Chapter 2

Application of new strongly convergent iterative methods to split equality problems

2.1 Introduction

Censor and Elfving [40] had been the researchers to introduce the split feasibility problem **(SFP)** in finite-dimensional spaces in 1994. Such problems arise in signal processing, specifically in phase retrieval and other image restoration problems. It has been found that the SFP can also be used in different areas such as computer tomography and intensity-modulated radiation therapy [41, 42, 43].

The split feasibility problem (SFP) given by

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q, \tag{2.1}$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. Some works on split feasibility problems in an infinite-dimensional real Hilbert space can be found in [44, 42, 45].

In 2011, Censor et al. [46] have introduced the following split variational inequality problem:

$$\text{find } \; x^* \in C \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C,$$

and

$$y^* = Ax^* \in Q \text{ that solves } \langle g(y^*), y - y^* \rangle \geq 0 \text{ for all } y \in Q,$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ is a bounded linear operator and $f : H_1 \to H_1$, $g : H_2 \to H_2$ are the given operators.

In 2011, Moudafi [47] has extended the split variational inequality problem [46] and has proposed the following split monotone variational inclusion problem **(SMVIP)**:

$$\text{find } \; x^* \in H_1 \text{ such that } f(x^*) + B_1(x^*) \ni 0,$$

and

$$y^* = Ax^* \in H_2 \text{ that solves } g(y^*) + B_2(y^*) \ni 0, \tag{2.2}$$

where $B_i : H_i \to 2^{H_i}$, for $i = 1, 2$; are multi-valued mappings on the real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator and $f : H_1 \to H_1$, $g : H_2 \to H_2$ are two given single-valued operators. Also, an algorithm for finding the solution of SMVIP (2.2) was introduced and the weak convergence of the proposed algorithm was proved.

In 2013, Kazmi and Rizvi[48] have introduced the split variational inclusion problem **(SVIP)**:

$$\text{find } x^* \in H_1 \text{ such that } B_1(x^*) \ni 0,$$

and

$$y^* = Ax^* \in H_2 \text{ that solves } B_2(y^*) \ni 0, \tag{2.3}$$

where $B_i : H_i \to 2^{H_i}$, for $i = 1, 2$; are multi-valued mappings on the real Hilbert spaces and $A : H_1 \to H_2$ is a bounded linear operator. The problem (2.3) is a special case of split monotone variational inclusion problem. They have also proposed strongly convergent iterative method to find the common solution of split variational inclusion problem and fixed point problem.

In 2013, Moudafi [49] has introduced the following split equality problem **(SEP)**:

$$\text{find } x^* \in C \text{ and } y^* \in Q \text{ such that } Ax^* = By^*, \tag{2.4}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators and C, Q are nonempty closed convex subsets of real Hilbert spaces H_1, H_2, respectively, and H_3 is also a Hilbert space. Obviously, if $B = I$ and $H_2 = H_3$ then SEP reduces to SFP.

In 2014, Moudafi [50] has analyzed the following split equality fixed point problem **(SEFP)**:

$$\text{find } x^* \in \mathcal{F}(R_1) \text{ and } y^* \in \mathcal{F}(R_2) \text{ such that } Ax^* = By^*, \tag{2.5}$$

where $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators, and $R_i : H_i \to H_i$ for $i = 1, 2$ are two nonlinear operators such that $\mathcal{F}(R_1) \neq \emptyset$ and $\mathcal{F}(R_2) \neq \emptyset$. Also, he proposed iterative method for solving **SEFP**:

$$\begin{cases} x_{n+1} = R_1(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = R_2(y_n + \gamma_n B^*(Ax_{n+1} - By_n)) \ \forall n > 0, \end{cases}$$

where $\{\gamma_n\}$ is a positive non-decreasing sequence such that $\gamma_n \in \left(\epsilon, \min\left(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}\right) - \epsilon\right)$ for small enough $\epsilon > 0$, where λ_A and λ_B denotes the spectral radius of A^*A and B^*B respectively. In this iterative method, computation of the norm of operators used is required, which can be tedious task sometimes.

In 2015, to solve the split equality fixed point problem (2.5) for quasi-nonexpansive mappings, Zhao [51] has proposed the following iteration algorithm which does not require the computation of the operator norms:

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n u_n + (1-\beta_n)R_1 u_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \beta_n v_n + (1-\beta_n)R_2 v_n, \quad \forall n \geq 0, \end{cases}$$

where the step-size γ_n is chosen as follows:

$$\gamma_n \in \left(\epsilon, \frac{\beta_n \parallel Ax_n - By_n \parallel}{\parallel A^*(Ax_n - By_n) \parallel^2 + \parallel B^*(Ax_n - By_n) \parallel^2} - \epsilon\right), \ n \in \Pi.$$

Otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\Pi = \{n \in \mathbb{N} : Ax_n - By_n \neq 0\}$ and $\alpha_n \subset (\delta, 1-\delta)$ and $\beta_n \subset (\eta, 1-\eta)$ for small enough $\delta, \eta \geq 0$.

