma(t)\langle B_2x^* - B_2x(t), z(t) - x^*\rangle. \end{aligned} \tag{4.9}$$

From (4.8) and (4.9), and the fact that B_2 is monotone, we obtain

$$\begin{aligned} \frac{d}{dt}\|x(t) - x^*\|^2 \leq &- 2\|x(t) - z(t)\|^2 + 2\gamma(t)\langle B_1x^* - B_1x(t), z(t) - x^*\rangle \\ &+ 2\gamma(t)\langle B_2x^* - B_2x(t), z(t) - x^*\rangle \\ &+ 2\gamma(t)\langle x(t) - x^*, B_2x(t) - B_2z(t)\rangle \\ \leq &- 2\|x(t) - z(t)\|^2 + 2\gamma(t)\langle B_1x^* - B_1x(t), z(t) - x^*\rangle \\ &+ 2\gamma(t)\langle B_2z(t) - B_2x(t), z(t) - x^*\rangle \\ &+ 2\gamma(t)\langle x(t) - x^*, B_2x(t) - B_2z(t)\rangle \\ \leq &- 2\|x(t) - z(t)\|^2 + 2\gamma(t)\langle B_1x^* - B_1x(t), z(t) - x^*\rangle \\ &+ 2\gamma(t)\langle B_2z(t) - B_2x(t), z(t) - x(t)\rangle. \end{aligned} \tag{4.10}$$

Since B_1 is β-cocoercive, so for any $\epsilon > 0$, we have

$$\begin{aligned}
&2\gamma(t)\langle z(t) - x^*, B_1x^* - B_1x(t)\rangle \\
&= 2\gamma(t)\langle x(t) - x^*, B_1x^* - B_1x(t)\rangle + 2\gamma(t)\langle z(t) - x(t), B_1x^* - B_1x(t)\rangle \\
&\leq -2\gamma(t)\beta\|B_1x(t) - B_1x^*\|^2 + 2\langle z(t) - x(t), \gamma(t)(B_1x^* - B_1x(t))\rangle \\
&= -2\gamma(t)\beta\|B_1x(t) - B_1x^*\|^2 + \epsilon\|x(t) - z(t)\|^2 + \frac{\gamma^2(t)}{\epsilon}\|B_1x(t) - B_1x^*\|^2 \\
&\quad - \epsilon\left\|x(t) - z(t) - \frac{\gamma(t)}{\epsilon}(B_1x(t) - B_1x^*)\right\|^2 \\
&= \epsilon\|x(t) - z(t)\|^2 - \gamma(t)\left(2\beta - \frac{\gamma(t)}{\epsilon}\right)\|B_1x(t) - B_1x^*\|^2 \\
&\quad - \epsilon\left\|x(t) - z(t) - \frac{\gamma(t)}{\epsilon}(B_1x(t) - B_1x^*)\right\|^2. \qquad (4.11)
\end{aligned}$$

From (4.10) and (4.11), we obtain

$$\begin{aligned}
\frac{d}{dt}\|x(t) - x^*\|^2 \leq (\epsilon - 2)\|x(t) - z(t)\|^2 &- \frac{\gamma(t)}{\epsilon}(2\beta\epsilon - \gamma(t))\|B_1x(t) - B_1x^*\|^2 \\
&- \epsilon\left\|x(t) - z(t) - \frac{\gamma(t)}{\epsilon}(B_1x(t) - B_1x^*)\right\|^2 \\
&+ 2\gamma(t)\langle B_2z(t) - B_2x(t), z(t) - x(t)\rangle.
\end{aligned}$$

Since B_2 is L-Lipschitz continuous, the above inequality can be rewritten as

$$\begin{aligned}
\frac{d}{dt}\|x(t) - x^*\|^2 &\leq (\epsilon - 2)\|x(t) - z(t)\|^2 - \frac{\gamma(t)}{\epsilon}(2\beta\epsilon - \gamma(t))\|B_1x(t) - B_1x^*\|^2 \\
&\quad - \epsilon\left\|x(t) - z(t) - \frac{\gamma(t)}{\epsilon}(B_1x(t) - B_1x^*)\right\|^2 + 2\gamma(t)L\|z(t) - x(t)\|^2 \\
&= (\epsilon + 2\gamma(t)L - 2)\|x(t) - z(t)\|^2 - \frac{\gamma(t)}{\epsilon}(2\beta\epsilon - \gamma(t))\|B_1x(t) - B_1x^*\|^2 \\
&\quad - \epsilon\left\|x(t) - z(t) - \frac{\gamma(t)}{\epsilon}(B_1x(t) - B_1x^*)\right\|^2 \\
&= -2L\left(\frac{2-\epsilon}{2L} - \gamma(t)\right)\|x(t) - z(t)\|^2 - \frac{\gamma(t)}{\epsilon}(2\beta\epsilon - \gamma(t))\|B_1x(t) - B_1x^*\|^2 \\
&\quad - \epsilon\left\|x(t) - z(t) - \frac{\gamma(t)}{\epsilon}(B_1x(t) - B_1x^*)\right\|^2. \qquad (4.12)
\end{aligned}$$

In order to find the largest interval for γ insuring that the first and second terms on the right hand side of (4.12) are negative, which is acquired by the choice of $\chi(\beta, L) = \frac{2-\epsilon}{2L} = 2\beta\epsilon$, and this yields $\epsilon = \frac{2}{1+4\beta L}$. So, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|x(t)-x^*\|^2 \\
&\leq -2L\left(\chi(\beta,L)-\gamma(t)\right)\|x(t)-z(t)\|^2 - \frac{2\beta\gamma(t)}{\chi(\beta,L)}(\chi(\beta,L)-\gamma(t))\|B_1x(t)-B_1x^*\|^2 \\
&- \frac{\chi(\beta,L)}{2\beta}\|x(t)-z(t)-\frac{2\beta\gamma(t)}{\chi(\beta,L)}(B_1x(t)-B_1x^*)\|^2 \qquad (4.13) \\
&\leq -2L\left(\chi(\beta,L)-\gamma(t)\right)\|x(t)-z(t)\|^2.
\end{aligned}$$

Hence, $\frac{d}{dt}\|x(t)-x^*\|^2 \leq 0$ and thus the mapping $t \mapsto \|x(t)-x^*\|$ is monotonically decreasing. From Lemma 4.3.4, taking $\theta = 1$, we get for $\tau > 0$.

$$\int_0^\tau L\left(\chi(\beta,L)-\gamma(t)\right)\|x(t)-z(t)\|^2 dt < \infty,$$

i.e.,

$$\int_0^\tau \left(\chi(\beta,L)-\gamma(t)\right)\|x(t)-z(t)\|^2 dt < \infty, \; as \; L > 0.$$

□

Lemma 4.3.6. Assume that $x : [0,\infty) \to H$ is the unique strong global solution of the dynamical system (4.3) and the map $t \mapsto z(t)$ is given by the dynamical system (4.3), and $\gamma : [0,\infty) \to (0, \chi(\beta, L))$ is locally absolutely continuous. Then the map $t \mapsto z(t)$ is locally absolutely continuous, and

$$\|\dot{z}(t)\| \leq \left(\left(1+\frac{\dot{\gamma}(t)}{\gamma(t)}\right) + \gamma(t)L + \gamma(t)\left(\frac{1}{\beta}+L\right)\sqrt{1+(\gamma(t))^2L^2}\right)\|x(t)-z(t)\|, \qquad (4.14)$$

for almost every $t \in [0, \infty)$.