In 2016, Chang et al.[52] have introduced and studied the split equality variational inclusion problems in the setting of Banach spaces. The split equality variational inclusion problem **(SEVIP)** is defined as follows:

$$\text{find } x^* \in T_1^{-1}(0) \text{ and } y^* \in T_2^{-1}(0) \text{ such that } Ax^* = By^*, \tag{2.6}$$

where $T_i : H_i \to 2^{H_i}$, $i = 1, 2$ are maximal monotone operators, $A : H_1 \to X$ and $B : H_2 \to X$ are bounded linear operators. Here, H_i, $i = 1, 2$ are real Hilbert spaces and X is a real Banach space. If we consider $X = H_3$, where H_3 is a real Hilbert space, then the main result of Chang et al. [52] goes as follows:

Theorem 2.1.1. Denote $C_1 = H_1$, $Q_1 = H_2$. For given $x_1 \in C_1$ and $y_1 \in Q_1$, let the iterative sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} u_n = J_\lambda^{T_1}(x_n - \gamma_n A^*(Ax_n - By_n)), \\ v_n = J_\lambda^{T_2}(y_n + \gamma_n B^*(Ax_n - By_n)), \\ C_{n+1} \times Q_{n+1} = \{(x,y) \in C_n \times Q_n : \quad \| u_n - x \|^2 + \| v_n - y \|^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq \| x_n - x \|^2 + \| y_n - y \|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \\ y_{n+1} = P_{Q_{n+1}} x_1. \end{cases} \tag{2.7}$$

If the solution set $S := \{(p,q) \in H_1 \times H_2 : (p,q) \in T_1^{-1} \times T_2^{-1} \textit{ and } Ap = Bq\}$ of SEVIP (2.6) is nonempty and the following condition is satisfied

$$0 < \gamma_n < \frac{2}{\|A\|^2 + \|B\|^2},$$

then the sequence $\{(x_n, y_n)\}$ converges strongly to some point $(x^*, y^*) \in S$, where $\|A\|$ and $\|B\|$ are the norms of the operators A and B, respectively.

The inertial term has been first used to define the heavy ball method proposed by Polyak [53] to minimize the convex smooth function f, which is considered as a discretization of time dynamical system, given by

$$\ddot{x}(t) + \alpha_1 \dot{x}(t) + \alpha_2 \nabla f(x(t)) = 0,$$

where $\alpha_1(> 0)$ and $\alpha_2(> 0)$ are free model parameters of the equation. Inertial term gives the advantage to use two previous terms to define the next iterate of the algorithm, which in turn increases the convergence speed of the algorithm. This term was further used by Alvarez and Attouch [54] to define the inertial proximal point algorithm for solving the problem of finding zero of a maximal monotone operator

T, which is as follows:

$$x_{n+1} = J_{\lambda_n}^T(x_n + \theta_n(x_n - x_{n-1})),$$

where $J_{\lambda_n}^T$ is the resolvent of T with parameter $\lambda_n > 0$ and the inertia is induced by the term $\theta_n(x_n - x_{n-1})$, with $\theta_n \in [0, 1)$. Since their introduction one can notice an increasing interest in inertial algorithms having inertial term particularity, see [55, 56, 57].

In this chapter, the following problem has been considered

$$(\mathbf{P}) \qquad \text{find } z^* \in T^{-1}(0) \cap (\cap_{i=1}^m \mathcal{F}(R_i)) \text{ such that } F(z^*) = 0, \qquad (2.8)$$

where $F : H \to \mathbb{R}$ is a nonnegative lower semicontinuous (l.s.c.) function defined on H, $T : H \to 2^H$ is a maximal monotone operator and each $R_i : H \to H$, $i = 1, 2, \ldots, m$ is a quasi-nonexpansive mapping such that $\cap_{i=1}^m \mathcal{F}(R_i) \neq \emptyset$. Throughout the chapter, we assume that solution set of the problem **(P)** is denoted by Ω, i.e., $\Omega = \{z \in H : z \in T^{-1}(0) \cap (\cap_{i=1}^m \mathcal{F}(R_i)) \text{ and } F(z) = 0\}$.

One can see that problem **(P)** is unification of the following three problems:

(i) finding zero of nonnegative function F;

(ii) finding zero of set-valued operator T;

(iii) finding common fixed points of operators $R_1, R_2, \ldots, R_m$.

An important particular case of problem (**P**) is split equality variational inclusion fixed point problem which can be expressed as

$$\text{find } x^* \in T_1^{-1}(0) \cap (\cap_{i=1}^m \mathcal{F}(M_i))$$

and

$$y^* \in T_2^{-1}(0) \cap (\cap_{i=1}^m \mathcal{F}(N_i)) \text{ such that } Ax^* = By^*, \tag{2.9}$$

where $T_i : H_i \to 2^{H_i}$, for $i = 1, 2$ are maximal monotone operators, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ are bounded linear operators. For integers $1 \leq i \leq m$, $M_i : H_1 \to H_1$ and $N_i : H_2 \to H_2$ are two finite families of quasi-nonexpansive mappings.
If we suppose that $M_i = N_i = 0$, $\forall 1 \leq i \leq m$, then the split equality variational inclusion fixed point problem get converted to split equality variational inclusion problem, which has been studied earlier by Chang et al. [52] and Chuang [58]. Also, if we assume that $B = I$ and $H_3 = H_2$, then the above problem (2.9) gets converted to split variational inclusion fixed point problem, which has been studied by Majee et al. [59].

The main purpose of this chapter is to propose three iterative methods for solving problem **(P)** and to study the convergence analysis of the proposed iterative methods in a real Hilbert space setting. Our results unify some known results.

The remaining parts of this chapter are organized as follows: some lemmas and definitions required for proving main results are presented in section 2.2. Three iterative methods for solving problem **(P)** are introduced in section 2.3. Strong convergence of the proposed iterative methods are discussed in section 2.3. The applications of our results to the split equality variational inclusion fixed point problem and split equality equilibrium fixed point problem are given in section 2.4. Further, the efficiency of our iterative methods are demonstrated in section 2.5.

2.2 Preliminaries

Let $R : H \to H$ be a mapping. An element $z \in H$ is said to be a fixed point of R if $z = Rz$. We use $\mathcal{F}(R)$ to denote the set of all fixed points of R.

Definition 2.2.1. A map $R : H \to H$ is called

(i) nonexpansive if

$$\|Rx - Ry\| \leq \| x - y \| \text{ for all } x, y \in H,$$

(ii) quasi-nonexpansive if

$$\mathcal{F}(R) \neq \emptyset \text{ and } \|Rx - Rp\| \leq \| x - p \| \text{ for all } x \in H \text{ and } p \in \mathcal{F}(R).$$

(iii) demi-closed at zero if

$$\lim_{n\to\infty} \|z_n - Rz_n\| = 0 \text{ and } z_n \rightharpoonup z^* \text{ imply that } z^* = Rz^* \text{ for any sequence } \{z_n\} \in H.$$

Throughout this paper, the symbols $\mathbb{N}$ and $\mathbb{R}$ stand for the set of all natural numbers and set of real numbers, respectively. Also, we use the symbol I for the identity operator on H.

Lemma 2.2.1. [60, Lemma 1.1] Let H be a real Hilbert space. For each $x_1, x_2, \cdots, x_m \in H$ and $\alpha_1, \alpha_2, \cdots, \alpha_m \in [0, 1]$ with $\sum_{i=1}^{m} \alpha_i = 1$, the equality

$$\| \alpha_1 x_1 + \cdots + \alpha_m x_m \|^2 = \sum_{i=1}^{m} \alpha_i \| x_i \|^2 - \sum_{1 \leq i,j \leq m} \alpha_i \alpha_j \| x_i - x_j \|^2,$$

holds.

Lemma 2.2.2. [61, Lemma 2.5] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - a_n)s_n + a_n b_n \text{ for all } n \in \mathbb{N},$$

where $\{a_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that

(a) $\sum_{n=1}^{\infty} a_n = \infty$ and

(b) either $\limsup_{n\to\infty} b_n \leq 0$ or $\sum_{n=1}^{\infty} |a_n b_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

2.3 Iterative Schemes and Their Convergence

In this section, we introduce strongly convergent iterative schemes for finding the solution of problem **(P)**. We have the following assumptions on the operators F, T and R_i:

Assumption 2.3.1. (i) $F : H \to \mathbb{R}$ is a nonnegative lower semicontinuous function;

(ii) $T : H \to 2^H$ is a maximal monotone operator;

(iii) for each $i \in \{1, 2, ..., m\}$, $R_i : H \to H$ is a quasi-nonexpansive mapping.

Now, we introduce our iterative algorithms for solving the problem (**P**) as follows:

Algorithm 2.3.1. (1) Initialization: Denote $D_1 = H$ and select $z_1 \in D_1$ arbitrarily. (2) Iterative step: Select $\{\mu_n\}$ and $\{\delta_{i,n}\}$ as iteration parameters and compute the

$(n+1)^{th}$ iteration as follows:

$$\begin{cases} s_n = J_\lambda^T(z_n - \mu_n d_n), \\ t_n = \delta_{0,n} s_n + \sum\limits_{i=1}^{m} \delta_{i,n} R_i(s_n), \\ D_{n+1} = \{z \in D_n : \|t_n - z\|^2 \leq \|z_n - z\|^2\}, \\ z_{n+1} = P_{D_{n+1}} z_1, \qquad n \in \mathbb{N}, \end{cases} \tag{2.10}$$

where d_n is a search direction, $\lambda > 0$ and $\{\delta_{i,n}\}$ is a sequence such that $\delta_{i,n} \in (0,1), \liminf\limits_{n} \delta_{i,n} > 0, \sum\limits_{i=0}^{m} \delta_{i,n} = 1$. The step size μ_n is selected as follows:

$$\mu_n = \begin{cases} \dfrac{\beta_n F(z_n)}{\|d_n\|^2}, & if\ d_n \neq 0 \\ 0, & otherwise, \end{cases} \tag{2.11}$$

where $\beta_n \in (0,2)$.