Proof. Note that the maps $t \mapsto x(t)$ and $t \mapsto \gamma(t)$ are absolutely continuous on $[0, b]$, for $b > 0$. Since B_1 is β-cocoercive and B_2 is monotone and continuous, so $(B_1+B_2)x$ is absolutely continuous on $[0, b]$, and $t \mapsto y(t) := x(t)-\gamma(t)(B_1+B_2)x(t)$ is also absolutely continuous on $[0, b]$. Let $t_1, t_2 \in [0, b]$. By nonexpansivety of $J_{\gamma A}$,

$$\begin{aligned}\|z(t_2) - z(t_1)\| &= \|J_{\gamma(t_2)A}y(t_2) - J_{\gamma(t_1)A}y(t_1)\| \\ &\leq \|J_{\gamma(t_2)A}y(t_2) - J_{\gamma(t_2)A}y(t_1)\| + \|J_{\gamma(t_2)A}y(t_1) - J_{\gamma(t_1)A}y(t_1)\| \\ &\leq \|y(t_2) - y(t_1)\| + \|J_{\gamma(t_2)A}y(t_1) - J_{\gamma(t_1)A}y(t_1)\|. \end{aligned} \tag{4.15}$$

One can obtain from [11, Proposition 23.28] that $\|J_{\gamma A}x-J_{\lambda A}x\| \leq |\gamma-\lambda|\|A_\gamma x\| \; \forall \gamma, \lambda > 0$ and $\forall x \in H$. So, from (4.15), for every $t_1, t_2 \in [0, b]$ we have

$$\|z(t_2) - z(t_1)\| \leq \|y(t_2) - y(t_1)\| + |\gamma(t_2) - \gamma(t_1)|\|A_{\gamma(t_2)}(y(t_2))\|.$$

The mapping $\gamma \mapsto \|A_\gamma(y(t))\|$ is nonincreasing and Yosida approximation of a maximal monotone operator is Lipschitz continuous [11, Corollary 23.10]. Also, γ is continuous on $[0, b]$, so $\exists\ \gamma_{\min}, \gamma_{\max} \in (0, \chi(\beta, L))$ such that $\gamma_{\min} \leq \gamma(\cdot) \leq \gamma_{\max}$. Hence

$$\begin{aligned}\|z(t_2) - z(t_1)\| &\leq \|y(t_2) - y(t_1)\| + |\gamma(t_2) - \gamma(t_1)|\|A_{\gamma(t_2)}(y(t_2))\| \\ &\leq \|y(t_2) - y(t_1)\| + |\gamma(t_2) - \gamma(t_1)|\|A_{\gamma_{\min}}(y(t_2))\| \\ &\leq \|y(t_2) - y(t_1)\| + |\gamma(t_2) - \gamma(t_1)| \left(\|A_{\gamma_{\min}}(0)\| + \frac{1}{\gamma_{\min}}\|y(t_2)\|\right). \end{aligned}$$

Therefore, z is absolutely continuous on $[0, b]$.

By the dynamical system (4.3) for $t_1, t_2 \in [0, b]$, $t_1 \neq t_2$, and monotonicity of A, we obtain

$$0 \leq \left\langle z(t_1) - z(t_2), \frac{x(t_1) - z(t_1)}{\gamma(t_1)} - (B_1 + B_2)x(t_1) - \frac{x(t_2) - z(t_2)}{\gamma(t_2)} + (B_1 + B_2)x(t_2) \right\rangle,$$

which implies that

$$\begin{aligned} &\left\| \frac{z(t_1) - z(t_2)}{t_1 - t_2} \right\|^2 \leq \\ &\left\langle \frac{z(t_1) - z(t_2)}{t_1 - t_2}, \frac{x(t_1) - x(t_2)}{t_1 - t_2} + \frac{\gamma(t_2) - \gamma(t_1)}{t_1 - t_2} \frac{x(t_2) - z(t_2)}{\gamma(t_2)} \right. \\ &\left. + \gamma(t_1) \frac{(B_1 + B_2)x(t_2) - (B_1 + B_2)x(t_1)}{t_1 - t_2} \right\rangle, \end{aligned} \tag{4.16}$$

then by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left\| \frac{z(t_1) - z(t_2)}{t_1 - t_2} \right\| \leq \\ &\left\| \frac{x(t_1) - x(t_2)}{t_1 - t_2} + \frac{\gamma(t_2) - \gamma(t_1)}{t_1 - t_2} \frac{x(t_2) - z(t_2)}{\gamma(t_2)} + \gamma(t_1) \frac{(B_1 + B_2)x(t_2) - (B_1 + B_2)x(t_1)}{t_1 - t_2} \right\|. \end{aligned}$$

Taking the limit as $t_1 \to t_2 = t(\text{say})$, we have for almost every $t \in [0, \infty)$

$$\begin{aligned} &\|\dot{z}(t)\| \\ &\leq \left\| \dot{x}(t) - \frac{\dot{\gamma}(t)}{\gamma(t)}(x(t) - z(t)) - \gamma(t) \frac{d}{dt}(B_1 + B_2)(x(t)) \right\| \\ &= \left\| \left(1 + \frac{\dot{\gamma}(t)}{\gamma(t)}\right)(z(t) - x(t)) + \gamma(t)(B_2 x(t) - B_2 z(t)) - \gamma(t) \frac{d}{dt}(B_1 + B_2)(x(t)) \right\|. \end{aligned} \tag{4.17}$$

Since B_1 is $\frac{1}{\beta}$-Lipschitz continuous and B_2 is L-Lipschitz continuous, so by Remark 3.3.1(b), we get $\|\frac{d}{dt}(B_1 + B_2)x(t)\| \leq \|\frac{d}{dt}B_1 x(t)\| + \|\frac{d}{dt}B_2 x(t)\| \leq (\frac{1}{\beta} + L)\|\dot{x}(t)\|$ for almost every $t \in [0, \infty)$. Moreover, since B_2 is monotone and Lipschitz continuous, so for almost every $t \in [0, \infty)$

$$\begin{aligned}\|\dot{x}(t)\|^2 &= \|x(t) - z(t)\|^2 + (\gamma(t))^2\|B_2x(t) - B_2z(t)\|^2 \\ &\quad + 2\gamma(t)\langle x(t) - z(t), B_2z(t) - B_2x(t)\rangle \\ &\leq \left(1 + (\gamma(t))^2L^2\right)\|x(t) - z(t)\|^2. \end{aligned} \tag{4.18}$$

Also,

$$\begin{aligned}&\left\|\left(1 + \frac{\dot{\gamma}(t)}{\gamma(t)}\right)(z(t) - x(t)) + \gamma(t)(B_2x(t) - B_2z(t))\right\| \\ &\leq \left(\left(1 + \frac{\dot{\gamma}(t)}{\gamma(t)}\right) + (\gamma(t))L\right)\|x(t) - z(t)\|.\end{aligned}$$

Therefore (4.17) gives

$$\|\dot{z}(t)\| \leq \left(\left(1 + \frac{\dot{\gamma}(t)}{\gamma(t)}\right) + \gamma(t)L + \gamma(t)\left(\frac{1}{\beta} + L\right)\sqrt{1 + (\gamma(t))^2L^2}\right)\|x(t) - z(t)\|.$$

□

Now, we prove the main result of this section.

Theorem 4.3.1. Let $\gamma : [0, \infty) \to [\eta, \chi(\beta, L) - \bar{\eta}]$ be a locally absolutely continuous function, for any $\eta, \bar{\eta} > 0$, and $\dot{\gamma} \in L^\infty([0, \infty))$. *Assume that* $\operatorname{Zer}(A + B_1 + B_2) \neq \emptyset$. Then we have the following:

(a) The trajectories $x(t)$ and $z(t)$ governed by dynamical system (4.3) converges weakly to an element in $\operatorname{Zer}(A + B_1 + B_2)$ as $t \to \infty$.

(b) If $x^* \in \operatorname{Zer}(A + B_1 + B_2)$. Then we have

   (i) $\int_0^\infty \|B_1x(t) - B_1x^*\|^2 dt < \infty$.

(ii) $\lim_{t\to\infty} B_1 x(t) = B_1 x^*$.

Proof. (a) From Lemma 4.3.5, we get that the mapping $t \mapsto \|x(t) - z(t)\|^2$ from $[0,\infty)$ to itself, resides in $L^1[0,\infty)$. Moreover, by Lemma 4.3.6, (4.18) and the Cauchy- Schwarz inequality, we deduce that for almost every $t \in [0,\infty)$

$$\begin{aligned}
&\frac{d}{dt}\|x(t) - z(t)\|^2 \\
&= 2\langle x(t) - z(t), \dot{x}(t) - \dot{z}(t)\rangle \\
&\leq 2(\|\dot{x}(t)\| + \|\dot{z}(t)\|)\|x(t) - z(t)\| \\
&\leq 2\left(\left(1 + \frac{\dot{\gamma}(t)}{\gamma(t)}\right) + \gamma(t)L + \left(1 + \gamma(t)\left(\frac{1}{\beta} + L\right)\right)\sqrt{1 + (\gamma(t))^2 L^2}\right)\|x(t) - z(t)\|^2 \\
&\leq 2\left(\left(1 + \frac{\|\dot{\gamma}\|_{L^\infty([0,\infty))}}{\eta}\right) + 1 + 6\sqrt{2}\right)\|x(t) - z(t)\|^2.
\end{aligned}$$

By Lemma 3.2.4, we get that $\lim_{t\to\infty}\|x(t) - z(t)\|^2 = 0$, and hence by (4.18), $\lim_{t\to\infty}\dot{x}(t) = 0$.

Next, we show that every weak sequential cluster point of $x(t)$ is in $\mathrm{Zer}(A + B_1 + B_2)$.