Algorithm 2.3.2. (1) Initialization: Denote $D_1 = H$ and select $z_0, z_1 \in D_1$ arbitrarily.
(2) Iterative step: Select $\{\mu_n\}$ and $\{\delta_{i,n}\}$ as iteration parameters and compute the $(n+1)^{th}$ iteration as follows:

$$\begin{cases} w_n = z_n + \alpha_n(z_n - z_{n-1}), \\ s_n = J_\lambda^T(w_n - \mu_n d_n), \\ t_n = \delta_{0,n} s_n + \sum\limits_{i=1}^{m} \delta_{i,n} R_i(s_n), \\ D_{n+1} = \{z \in D_n : \|t_n - z\|^2 \leq \|z_n - z\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z, z_n - z_{n-1}\rangle\}, \\ z_{n+1} = P_{D_{n+1}} z_n, \qquad n \in \mathbb{N}, \end{cases} \tag{2.12}$$

where d_n is a search direction, $\lambda > 0$ and $\{\delta_{i,n}\}$ is a sequence such that $\delta_{i,n} \in (0,1)$, $\liminf_n \delta_{i,n} > 0$, $\sum\limits_{i=0}^{m} \delta_{i,n} = 1$. The step size μ_n is selected as (2.11). Also, $\alpha_n \in [0,\alpha]$ for some $\alpha \in [0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n \|z_n - z_{n-1}\| < \infty$.

Algorithm 2.3.3. (1) Initialization: Denote $D_1 = H$ and select $z_1 \in D_1$ arbitrarily.

(2) Iterative step: Select $\{\mu_n\}$ and $\{\delta_{i,n}\}$ as iteration parameters and compute the $(n+1)^{th}$ iteration as follows:

$$\begin{cases} s_n = J_\lambda^T(z_n - \mu_n d_n), \\ t_n = \delta_{0,n} s_n + \sum\limits_{i=1}^{m} \delta_{i,n} R_i(s_n), \\ D_{n+1} = \{z \in D_n : \|t_n - z\|^2 \leq \|z_n - z\|^2\}, \\ z_{n+1} = P_{D_{n+1}} z_n, \qquad n \in \mathbb{N}, \end{cases} \tag{2.13}$$

where d_n is a search direction, $\lambda > 0$ and $\{\delta_{i,n}\}$ is a sequence such that $\delta_{i,n} \in (0,1)$, $\liminf_n \delta_{i,n} > 0$, $\sum\limits_{i=0}^{m} \delta_{i,n} = 1$. The step size μ_n is selected as (2.11).

Remark 2.3.1.

1. In Algorithm 2.3.2, we have used two previous terms to define the next iterate of the algorithm, which in turn increases the convergence speed of the algorithm.

2. In Algorithm 2.3.3, projection of z_n is taken on the set D_{n+1} instead of z_1 to calculate the $(n+1)^{th}$ term of the algorithm.

3. By choosing $\alpha_n = 0$, Algorithm 2.3.2 get converted to Algorithm 2.3.3.

In order to establish the strong convergence of Algorithms 2.3.1, 2.3.2 and 2.3.3, we need the following assumptions:

(A0) $\langle d_n, z_n - z\rangle \geq F(z_n)$ for all $n \in \mathbb{N}$ and for all $z \in \Omega$;

(A1) $0 < \mu \leq \mu_n < \bar{\mu}$ for all $n \in \mathcal{I}$;

(A2) $\inf_{n\in\mathcal{I}}[\beta_n(2-\beta_n)] > 0$.

Here $\mathcal{I}$ denotes the index set $\{n \in \mathbb{N} : d_n \neq 0\}$.

Remark 2.3.2. Any vector $d_n \in \partial F(z_n)$is an example of direction satisfying (A0). Since, $F(z) = 0$, we have by definition of the subdifferential of a proper function that

$$F(z_n) + \langle d_n, z - z_n\rangle \leq 0,$$

and thus (A0) is satisfied. On the other hand, from the definition of μ_n and Assumption (A0), we easily observe if $n \notin \mathcal{I}$, then $d_n = 0, F(z_n) = 0, \mu_n = 0$, and $s_n = J_\lambda^T z_n$.

Before presenting our main results, we need the following proposition:

Proposition 2.3.1. Let H be a real Hilbert space and Assumptions 2.3.1 holds with $I - R_i$ being demi-closed at zero and $\Omega \neq \emptyset$. Assume that (A0) and (A2) hold. Let $\{z_n\}$ be the sequence generated by Algorithm 2.3.1 or Algorithm 2.3.3. Then $\Omega \subseteq D_n$, for all $n \in \mathbb{N}$.

Proof. Let z be any point in Ω. Here $z \in T^{-1}(0) = \mathcal{F}(J_\lambda^T) \subset H = D_1$. Hence, $z \in D_1$. If for some $n \geq 2, z \in D_n$, we show that $z \in D_{n+1}$. From (2.10), assumption (A2), and the fact that J_λ^T is firmly nonexpansive, we have

$$\begin{aligned}
\|s_n - z\|^2 &= \left\|J_\lambda^T(z_n - \mu_n d_n) - J_\lambda^T(z)\right\|^2 \\
&\leq \|z_n - \mu_n d_n - z\|^2 \\
&= \|z_n - z\|^2 + \mu_n^2 \|d_n\|^2 - 2\mu_n\langle z_n - z, d_n\rangle \qquad (2.14)\\
&= \|z_n - z\|^2 + \frac{\beta_n^2[F(z_n)]^2}{\|d_n\|^2} - 2\langle z_n - z, \frac{\beta_n[F(z_n)]}{\|d_n\|^2} d_n\rangle \\
&= \|z_n - z\|^2 - \frac{\beta_n[F(z_n)]}{\|d_n\|^2}[2\langle z_n - z, d_n\rangle - \beta_n F(z_n)] \\
&\leq \|z_n - z\|^2 - \frac{\beta_n[F(z_n)]}{\|d_n\|^2}[2F(z_n) - \beta_n F(z_n)] \\
&= \|z_n - z\|^2 - \beta_n(2 - \beta_n)\frac{[F(z_n)]^2}{\|d_n\|^2} \qquad (2.15)\\
&\leq \|z_n - z\|^2 . \qquad (2.16)
\end{aligned}$$

From (2.10) and Lemma 2.2.1, we have

$$
\begin{aligned}
\|t_n - z\|^2 &= \| \, \delta_{0,n} s_n + \sum_{i=1}^{m} \delta_{i,n} R_i(s_n) - z \, \|^2 \\
&= \| \, \delta_{0,n}(s_n - z) + \sum_{i=1}^{m} \delta_{i,n}(R_i(s_n) - z) \, \|^2 \\
&\leq \delta_{0,n} \, \|s_n - z\|^2 + \sum_{i=1}^{m} \delta_{i,n} \, \| \, (R_i(s_n) - R_i z) \, \|^2 - \sum_{1 \leq i \leq m} \delta_{0,n}\delta_{i,n}\|s_n - R_i(s_n)\|^2 \\
&\leq \delta_{0,n} \, \|s_n - z\|^2 + \sum_{i=1}^{m} \delta_{i,n} \, \| \, s_n - z \, \|^2 - \sum_{1 \leq i \leq m} \delta_{0,n}\delta_{i,n}\|s_n - R_i(s_n)\|^2 \\
&= \|s_n - z\|^2 - \delta_{0,n} \sum_{1 \leq i \leq m} \delta_{i,n}\|s_n - R_i(s_n)\|^2 \qquad (2.17) \\
&\leq \|s_n - z\|^2 \qquad (2.18) \\
&\leq \|z_n - z\|^2. \qquad (2.19)
\end{aligned}
$$

Hence, $z \in D_{n+1}$ and so $\Omega \subseteq D_{n+1}, \forall n \geq 1$. □

Now, we are ready to establish the strong convergence of Algorithm 2.3.1 for solving problem (**P**).

Theorem 2.3.1. Let H be a real Hilbert space and Assumptions 2.3.1 holds with $I - R_i$ being demi-closed at zero and $\Omega \neq \emptyset$. Assume that (A0)-(A2) hold. Let $\{z_n\}$ be the sequence generated by Algorithm 2.3.1. Then the sequence $\{z_n\}$ converges strongly to some point $z^* \in \Omega$.

Proof. Since $D_n, n \geq 1$ is a nonempty closed convex subset of H, therefore sequence $\{z_n\}$ is well defined.

We proceed the proof in the following steps:

Step 1: $\{z_n\}$ is Cauchy sequence.

By Proposition 2.3.1, we get $\Omega \subseteq D_{n+1}, \quad \forall n \geq 0, \quad D_{n+1} \subseteq D_n$ and $z_{n+1} = P_{D_{n+1}} z_1$.

Note that for any $z \in \Omega$,

$$\|z_{n+1} - z_1\| \leq \|z - z_1\|.$$

Hence, $\{z_n\}$ is a bounded sequence. Moreover, it follows from (2.10) that

$$\|z_n - z_1\| \leq \|z_{n+1} - z_1\|, \ \forall n \geq 1.$$

So, $\{\| z_n - z_1 \|\}$ is a convergent sequence.