For this, let $x \in H$ be a weak sequential cluster point of $x(t)$ and $\{t_n\}$ be a sequence in $[0,\infty)$ such that $t_n \to \infty$ and $x(t_n) \rightharpoonup x$ as $n \to \infty$. Also, we have $z(t_n) \rightharpoonup x$ as $n \to \infty$, due to the fact that $\lim_{t\to\infty}\|x(t) - z(t)\| = 0$. Since $z(t) = J_{\gamma(t)A}(x(t) - \gamma(t)(B_1 + B_2)x(t))$, so we have $u(t) := \frac{1}{\gamma(t)}(x(t) - z(t)) - (B_1 + B_2)x(t) + (B_1 + B_2)z(t) \in (A + B_1 + B_2)z(t)$. Using the fact that B is Lipschitz continuous, $\gamma(t) \geq \eta > 0$ and $\lim_{t\to\infty}\|x(t) - z(t)\| = 0$, we obtain $\lim_{t\to\infty} u(t) = 0$. Since, B_2 is monotone with $\mathcal{D}(B_2) = H$, and B_1 is cocoercive with full domain, $A + B_1 + B_2$ is maximally monotone. By sequential weak-strong closeness of the graph of maximal monotone operator, we have

$(x, 0) \in \mathcal{G}(A + B_1 + B_2)$, hence $x \in \text{Zer}(A + B_1 + B_2)$.

From Lemma 4.3.5, $\lim_{t\to\infty} \|x(t) - x^*\|$ exists for all $x^* \in \text{Zer}(A + B_1 + B_2)$. Therefore, by Lemma 3.2.5, $x(t)$ (and, hence $z(t)$) converges weakly to a zero of $A + B_1 + B_2$ as $t \to \infty$.

(b) From (4.13), we get

$$\frac{d}{dt}\|x(t) - x^*\|^2 + \frac{2\beta\gamma(t)}{\chi(\beta, L)}(\chi(\beta, L) - \gamma(t))\|B_1x(t) - B_1x^*\|^2 < 0.$$

Integrating the above inequality and using that $\eta \leq \gamma(t) \leq \chi(\beta, L) - \bar{\eta}$, we get

$$\int_0^\infty \|B_1x(t) - B_1x^*\|^2 dt < \infty.$$

From Lemma 4.3.5 and (4.18), we obtain that the map $t \mapsto \|\dot{x}(t)\| \in L^2[0, \infty)$ and B_1 is $\frac{1}{\beta}$-Lipschitz, so by Remark 3.3.1(b) we deduce that $t \mapsto \frac{d}{dt}B_1x(t) \in L^2([0, \infty], H)$. So we have for all $t \geq 0$

$$\begin{aligned} \frac{d}{dt}\left(\|B_1x(t) - Bx^*\|^2\right) &= 2\left\langle \frac{d}{dt}B_1x(t), B_1x(t) - B_1x^*\right\rangle \\ &\leq \left\|\frac{d}{dt}B_1x(t)\right\|^2 + \|B_1x(t) - B_1x^*\|^2. \end{aligned} \tag{4.19}$$

From Lemma 3.2.4 and (4.19), we arrive that $\lim_{t\to\infty} B_1x(t) = B_1x^*$.

$\square$

Theorem 4.3.2. Let $T_\gamma : H \to H$ be a map defined by

$$T_\gamma(x) = (I - \gamma B_2) \circ J_{\gamma A} \circ (I - \gamma(B_1 + B_2))x + \gamma B_2 x, \tag{4.20}$$

where $\gamma \in (0, \chi(\beta, L))$. Then for $x_0 \in H$, the trajectory of the dynamical system

$$\begin{cases} \dot{x}(t) = (T_\gamma - I)x(t), \\ x(0) = x_0, \end{cases} \tag{4.21}$$

converges weakly to an element of $\mathcal{F}(T_\gamma) = \mathrm{Zer}(A + B_1 + B_2)$.

Proof. For $\gamma(t) = \gamma(> 0)$, for all $t \in [0, \infty)$, dynamical system (4.3) get converted to

$$\begin{cases} z(t) = J_{\gamma A}(I - \gamma(B_1 + B_2))x(t), \\ \dot{x}(t) = z(t) - x(t) + \gamma(B_2 x(t) - B_2 z(t)), \\ x(0) = x_0, \end{cases}$$

which is equivalent to the dynamical system

$$\begin{cases} \dot{x}(t) = (T_\gamma - I)x(t), \\ x(0) = x_0, \end{cases}$$

where

$$T_\gamma := (I - \gamma B_2) \circ J_{\gamma A} \circ (I - \gamma(B_1 + B_2)) + \gamma B_2.$$

Hence from Theorem 4.3.1, trajectory of the dynamical system (4.21) converges to an element of $\mathcal{F}(T_\gamma)$, as $\mathcal{F}(T_\gamma) = \mathrm{Zer}(A + B_1 + B_2)$ [79, Proposition 2.1]. □

Remark 4.3.2. (a) If $B_2 = 0$, $B_1 = B$ and $\gamma(t) = \gamma\ \forall t \in [0, \infty)$, then Theorem 4.3.1 reduces to Theorem 12 of Bot et al. [33] for $\lambda(t) = I\ \forall t \in [0, \infty)$. One can observe that the the step-size γ is less than 2β in [33, Theorem 12], which is less than 4β.

(b) In Theorem 4.3.1, consider the case:

(i) $A = \partial f$, where $f : H \to \mathbb{R} \cup \{\infty\}$ is a proper lower semicontinuous convex function;

(ii) $B_1 = B$, $B_2 = 0$;

(iii) $\gamma(t) = \gamma \ \forall t \in [0, \infty)$.

Then for $x_0 \in H$, the dynamical system (4.3) reduces to

$$\begin{cases} \dot{x}(t) = \mathrm{prox}_{\gamma f}(x(t) - \gamma B x(t)) - x(t), \\ x(0) = x_0. \end{cases} \tag{4.22}$$

Therefore, one can see that the dynamical system (4.22) is equivalent to the FB-type dynamical system (88) of [32]. Now consider $B = \nabla g$, where g is a convex differentiable function, then from [32, Theorem 5.2] the trajectory of the dynamical system (4.22) converges for every $\gamma > 0$.

(c) If $B_1 = 0$, $B_2 = B$, then Theorem 4.3.1 get converted to Theorem 2 of Banert et al. [38].

4.4 Forward-backward-half forward dynamical system for non self-adjoint linear operator

In this section, we study weak convergence of trajectory generated by a dynamical system in the framework of variable metric, which need not be self-adjoint.

Let $M : H \to H$ be a linear bounded operator. Define

$$U := (M + M^*)/2 \text{ and } V := (M - M^*)/2 \tag{4.23}$$

Note that U and V are the self-adjoint and skew symmetric components of M, respectively. For $x_0 \in H$, we consider the following dynamical system:

$$\begin{cases} z(t) = J_{M^{-1}A}(I - M^{-1}(B_1 + B_2))x(t), \\ \dot{x}(t) = z(t) - x(t) + U^{-1}(B_2x(t) - B_2z(t)) - V(x(t) - z(t)), \\ x(0) = x_0, \end{cases} \tag{4.24}$$

where the operators A, B_1 and B_2 satisfy Assumption 4.3.1 with $\mathcal{D}(B_2) = H$. To implement the non self-adjoint variable metric, we use some resolvent identities, which are valid in the metric $\langle \cdot, \cdot \rangle_U$.

Lemma 4.4.1. [79, Proposition 3.1] Let $A : H \to 2^H$ be a maximal monotone operator and $M : H \to H$ be an invertible linear bounded operator. Define U and V by (4.23). Suppose that $\exists\, \zeta > 0$ such that $\zeta\|x\|^2 \leq \langle Ux, x\rangle$ for all $x \in H$. Define $\langle \cdot, \cdot \rangle_U$ and $\| \cdot \|_U$ by

$$\langle Ux, y\rangle = \langle \cdot, \cdot \rangle_U, (x, y) \in H \times H \text{ and } \|x\|_U = \sqrt{\langle Ux, x\rangle}, x \in H,$$

respectively. Then

$$J_{M^{-1}A} = J_{U^{-1}(A+V)}(I + U^{-1}V).$$

Particularly, $J_{M^{-1}A} : H \to H$ is everywhere defined, single-valued and

$$\|J_{M^{-1}A}x - J_{M^{-1}A}y\|_U^2 \leq \langle J_{M^{-1}A}x - J_{M^{-1}A}y, Mx - My\rangle \; \forall x, y \in H.$$

Also $U^{-1}M^*J_{M^{-1}A}$ is firmly nonexpansive in $(H, \langle\cdot,\cdot\rangle_U)$.