Note that $z_k = P_{D_k} z_1, \forall k \geq 1$. By the definition of projection and by item (iii) of Lemma 1.2.1, we have

$$\begin{aligned} \| z_n - z_k \|^2 + \| z_k - z_1 \|^2 &= \| z_n - P_{D_k} z_1 \|^2 + \| P_{D_k} z_1 - z_1 \|^2 \\ &\leq \| z_n - z_1 \|^2, \end{aligned}$$

and so,

$$\lim_{n,k\to\infty} \| z_n - z_k \|^2 \leq \lim_{n\to\infty} \| z_n - z_1 \|^2 - \lim_{k\to\infty} \| z_k - z_1 \|^2 = 0,$$

which proves that $\{z_n\}$ is a Cauchy sequence in H.

Without loss of generality, we can assume that $z_n \to z^*$.

Step 2: $z^* \in \Omega$.

Since $z_{n+1} \in D_{n+1}$, it follows from (2.10) that

$$\| t_n - z_{n+1} \| \leq \| z_n - z_{n+1} \| .$$

Hence, $\lim_{n\to\infty} \| t_n - z_{n+1} \| = 0$ and so, $t_n \to z^*$.

Since for $z \in \Omega$, from (2.16) and (2.19), we have $\|t_n - z\|^2 \leq \|s_n - z\|^2 \leq \|z_n - z\|^2$, hence the sequences $\{\|s_n - z\|\}$, $\{\|t_n - z\|\}$ and $\{\|z_n - z\|\}$ have the same limit.

From (2.17), we have

$$\|t_n - z\|^2 \leq \|s_n - z\|^2 - \delta_{0,n} \sum_{1\leq i\leq m} \delta_{i,n}\|s_n - R_i(s_n)\|^2.$$

Let $\nu_i = \inf_{n\in\mathbb{N}} \delta_{i,n},\ \forall i \in \{0, 1, \dots, m\}$. Hence,

$$\nu_0 \sum_{1\leq i\leq m} \nu_i\|R_i(s_n) - s_n\|^2 \leq \|s_n - z\|^2 - \|t_n - z\|^2 \to 0,\ \text{ as } n \to \infty, \tag{2.20}$$

which implies that $\|R_i(s_n) - s_n\| \to 0$, as $n \to \infty$. From (2.15), we have

$$\lim_{n\to\infty} \beta_n(2 - \beta_n)\frac{[F(z_n)]^2}{\|d_n\|^2} \leq \lim_{n\to\infty} \|z_n - z\|^2 - \lim_{n\to\infty} \|s_n - z\|^2 = 0. \tag{2.21}$$

Hence,

$$\lim_{n\to\infty} \frac{[F(z_n)]^2}{\|d_n\|^2} = 0. \tag{2.22}$$

Also, since $0 < \mu \leq \mu_n = \beta_n \frac{F(z_n)}{\|d_n\|^2}$, for all $n \in \mathbb{N}$. So, $0 \leq \mu_n\|d_n\| = \beta_n \frac{F(z_n)}{\|d_n\|}$. Hence, from (2.22) and (A2), $\mu_n\|d_n\| \to 0$. So, $\|d_n\| \to 0$ as $\mu_n \geq \mu > 0$ and accordingly

$$F(z_n) = \frac{F(z_n)}{\|d_n\|}\|d_n\| \to 0 \text{ as } n \to \infty.$$

So, $F(z^*) = 0$, as F is a positive lower semicontinuous function and $z_n \to z^*$. Also,

$$\lim_{n\to\infty} \| z_n - s_n \| \leq \lim_{n\to\infty} \|z_n - t_n\| + \lim_{n\to\infty} \|t_n - s_n\| = 0. \tag{2.23}$$

Now,

$$\begin{aligned} \| z_n - J_\lambda^T z_n \| &\leq \| z_n - s_n \| + \| s_n - J_\lambda^T z_n \| \\ &= \| z_n - s_n \| + \| J_\lambda^T(z_n - \mu_n d_n) - J_\lambda^T z_n \| \\ &\leq \| z_n - s_n \| + \| \mu_n d_n \| \end{aligned}$$

$$\begin{aligned}
&= \| z_n - s_n \| + \left\| \frac{\beta_n F(z_n)}{\| d_n \|^2} d_n \right\| \\
&= \| z_n - s_n \| + \left\| \frac{\beta_n F(z_n)}{\|d_n\|^2} \right\| \|d_n\| \\
&= \| z_n - s_n \| + \left\| \frac{\beta_n F(z_n)}{\|d_n\|} \right\| .
\end{aligned}$$

So, from (2.22) and (2.23), we get that

$$\| z_n - J_\lambda^T z_n \| \to 0 \quad \text{as } n \to \infty.$$

Thus we have $z^* = J_\lambda^T z^*$.

Step 3: Next, we show that $z^* \in \mathcal{F}(R_i)$. Since $\lim_{n\to\infty} \|s_n - R_i(s_n)\| = 0$ and $s_n \to z^*$. Using the fact that $I - R_i$ is demi-closed, we get $z^* \in \mathcal{F}(R_i)$ (for each $i = 1, 2, \ldots, m$). Hence, $z^* \in \mathcal{F}(R_i)$, for each $i = 1, 2, \ldots, m$.

Therefore, we conclude that $z^* \in \Omega$ and $z_n \to z^*$. □

We now study the convergence analysis of Algorithm 2.3.2 for solving problem (**P**).

Theorem 2.3.2. Let H be a real Hilbert space and Assumptions 2.3.1 holds with $I - R_i$ being demi-closed at zero and $\Omega \neq \emptyset$. Assume that (A0)-(A2) hold. Let $\{z_n\}$ be the sequence generated by Algorithm 2.3.2. Then the sequence $\{z_n\}$ converges strongly to some point $z^* \in \Omega$.

Proof. We proceed the proof in the following steps:

Step 1: $\Omega \subseteq D_{n+1}$

For any $z \in \Omega$, we have $z \in T^{-1}(0) = \mathcal{F}(J_\lambda^T) \subset H = D_1$. Hence, $z \in D_1$. If for some $n \geq 2,\ z \in D_n$, we show that $z \in D_{n+1}$. From (2.12), and (2.11) we have

$$\begin{aligned}
\|s_n - z\|^2 &= \|J_\lambda^T(w_n - \mu_n d_n) - J_\lambda^T(z)\|^2 \\
&\leq \|w_n - \mu_n d_n - z\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|z_n + \alpha_n(z_n - z_{n-1}) - \mu_n d_n - z\|^2 \\
&= \|z_n - \mu_n d_n - z\|^2 + \alpha_n^2 \|z_n - z_{n-1}\| + 2\langle z_n - \mu_n d_n - z, \alpha_n(z_n - z_{n-1})\rangle \\
&= \|z_n - z\|^2 + \mu_n^2 \|d_n\|^2 - 2\langle z_n - z, \mu_n d_n\rangle + \alpha_n^2 \|z_n - z_{n-1}\|^2 \\
&\quad +2\alpha_n \langle z_n - z, z_n - z_{n-1}\rangle - 2\langle \mu_n d_n, \alpha_n(z_n - z_{n-1})\rangle \\
&= \|z_n - z\|^2 + \alpha_n^2 \|z_n - z_{n-1}\|^2 + 2\alpha_n \langle z_n - z, z_n - z_{n-1}\rangle \\
&\quad +\mu_n^2 \|d_n\|^2 - 2\langle \mu_n d_n, z_n - z + \alpha_n(z_n - z_{n-1})\rangle \qquad (2.24) \\
&= \|z_n - z\|^2 + \alpha_n^2 \|z_n - z_{n-1}\|^2 + 2\alpha_n \langle z_n - z, z_n - z_{n-1}\rangle \\
&\quad -\frac{\beta_n [F(z_n)]}{\|d_n\|^2}[2\langle w_n - z, d_n\rangle - \beta_n F(z_n)] \\
&\le \|z_n - z\|^2 + \alpha_n^2 \|z_n - z_{n-1}\|^2 + 2\alpha_n \langle z_n - z, z_n - z_{n-1}\rangle. \qquad (2.25)
\end{aligned}
$$

From (2.17), we have

$$
\begin{aligned}
\|t_n - z\|^2 &\le \|s_n - z\|^2 - \delta_{0,n} \sum_{1\le i\le m} \delta_{i,n} \|R_i(s_n) - s_n\|^2 \qquad (2.26) \\
&\le \|s_n - z\|^2. \qquad (2.27)
\end{aligned}
$$

From (2.25) and (2.27), we obtain

$$
\|t_n - z\|^2 \le \|z_n - z\|^2 + \alpha_n^2 \|z_n - z_{n-1}\|^2 + 2\alpha_n \langle z_n - z, z_n - z_{n-1}\rangle.
$$

By the definition of D_{n+1}, we get $z \in D_{n+1}$ and so $\Omega \subseteq D_{n+1}, \forall n \ge 1$.

Since $D_n, n \ge 1$ is a nonempty closed convex subset of H, therefore sequence $\{z_n\}$ is well defined sequence.

Step 2: $\{z_n\}$ is Cauchy sequence.

By Proposition 2.3.1, we get $\Omega \subseteq D_{n+1}, \quad \forall n \ge 0, \quad D_{n+1} \subseteq D_n$ and, from (2.12), $z_{n+1} = P_{D_{n+1}} z_n$.

Note that for any $z \in \Omega$,

$$\|z_{n+1} - z_1\| \leq \|z - z_1\|.$$

Hence, $\{z_n\}$ is a bounded sequence. Moreover, it follows from (2.12) that

$$\|z_n - z_1\| \leq \|z_{n+1} - z_1\|, \ \forall n \geq 1.$$

So, $\{\| z_n - z_1 \|\}$ is a convergent sequence.