Theorem 4.4.1. Let A, B_1 and B_2 be operators satisfying Assumption 4.3.1 with $\mathcal{D}(B_2) = H$ and $\operatorname{Zer}(A + B_1 + B_2) \neq \emptyset$ and let $M : H \to H$ be an invertible linear bounded operator. Define U and V by (4.23). Suppose that $\exists\ \zeta > 0$ such that $\zeta\|x\|^2 \leq \langle Ux, x\rangle$ for all $x \in H$. Assume that $K > 0$ such that

$$K < \ \zeta - \frac{1}{4\beta} \tag{4.25}$$

and K is the Lipschitz constant of $B_2 - V$. Suppose that $\operatorname{Zer}(A+B_1+B_2) \neq \emptyset$. Then trajectory $x(t)$ of dynamical system (4.24) converges weakly to a zero of $A+B_1+B_2$.

Proof. Note that operator U is invertible. Hence problem (4.1) can be written as

$$\text{find } x \in H \text{ such that } \ 0 \in U^{-1}(A + V)x + U^{-1}B_1x + U^{-1}(B_2 - V)x. \tag{4.26}$$

Since both V and $-V$ are monotone and Lipschitz, it follows that the operator $\mathbf{A} := U^{-1}(A + V)$ is maximal monotone. From [84, Proposition 1.5], we see that $\mathbf{B}_1 := U^{-1}B_1$ is $\zeta\beta$-cocoercive. Note that $\mathbf{B}_2 := U^{-1}(B_2 - V)$ is monotone and $\zeta^{-1}K$- Lipschitz continuous in $(H, \langle\cdot,\cdot\rangle_U)$ [79, Theorem 3.2].

Choose the step-size $\gamma(t) = 1$. Hence, from Lemma 4.4.1, the dynamical system (4.24) reduces to the following dynamical system:

$$\begin{cases} z(t) = J_{\mathbf{A}}(I - (\mathbf{B}_1 + \mathbf{B}_2))x(t), \\ \dot{x}(t) = z(t) - x(t) + (\mathbf{B}_2x(t) - \mathbf{B}_2z(t)), \\ x(0) = x_0, \end{cases} \tag{4.27}$$

where $x_0 \in H$. Note that the step-size condition of the dynamical system (4.27) reduces to

$$\gamma(t) = 1 < \frac{4\beta\zeta}{1 + 4\beta K},$$

which implies that

$$K < \zeta - \frac{1}{4\beta},$$

which is the second condition of (4.25). The inclusion (4.26) and the dynamical system (4.27) meet the conditions of Theorem 4.3.1 under the metric $(H, \|\cdot\|_U)$ as $\mathbf{A} + \mathbf{B}_2 = U^{-1}(A + B_2)$ is maximally monotone in $(H, \|\cdot\|_U)$. Therefore, from Theorem 4.3.1, we conclude that trajectory of the dynamical system (4.24) converges to a solution of $\mathrm{Zer}(A + B_1 + B_2)$. $\square$

Remark 4.4.1. (i) One can obtain FBF dynamical system [38] in the framework of non-self adjoint variable metric by considering $B_1 = 0$, and by taking $\beta \to \infty$, from second condition of (4.25), we get that $K < \zeta$. In the similar manner, FB dynamical system [33] in variable metric sense can be obtained by taking $B_2 = 0$. In this case, for the step size, we take $L = 0$, then $K = \|V\|$ and, hence, the second condition of (4.25) reduces to $\|V\| < \zeta - \frac{1}{4\beta}$.

(ii) In particular, if we take $M = \frac{I}{\gamma(t)}$, the dynamical system (4.24) reduces to (4.3) when the step-sizes are constant.

(iii) One can write the dynamical system (4.27) as follows:

$$\begin{cases} \dot{x}(t) = (T_1 - I)x(t), \\ x(0) = x_0, \end{cases} \tag{4.28}$$

where

$$T_1 = (I - \mathbf{B}_2) \circ J_{\mathbf{A}} \circ (I - (\mathbf{B}_1 + \mathbf{B}_2) + \mathbf{B}_2.$$

One can observe that dynamical system (4.28) converges to a fixed point of T_1, as $\mathcal{F}(T_1) = \mathrm{Zer}(A + B_1 + B_2)$ [79, Proposition 2.1].

In the dynamical system (4.24), it is assumed that the operator U is invertible. Now, we study how to remove this costly inversion by multiplying operator M. Using this idea, we get an operator of the class $\mathfrak{T}$ in $(H, \|\cdot\|_U)$, and another operator of the same class in $(H, \|\cdot\|)$ having the same set of fixed points [79, Proposition 4.1]. In the next result, we prove the weak convergence of the dynamical system's trajectory for an operator of the class $\mathfrak{T}$.

Proposition 4.1. *Let $R : H \to H$ be an operator of class $\mathfrak{T}$. Suppose that $\mathbb{F} = \mathcal{F}(R) \neq \emptyset$. Let the dynamical system be defined as*

$$\begin{cases} \dot{x}(t) = \lambda(t)(R - I)x(t), \\ x(0) = x_0, \end{cases} \tag{4.29}$$

where $x_0 \in H$ and $\lambda : [0, \infty) \to [0, 2]$ is a Lebesgue measurable function. Then we have the following:

(a) $\lim\limits_{t\to\infty} \|x(t) - \bar{x}\|^2$ exists for all $\bar{x} \in \mathbb{F}$, $\int_0^\infty \lambda(t)(2 - \lambda(t))\|(R - I)x(t)\|^2 dt < \infty$ and W is non-empty, where W denotes the set of all weak sequential cluster points of $x(t)$.

(b) If $0 < \liminf \lambda(t) \leq \limsup \lambda(t) < 2$, then $\int_0^\infty \|\dot{x}(t)\|^2 dt < \infty$.

(c) If every weak sequential cluster point of $x(t)$, $t \in [0, \infty)$ resides in $\mathbb{F}$, then $x(t)$ converges weakly to a point $\bar{x}$ in $\mathbb{F}$.

Proof. Let $\bar{x} \in \mathbb{F}$. From Lemma 3.2.1, we obtain

$$\|\dot{x}(t) + x(t) - \bar{x}\|^2 \leq \|x(t) - \bar{x}\|^2 - \lambda(t)(2 - \lambda(t))\|(R - I)x(t)\|^2,$$

which implies that

$$2\langle \dot{x}(t), x(t) - \bar{x} \rangle \leq -\|\dot{x}(t)\|^2 - \lambda(t)(2 - \lambda(t))\|(R - I)x(t)\|^2.$$

Hence

$$2\langle \dot{x}(t), x(t) - \bar{x} \rangle \leq (\theta - 1)\|x(t) - \bar{x}\|^2 - \lambda(t)(2 - \lambda(t))\|(R - I)x(t)\|^2$$

with $\theta = 1$.

(a) Set $\alpha(t) = \|x(t) - \bar{x}\|^2$, $\beta(t) = \lambda(t)(2 - \lambda(t))\|(R - I)x(t)\|^2$ and $\bar{\epsilon}(t) = 0$. Then from Lemma 4.3.4, we obtain that $\lim\limits_{t\to\infty} \|x(t) - \bar{x}\|^2$ exists and $\int_0^\infty \lambda(t)(2 - \lambda(t))\|(R - I)x(t)\|^2 dt < \infty$. Since $\lim\limits_{t\to\infty} \|x(t) - \bar{x}\|^2$ exists, it shows that $x(t)$ is bounded and hence there exists at least one weak sequential cluster point.

(b) This part follows from (a) and (4.29).

(c) The proof follows from the proof of continuous version of Opial lemma.

$\square$

Next, suppose that A, B_1 and B_2 are the operators satisfying Assumption 4.3.1. For $x_0 \in H$, we introduce the following dynamical system:

$$\begin{cases} z(t) = J_{M^{-1}A}(I - M^{-1}(B_1 + B_2))x(t), \\ \dot{x}(t) = \phi(t)(B_2 x(t) - B_2 z(t) + M(z(t) - x(t))), \\ x(0) = x_0, \end{cases} \tag{4.30}$$

where $\phi : [0, \infty) \to (0, \infty)$ is a map.

In what follows, we show the weak convergence of the trajectories generated by a dynamical system (4.30).

Theorem 4.4.2. Let A, B_1 and B_2 be the operators satisfying Assumption 4.3.1 with $\mathcal{D}(B_2) = H$, and $\mathrm{Zer}(A + B_1 + B_2) \neq \emptyset$. Let $M : H \to H$ be an invertible bounded linear mapping with components U and V defined by (4.23). Suppose that $\exists\, \zeta > 0$ such that

$$\zeta \|x\|^2 \leq \langle Ux, x \rangle \textit{ for all } x \in H \textit{ and } K^2 < \frac{\zeta}{1+\epsilon}\left(\frac{\zeta}{1+\epsilon} - \frac{1}{2\beta}\right), \tag{4.31}$$

where K is the Lipschitz constant of $B_2 - V$. Suppose that $\phi : [0, \infty) \to (0, \infty)$ is a map satisfying the condition:

$$0 < \|U\| \liminf \phi(t) \leq \|U\| \limsup \phi(t) < 1.$$

Then trajectory $x(t)$ of the dynamical system (4.30) converges weakly to an element in $\mathrm{Zer}(A + B_1 + B_2)$ as $t \to \infty$.

Proof. Let $\mathbf{T} : H \to H$ be an operator defined by

$$\mathbf{T}(x) = z + U^{-1}(B_2x - B_2z - V(x - z)) \ \forall x \in H,$$

where $z = J_{M^{-1}A}(x - M^{-1}(B_1 + B_2)x)$. Here $\mathbf{T} := (I - \mathbf{B}_2) \circ J_{\mathbf{A}} \circ (I - (\mathbf{B}_1 + \mathbf{B}_2) + \mathbf{B}_2$, where $\mathbf{A} := U^{-1}(A+V), \mathbf{B}_1 := U^{-1}B_1$, and $\mathbf{B}_2 := U^{-1}(B_2 - V)$ satisfy the following properties

(i) $\mathbf{A}$ is maximal monotone,

(ii) $\mathbf{B}_1$ is $\zeta\beta$-cocoercive, and $\mathbf{B}_2$ is monotone and $\zeta^{-1}K$-Lipschitz.

Hence, from [79, Proposition 2.1], $\mathbf{T}$ is quasi-nonexpansive map endowed with the inner product $\langle \cdot, \cdot \rangle_U$. Note that

$$\begin{aligned} (I - \mathbf{T})(x) &= (x - z) + U^{-1}V(x - z) - U^{-1}(B_2x - B_2z) \\ \Leftrightarrow U(I - \mathbf{T})(x) &= (U + V)(x - z) + B_2z - B_2x \\ &= M(x - z) + B_2z - B_2x. \end{aligned} \tag{4.32}$$

Since $\mathbf{T}$ is quasi-nonexpansive in $(H, \|\cdot\|_U)$, so from [83, Proposition 2.2], $\mathcal{S} := (I + \mathbf{T})/2$ is an element of the class $\mathfrak{T}$ in $(H, \|\|_U)$. Therefore, from Lemma 4.2.2 and (4.32), we get the operator

$$\mathcal{R} := I - \|U\|^{-1}U(I - \mathcal{S}) = I - \frac{\|U\|^{-1}}{2}U(I - \mathbf{T}) \tag{4.33}$$

is an element of the class $\mathfrak{T}$ in $(H, \|\cdot\|)$ and $\mathcal{F}(\mathcal{S}) = \mathcal{F}(\mathcal{R}) = \mathrm{Zer}(U(I - \mathcal{T})) = \mathcal{F}(\mathbf{T}) = \mathrm{Zer}(A + B_1 + B_2)$, for every $t \in [0, \infty)$. From (4.30), (4.32) and (4.33), we have

$$\dot{x}(t) = -\phi(t)(M(x(t) - z(t)) + B_2z(t) - B_2x(t))$$

$$= 2\|U\|\phi(t)(\mathcal{R}x(t) - x(t)).$$

$$= \lambda(t)(\mathcal{R}x(t) - x(t)),$$

where $\lambda(t) = 2\|U\|\phi(t)$. As $0 < \|U\| \liminf \phi(t) \leq \|U\| \limsup \phi(t) < 1$, i.e., $0 < \liminf \lambda(t) \leq \limsup \lambda(t) < 2$, it follows from Proposition 4.1(a) that $\|x(t) - \mathcal{R}x(t)\|^2$ is an integrable function, $\dot{x}(t) \in L^2([0, \infty); H)$ and $x(t)$ converges weakly in $(H, \|\cdot\|)$ to a solution in

$$\mathcal{F}(\mathcal{R}) = \mathcal{F}(\mathbf{T}) = \mathrm{Zer}(A + B_1 + B_2)$$

if and only if every weak sequential cluster point of $x(t),\ t \in [0, \infty)$ is a solution. Note that

$$\begin{aligned}
\|x(t) - \mathbf{T}x(t)\|_U^2 &= \langle U(x(t) - \mathbf{T}x(t)), x(t) - \mathbf{T}x(t)\rangle \\
&\leq \|U(x(t) - \mathbf{T}x(t))\|\|x(t) - \mathbf{T}x(t)\| \\
&= \|U^{-1}\|\|U(x(t) - \mathbf{T}x(t))\|^2 \\
&= 4\|U^{-1}\|U\|^2\|x(t) - \mathcal{R}x(t)\|^2 \\
&\leq 4\|U\|^2\zeta^{-1}\|x(t) - \mathcal{R}x(t)\|^2.
\end{aligned}$$

Integrating the above inequality, we obtain that

$$\int_0^\tau \|x(t) - \mathbf{T}x(t)\|_U^2 dt \leq \int_0^\tau 4\|U\|^2\zeta^{-1}\|x(t) - \mathcal{R}x(t)\|^2 dt < \infty \ \forall \tau \in [0, \infty). \quad (4.34)$$

Since $J_{\mathbf{A}}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ are Lipschitz continuous in $(H, \|\cdot\|_U)$, and hence $I - \mathbf{T}$ is Lipschitz continuous in $(H, \|\cdot\|_U)$. Let $\mathbf{L}$ be a Lipschitz constant of $I - \mathbf{T}$.
From Remark 3.3.1(b), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\|(I-\mathbf{T})x(t)\|_U^2\right)\\
&=\frac{d}{dt}\langle(I-\mathbf{T})x(t),U(I-\mathbf{T})x(t)\rangle\\
&=\left\langle(I-\mathbf{T})x(t),U^{-1}\frac{d}{dt}U(I-\mathbf{T})x(t)\right\rangle_U+\left\langle\frac{d}{dt}(I-\mathbf{T})x(t),U^{-1}U(I-\mathbf{T})x(t)\right\rangle_U\\
&\leq\frac{1}{2}\|(I-\mathbf{T})x(t)\|_U^2+\frac{1}{2}\|U^{-1}\frac{d}{dt}U(I-\mathbf{T})x(t)\|_U^2+\frac{1}{2}\mathbf{L}^2\|\dot{x}(t)\|_U^2+\frac{1}{2}\|(I-\mathbf{T})x(t)\|_U^2\\
&\leq\|(I-\mathbf{T})x(t)\|_U^2+\frac{1}{2}\mathbf{L}^2\|\dot{x}(t)\|_U^2+\zeta^{-2}\|U\|\mathbf{L}\|\dot{x}(t)\|_U^2.
\end{aligned}\tag{4.35}
$$

Hence, from (4.34), (4.36) and Lemma 3.2.4, we deduce that

$$
\lim_{t\to\infty}\|(I-\mathbf{T})x(t)\|_U^2=0.\tag{4.36}
$$

Furthermore, $\mathbf{T}$ matches with operator T_γ defined by (4.20) with $\gamma=1$ in terms of the operators $\mathbf{A},\mathbf{B}_1$ and $\mathbf{B}_2$.

Define $z(t):=J_{M^{-1}A}(x(t)-M^{-1}(B_1+B_2)x(t))$. From [79, Proposition 2.1], for every $x^*\in\operatorname{Zer}(A+B_1+B_2)$, we get

$$
\begin{aligned}
&\zeta^{-2}K^2\left(\Theta^2-1\right)\|x(t)-z(t)\|_U^2\;+\frac{2\beta\zeta}{\Theta}(\Theta-1)\|U^{-1}(B_1x(t)-B_1x^*)\|_U^2\\
&+\frac{\Theta}{2\beta\zeta}\|x(t)-z(t)-\frac{2\beta\zeta}{\Theta}U^{-1}(B_1x(t)-B_1x^*)\|_U^2\\
&\leq\|x(t)-x^*\|_U^2-\|\mathbf{T}x(t)-x^*\|_U^2\\
&=-\|\mathbf{T}x(t)-x(t)\|_U^2+2\langle\mathbf{T}x(t)-x(t),x^*-x(t)\rangle_U\\
&\leq-\|\mathbf{T}x(t)-x(t)\|_U^2+2\|\mathbf{T}x(t)-x(t)\|_U\|x^*-x(t)\|_U,
\end{aligned}\tag{4.37}
$$

where

$$\Theta = \frac{4\beta\zeta}{1+\sqrt{1+16\beta^2K^2}} \leq \zeta\min\{2\beta, K^{-1}\}. \tag{4.38}$$

Note $\Theta \geq 1+\epsilon$, for $\epsilon > 0$. From (4.38), we obtain

$$1+\epsilon \leq \frac{4\beta\zeta}{1+\sqrt{1+16\beta^2K^2}} \Leftrightarrow K^2 \leq \frac{\zeta}{1+\epsilon}\left(\frac{\zeta}{1+\epsilon} - \frac{1}{2\beta}\right) \text{ for all } t \in [0,\infty).$$

Hence, from (4.31), (4.38) and (4.37), we have

$$\begin{aligned}
&\epsilon\zeta^{-1}K^2\|x(t)-z(t)\|^2 \; + \epsilon\zeta\|U^{-1}(B_1x(t)-B_1x^*)\|^2 \\
&+ \frac{\Theta}{2\beta}\|x(t)-z(t) - \frac{2\beta\zeta}{\Theta}U^{-1}(B_1x(t)-B_1x^*)\|^2 \\
&\leq -\|\mathbf{T}x(t)-x(t)\|_U^2 + 2\|\mathbf{T}x(t)-x(t)\|_U\|x^*-x(t)\|_U.
\end{aligned} \tag{4.39}$$

Let x be a weak sequential cluster point of $x(t)$, so $\|x(t)-x^*\|$ is bounded and hence $\|x(t)-x^*\|_U$ is bounded. Hence, from (4.36) and (4.39), we obtain that $\lim_{t\to\infty}\|x(t)-z(t)\| = 0$. Also, since M is a bounded linear operator, hence

$$\|M(z(t)-x(t)\| \leq \|M\|\|z(t)-x(t)\| \to 0 \text{ as } t\to\infty.$$

Finally, we have

$$u(t) := M(z(t)-x(t)) - ((B_1+B_2)z(t) - (B_1+B_2)x(t)) \in (A+B_1+B_2)z(t).$$

Since $z(t)-x(t) \to 0$, and B_1+B_2 is continuous, so $u(t)\to 0$. By sequential weak-strong closeness of the graph of maximal monotone operator (see [11, Proposition 20.33]), we have $(x,0) \in \mathcal{G}(A+B_1+B_2)$ and hence $x \in \operatorname{Zer}(A+B_1+B_2)$. □

Remark 4.4.2. (i) As observed in Remark 4.4.1, by setting $B_1 = 0$ or $B_2 = 0$ in

(4.30), one can also derive non self-adjoint variable metric versions of FBF dynamical system [38] and FB dynamical system [33] without using the inversion of U.

4.5 Numerical Example

In this section, we discuss an example in support of the dynamical system (4.3).

Example 4.5.1. Let $H = \mathbb{R}^2$ be a real Hilbert space endowed with Euclidean inner product and $A : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be a maximal monotone operator defined by $A \equiv N_{B(0;1)}$, where $N_{B(0;1)}$ is the normal cone at $B(0;1)$ and $B(0;1) \subset \mathbb{R}^2$ is the closed unit ball centered at 0. Let $B_1, B_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the operators defined by

$$B_1(x) = \begin{cases} \left(1 - \frac{1}{\|x\|}\right) x, & if \|x\| > 1, \\ 0, & if \|x\| \leq 1, \end{cases}$$

and

$$B_2(x_1, x_2) = (x_1 - x_2, x_2 - x_1) \ \forall (x_1, x_2) \in \mathbb{R}^2.$$

Note that B_1 is β-cocoercive operator [11, Example 4.9], where $\beta = 1$. Further, we observe that B_2 is monotone and L-Lipschitz continuous, where $L = 2$. Clearly, $\chi(\beta, L) = \frac{4\beta}{1+4\beta L} = 0.44$. We can see that all the assumptions of dynamical system (4.3) are satisfied. Figure 4.3 and Figure 4.6 show the trajectories of the dynamical system (4.3) for function $\gamma : [0, \infty) \to (0, \chi(\beta, L))$ defined by $\gamma(t) = \frac{1}{t+5}$ and $\gamma(t) = \frac{1}{t+10}$, respectively.

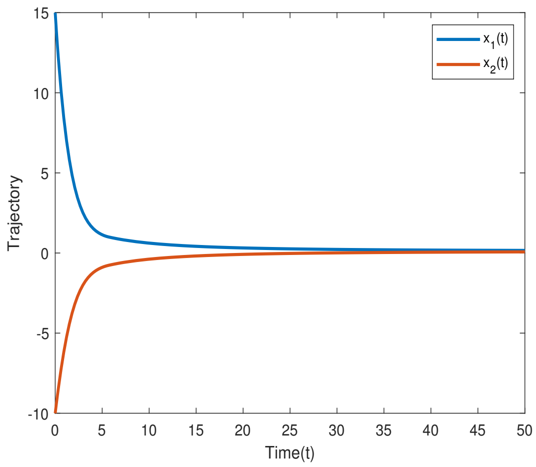

FIGURE 4.1: For $(x_1(0), x_2(0)) = (15, -10)$

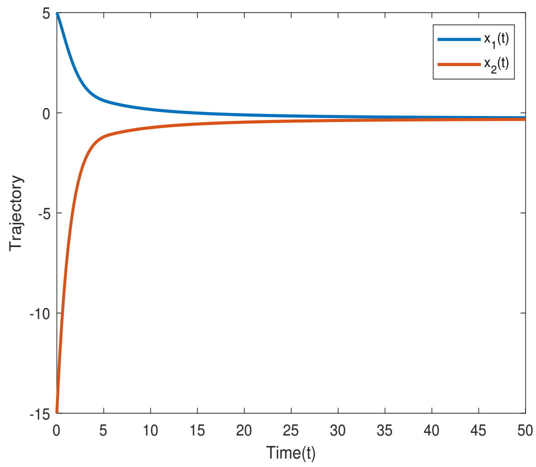

FIGURE 4.2: For $(x_1(0), x_2(0)) = (5, -15)$

FIGURE 4.3: Convergence of trajectories of dynamical system (4.3) for $\gamma(t) = 1/(t+5)$.

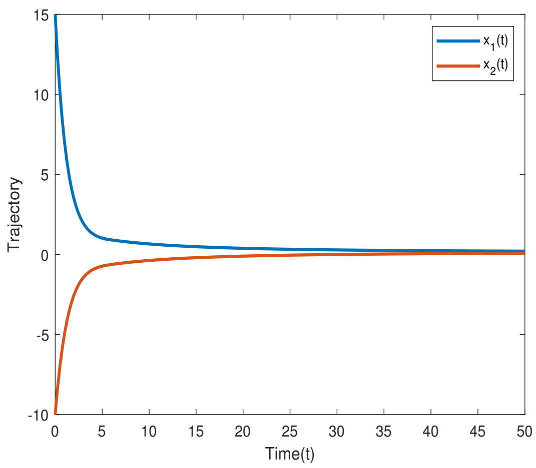

FIGURE 4.4: For $(x_1(0), x_2(0)) = (15, -10)$

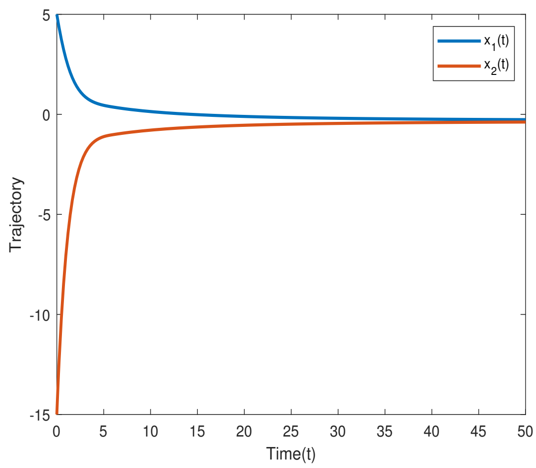

FIGURE 4.5: For $(x_1(0), x_2(0)) = (5, -15)$

FIGURE 4.6: Convergence of trajectories of the dynamical system (4.3) for $\gamma(t) = 1/(t+10)$.

4.6 Nash Equilibrium Problem

Prevalence of generalized Nash equilibrium problems (GNE) occur in diverse engineering applications, such as in demand-side management in the communication networks [85], charging/ discharging of electric vehicles [86], and smart grid [87]. In such examples, multiple selfish decision-makers or agents aspire to optimize their respective, yet inter-dependent, objective functions, concerned to coupled constraints. From a game-theoretic perspective, the objective is to design a distributed GNE algorithm employing accessible local information for each agent. Besides, in the framework of the cyber-physical system, agents play the games with their own dynamics [88, 89]. In this respect, each agent's strategy produces a dynamical system, and controllers must visualize physical processes moving to Nash equilibrium while assuring closed-loop stability. Thus, it is favorable to contemplate continuous-time methods, for which control-theoretic properties are effortlessly disentangled.

A plethora of different techniques has been presented to fetch GNE in a distributed manner [see [90, 91, 80]]. These works allude to a set of information and data, where every agent can access all other agents' decisions. A refinement of GNEs, whose solution is given by a variational inequality, constructed by gradients of the players' objective and common constraints, called the variational generalized Nash equilibrium (v-GNE) [92]. Variational inequality can be solved via operator splitting methods, which makes v-GNE computationally attractive [93, 11]. Franci et al. [80] studied the forward-backward-half forward algorithm to solve the Nash equilibrium problem associated with the monotone inclusion problem. The authors indicate that the algorithm is distributed so that each agent has information only to its local cost function and its local feasible set, but not the information about the central coordinator that updates and propagates dual variables. Continuous-time GNE problems sought for a network of single or double-integrator agents have been investigated in [94, 95]. In this context, in what follows, we develop a technique to solve above complex problem using our results discussed in section 4.4.

4.6.1 Notation

The set of real numbers (non-negative) is denoted by $\mathbb{R}$ ($\mathbb{R}_+$) and $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. $1_N(0_N)$ denotes the column vector of N ones (zeros). The induced inner product and induced norm by a positive definite ($\succ 0$), symmetric matrix Φ are $\langle \cdot, \cdot \rangle_\Phi := \langle \Phi \cdot, \cdot \rangle$, and $\| \cdot \|_\Phi := \langle \cdot, \cdot \rangle_\Phi^{1/2}$, respectively.

Let A be a matrix in $\mathbb{R}^{n\times m}$. The element at position i^{th} row and j^{th} column, and transpose are denoted by $[A]_{i,j}$, and A^T, respectively. The Kronecker product of the matrices A and B is represented by $A\otimes B$. Given N vectors $x_1, x_2, \ldots, x_N$, possibly of different dimensions, $x := \mathrm{col}(x_1, \ldots, x_N)$, and $x_{-i} := \mathrm{col}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$, for $i \in \{1, \ldots, N\}$.

Definition 4.6.1. For $x \in \mathbb{R}^n$ and a convex subset $S(\neq \emptyset)$ of $\mathbb{R}^n$, the normal cone and tangent cone operators to S at x are defined as

$$N_S(x) = \begin{cases} \{u \in \mathbb{R}^n : \sup\limits_{z\in S} u^T(z-x) \leq 0\}, \; if \; x \in S \\ \emptyset, \qquad otherwise, \end{cases}$$

and

$$T_S(x) = \begin{cases} \overline{\cup_{\delta>0} \frac{1}{\delta}(S-x)}, \; if \; x \in S \\ \emptyset, \qquad otherwise, \end{cases}$$

respectively.

Let $\Pi_S(x,u) := P_{T_S(x)}(u)$ be the projection of $u \in \mathbb{R}^n$ on the tangent cone of S at x. By Moreau's Decomposition theorem [11, Theorem 6.29] it gives that $u = P_{T_S(x)}(u) + P_{N_S(x)}(u)$ and $P_{T_S(x)}(u)^T P_{T_N(x)}(u) = 0$.

4.6.2 Mathematical Setup

Let $\mathcal{I} := \{1,\ldots,N\}$ denote the set of noncooperative agents, where every agent $i \in \mathcal{I}$ shall select its decision variable x_i from its local decision set $\Omega_i \subseteq \mathbb{R}^{n_i}$. The overall action space is represented by $\Omega = \times_{i\in\mathcal{I}}\Omega_i \subseteq \mathbb{R}^n$; $n = \sum_{i=1}^N n_i$. The stacked vector of all agents' outcomes and joint strategy of all the agents, excluding the agent i are denoted by $x = \mathrm{col}((x_i)_{i\in\mathcal{I}}) \in \Omega$, and $x_{-i} = \mathrm{col}((x_j)_{j\in\mathcal{I}-\{i\}})$, respectively.

The aim of each agent $i \in \mathcal{I}$ is to optimize its objective function $J_i(x_i, x_{-i})$ which depends on local variable x_i and decision variables of agents x_{-i}.

Now, considering the generalized games such that the agents are coupled by their feasible decision sets. We take affine coupling constraints, so the overall feasible set is given as

$$\mathcal{X} := \Omega \cap \{x \in \mathbb{R}^n | Ax \leq b\}, \tag{4.40}$$

with $A := [A_1,\ldots,A_N]$ and $b := \sum_{i=1}^N b_i$, where $A_i \in \mathbb{R}^{m\times n_i}$ and $b_i \in \mathbb{R}^m$ are the local data. Then, the game is denoted by the inter-dependent minimization problems:

$$\forall i \in \mathcal{I} : \begin{cases} \underset{y_i\in\mathbb{R}^{n_i}}{\operatorname{argmin}}\ J_i(y_i, x_{-i}), \\ \text{such that } (y_i, x_{-i}) \in \mathcal{X}. \end{cases} \tag{4.41}$$

In this work, we have considered the problem to find a generalized Nash equilibrium (GNE), which is defined as follows:

Definition 4.6.2. A collective strategy $x^* = \text{col}(x_1^*, \ldots, x_N^*) \in \mathcal{X}$ is a GNE of the game in (4.41), if

$$x_i^* \in \underset{y_i \in \mathbb{R}^{n_i}}{\operatorname{argmin}} \; J_i(y_i, x_{-i}^*) \; s.t. \; (y_i, x_{-i}^*) \in \mathcal{X}, \; for \; all \; i \in \mathcal{I}. \tag{4.42}$$

Now, we have the following assumptions on constraint sets and cost functions.

Assumption 4.6.1. (1) $\mathcal{X}(\neq \emptyset)$ satisfies Slater's constraint qualification;

(2) Ω_i is nonempty, closed and convex, for each $i \in \mathcal{I}$;

(3) J_i is continuously differentiable and for every x_{-i}, $J_i(\cdot, x_{-i})$ is convex.

For each agent i, Lagrangian of optimization problem (4.41) is

$$\mathcal{L}(x_i, \lambda_i; x_{-i}) = J_i(x_i, x_{-i}) + \lambda_i^T (Ax - b),$$

with dual variable $\lambda_i \in \mathbb{R}_+^m$. Let x_i^* be an optimal solution to (4.42), then $\exists \lambda_i^* \in \mathbb{R}_+^m$ such that Karush-Kuhn-Tucker conditions hold:

$$\begin{cases} 0_{n_i} = \nabla_{x_i} \mathcal{L}(x_i^*, \lambda_i^*; x_{-i}^*), x_i^* \in \Omega_i, i \in \mathcal{I}, \\ \langle \lambda_i^*, Ax^* - b \rangle = 0, -(Ax^* - b) \geq 0, \lambda_i^* \geq 0, \end{cases}$$

which implies that

$$\begin{cases} 0_{n_i} \in \nabla_{x_i} J_i(x_i^*; x_{-i}^*) + A_i^T \lambda_i^* + N_{\Omega_i}(x_i^*), i \in \mathcal{I}, \\ 0_m \in -(Ax^* - b) + N_{\mathbb{R}_+^m}(\lambda_i^*). \end{cases} \tag{4.43}$$

If Lagrangian multipliers are same for all i, then GNE is called variational GNE(v-GNE) [92]. A v-GNE of game (4.41), (4.40) is defined as $x^* \in \mathcal{X}$, a solution of variational inequality, given by

$$\langle F(x^*), x - x^* \rangle \geq 0 \ \forall x \in \mathcal{X}, \tag{4.44}$$

where F is pseudo-gradient of the game given by:

$$F(x) = \mathrm{col}(\nabla_{x_i} J(x_i, x_{-i}))_{i \in \mathcal{I}}. \tag{4.45}$$

The element $x^* \in \mathcal{X}$ solves (4.44) if and only if $\exists\ \lambda^* \in \mathbb{R}^m$ such that the Karush-Kuhn-Tucker conditions are fulfilled (see [93]),

$$\begin{cases} 0_n \in F(x^*) + A^T \lambda^* + N_\Omega(x^*), \\ 0_m \in -(Ax^* - b) + N_{\mathbb{R}^m_+}(\lambda^*). \end{cases} \tag{4.46}$$

Existence of the solution of (4.44) is assured by Assumption 4.6.1 (see [93]). From [92], if x^* is a solution of (4.44), so x^* with λ^* fulfills Karush-Kuhn-Tucker conditions (4.46), then x^* meets the Karush-Kuhn-Tucker conditions (4.43) with $\lambda_1^* = \lambda_2^* = \cdots = \lambda_n^* = \lambda^*$, therefore x^* is v-GNE of game (4.41). We need the following assumption on the The pseudo-gradient function F.

Assumption 4.6.2. The function F is κ-Lipschitz continuous and μ-strongly monotone.

4.6.3 Distributed Generalized Nash Equilibrium Seeking

This section is devoted to the game in (4.41), where each agent is related with a dynamical system:

$$\forall i \in I : \dot{x}_i = \Pi_{\Omega_i}(x_i, u_i), x_i(0) \in \Omega_i.$$

Our goal is to make the inputs u_i to obtain a v-GNE in fully distributed manner. Each agent i is allowed to know his local problem data, i.e., J_i, Ω_i, A_i and b_i. Let local decision x_i, a local copy $\lambda_i \in \mathbb{R}^m_+$ of dual variables, and a local auxiliary variable $z_i \in \mathbb{R}^m$ used to impose consensus of the dual variables are controlled by each agent i. Suppose that the agents interchange the data via an undirected weighted communication graph for getting consensus on the dual variables. The weighted adjacency matrix is denoted by $W = [w_{ij}]_{i,j} \in \mathbb{R}^{N\times N}$. We suppose $w_{ij} > 0$ if and only if (i, j) is an edge in the communication graph. Let $\mathcal{N}_i^j = \{j | w_{ij} > 0\}$ denote the set of neighbours of agent i in the graph.

Assumption 4.6.3. The matrix W is symmetric and irreducible.

The weighted Laplacian $\mathbf{L}$ is defined by $\mathbf{L} := \text{diag}\{d_1, \ldots, d_N\} - W$, where $d_i = \sum_{j=1}^N w_{ij}$. With Assumption 4.6.3, we have $\ker(\mathbf{L}) = \text{span}(1_N)$, $\mathbf{L}^T = \mathbf{L}$, and $\mathbf{L}$ is positive semi-definite with eigenvalues $0 = s_1 < s_2 \leq \cdots \leq s_n$, which are distinct and real. Furthermore, for the maximum degree of the graph $\mathcal{G}^\lambda$, we have $\Delta \leq s_n \leq 2\Delta$, where $\Delta := \max\{d_1, d_2, \ldots, d_N\}$. Define tensorized Laplacian $\overline{\mathbf{L}} := \mathbf{L} \otimes I_m$. Set $\bar{b} = \text{col}(b_1, \ldots, b_N)$, $x = \text{col}(x_1, \ldots, x_N)$ and similarly z and λ.

Supposing $\mathbf{A} = \mathrm{diag}(A_1, \ldots, A_N)$, define

$$\begin{cases} \mathcal{A}(x,z,\lambda) := N_\Omega \times \{0_{mN}\} \times N_{\mathbb{R}_+^{mN}}(\lambda), \\ \mathcal{B}_1(x,z,\lambda) := \mathrm{col}(F(x), 0_{mN}, \bar{\mathbf{L}}\lambda + \bar{b}), \\ \mathcal{B}_2(x,z,\lambda) := \mathrm{col}(\mathbf{A}^T\lambda, \bar{\mathbf{L}}\lambda, -\mathbf{A}x - \bar{\mathbf{L}}z). \end{cases} \tag{4.47}$$

Lemma 4.6.1. [90, 80] Let $\mathcal{A}, \mathcal{B}_1$ and $\mathcal{B}_2$ the operators defined by (4.47). Then the set $\mathrm{Zer}(\mathcal{A} + \mathcal{B}_1 + \mathcal{B}_2)$ is the set of v-GNE and it is non-empty.

Lemma 4.6.2. [90, 80] Let $\Phi \succ 0$ and F as in (4.45) satisfies Assumptions 4.6.1, 4.6.2, and (4.41). Let $\mathcal{A}, \mathcal{B}_1$ and $\mathcal{B}_2$ be the operators defined by (4.47). Then, we have the following:

(i) $\mathcal{B}_1$ is θ_{B_1}-cocoercive with $\theta_{B_1} \le \min\{1/2\Delta, \mu/\kappa^2\}$.

(ii) $\Phi^{-1}\mathcal{B}_1$ is $\alpha\theta_{B_1}$-cocoercive with $\alpha = 1/|\Phi^{-1}|$.

(iii) $\mathcal{B}_2$ is maximally monotone and $L_{\mathcal{B}_2} = (2|\mathbf{A}| + 2|\mathbf{L}|)$-Lipschitz continuous.

(iv)) $\mathcal{A}$ is maximally monotone.

Now, we introduce a distributed forward-backward-half forward dynamical system (FBHF) (4.49). In compact form, FBHF dynamical system can be written as:

$$\begin{cases} u(t) = J_{\Phi^{-1}\mathcal{A}}(I - \Phi^{-1}(\mathcal{B}_1 + \mathcal{B}_2))v(t), \\ \dot{v}(t) = u(t) - v(t) + \Phi^{-1}(\mathcal{B}_2 v(t) - \mathcal{B}_2 u(t)), \\ v(0) = v_0, \end{cases} \tag{4.48}$$

where $v(0) = v_0 = (x_0, z_0, \lambda_0) \in \Omega \times \mathbb{R}^{mN} \times \mathbb{R}_+^{mN}$ and Φ is the block-diagonal matrix with the step sizes:

$$\Phi := \mathrm{diag}(\rho^{-1}, \sigma^{-1}, \tau^{-1}).$$

Taking the coordinates as $u(t) = (\bar{x}(t), \bar{z}(t), \bar{\lambda}(t))$ and $v(t) = (x(t), z(t), \lambda(t))$, the updates are explicitly given in the following dynamical system. For each $i \in \mathcal{I}$,

$$\begin{cases} \bar{x}_i(t) = \Pi_{\Omega_i}(x_i(t), u_i(t)) \\ u_i(t) = -\rho_i(\nabla_{x_i(t)} J_i(x_i(t), x_{-i}(t)) - A_i^T \lambda_i(t) \\ \bar{z}_i(t) = z_i(t) + \sigma_i \sum_{j \in \mathcal{N}_i^\lambda} w_{ij}(\lambda_i(t) - \lambda_j(t)) \\ \bar{\lambda}_i(t) = \Pi_{\mathbb{R}_+^m}(\lambda_i(t), -\tau_i(A_i x_i(t) - b_i) + \tau \sum_{j \in \mathcal{N}_i^\lambda} w_{ij}[(z_i(t) - z_j(t)) - (\lambda_i(t) - \lambda_j(t)]) \\ \dot{x}_i(t) = \bar{x}_i(t) + \rho_i A_i^T(\lambda_i(t) - \bar{\lambda}_i(t)) \\ \dot{z}_i(t) = \bar{z}_i(t) + \sigma_i \sum_{j \in \mathcal{N}_i^\lambda} w_{ij}[(\lambda_i(t) - \lambda_j(t)) - \bar{\lambda}_i(t) - \bar{\lambda}_j(t)] \\ \dot{\lambda}_i(t) = \bar{\lambda}_i(t) + \tau_i(A_i \bar{x}_i(t) - x_i(t)) + \tau_i \sum_{j \in \mathcal{N}_i^\lambda} w_{ij}[(z_i(t) - z_j(t)) - (\bar{z}_i(t) - \bar{z}_j(t)] \\ x_i(0) \in \Omega_i, \lambda_i(0) \in \mathbb{R}_+^m \text{ and } z_i(0) \in \mathbb{R}^m. \end{cases} \tag{4.49}$$

To assure the convergence of dynamical system (4.49) to a v-GNE of the game in (4.41), we require the following assumption:

Assumption 4.6.4. $|\Phi^{-1}| \leq \min\{4\theta_{B_1}, 1/L_{B_2}\}$, with θ_{B_1} as in Lemma 4.6.2 and L_{B_2} as in Lemma 4.6.2.

Theorem 4.6.1. Let Assumptions 4.6.2 and 4.6.4 hold. The trajectory $(x(t), \lambda(t))$ generated by dynamical system (4.49) converges to $\mathrm{Zer}(\mathcal{A} + \mathcal{B}_1 + \mathcal{B}_2)$, and hence the primal variable converges to a v-GNE of the game in (4.41).

Proof. Note that one can write dynamical system (4.49) as dynamical system (4.48), whose convergence is guaranteed by Theorem 4.3.1 under the Assumption 4.6.4 because $\Phi^{-1}\mathcal{B}_1$ is cocoercive by Lemma 4.6.2. □

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