Note that $z_k = P_{D_k} z_{k-1}, \forall k \geq 1$. By the definition of projection and by item (iii) of Lemma 1.2.1, we have

$$\begin{aligned} \| z_n - z_k \|^2 + \| z_k - z_1 \|^2 &= \| z_n - P_{D_k} z_{k-1} \|^2 + \| P_{D_k} z_{k-1} - z_1 \|^2 \\ &\leq \| z_n - z_1 \|^2, \end{aligned}$$

and so,

$$\lim_{n,k \to \infty} \| z_n - z_k \|^2 \leq \lim_{n \to \infty} \| z_n - z_1 \|^2 - \lim_{k \to \infty} \| z_k - z_1 \|^2 = 0,$$

which proves that $\{z_n\}$ is a Cauchy sequence in H.

Without loss of generality, we can assume that $z_n \to z^*$.

Step 3: $z^* \in \Omega$.

Since $\{z_n\}$ is a Cauchy sequence, we have

$$\|w_n - z_n\| = \alpha_n \|z_n - z_{n-1}\| \to 0, \text{ as } n \to \infty. \tag{2.28}$$

From (2.28), we get

$$\|w_n - z_{n+1}\| \leq \|w_n - z_n\| + \|z_{n+1} - z_n\| \to 0, \text{ as } n \to \infty. \tag{2.29}$$

From (2.25), we have

$$\|s_n - z\|^2 - \|z_n - z\|^2 \leq \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z, z_n - z_{n-1}\rangle \to 0, \text{ as } n \to \infty. \tag{2.30}$$

From (2.24) and (2.12), we deduce

$$\begin{aligned}\|s_n - z\|^2 &\leq \|z_n - z\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z, z_n - z_{n-1}\rangle \\ &+ \mu_n^2\|d_n\|^2 - 2\langle \mu_n d_n, z_n - z + \alpha_n(z_n - z_{n-1})\rangle \\ &\leq \|z_n - z\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z, z_n - z_{n-1}\rangle \\ &+ \beta_n^2\frac{[F(z_n)]^2}{\|d_n\|^2} - 2\mu_n F(z_n) \\ &\leq \|z_n - z\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z, z_n - z_{n-1}\rangle \\ &- \beta_n(2 - \beta_n)\frac{[F(z_n)]^2}{\|d_n\|^2},\end{aligned} \tag{2.31}$$

which implies that

$$\begin{aligned}\beta_n(2 - \beta_n)\frac{[F(z_n)]^2}{\|d_n\|^2} &\leq \|z_n - z\|^2 - \|s_n - z\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z, z_n - z_{n-1}\rangle \\ &\to 0, \text{ as } n \to \infty.\end{aligned} \tag{2.32}$$

Hence, $\lim\limits_{n\to\infty} \frac{[F(z_n)]^2}{\|d_n\|^2} = 0$. Also, since $0 < \mu \leq \mu_n = \beta_n \frac{F(z_n)}{\|d_n\|^2}$, for all n. So, $0 \leq \mu_n\|d_n\| = \beta_n \frac{F(z_n)}{\|d_n\|}$ which implies that $\mu_n\|d_n\| \to 0$. So, $\|d_n\| \to 0$ as $\mu_n \geq \mu > 0$ and accordingly

$$F(z_n) = \frac{F(z_n)}{\|d_n\|}\|d_n\| \to 0, \text{ as } n \to \infty.$$

Since F is a positive lower semicontinuous function and $z_n \to z^*$, it follows that $F(z^*) = 0$. Also,

$$\begin{aligned}\|t_n - s_n\| &= \|\delta_{0,n} s_n + \sum_{i=1}^{m} \delta_{i,n} R_{i,n}(s_n) - s_n\| \\ &\leq \delta_{0,n}\|s_n - s_n\| + \sum_{i=1}^{m} \delta_{i,n}\|R_{i,n}(s_n) - s_n\|.\end{aligned}$$

So, $\lim_{n\to\infty} \|t_n - s_n\| \to 0$. Since $z_{n+1} \in D_{n+1} \subset D_n$, from (2.29), we obtain

$$\begin{aligned}&\|w_n - s_n\| \\ &\leq \|w_n - z_{n+1}\| + \|t_n - s_n\| + \|t_n - z_{n+1}\| \\ &\leq \|w_n - z_{n+1}\| + \|t_n - s_n\| + \sqrt{\|z_n - z_{n+1}\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z_{n+1}, z_n - z_{n-1}\rangle} \\ &\leq \|w_n - z_{n+1}\| + \|t_n - s_n\| + \sqrt{\|z_n - z_{n+1}\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\|z_n - z_{n+1}\|\|z_n - z_{n-1}\|} \\ &\to 0, \text{ as } n \to \infty. \qquad (2.33)\end{aligned}$$

From (2.28) and (2.33), we have

$$\begin{aligned}\| z_n - J_\lambda^T z_n \| &\leq \|z_n - s_n\| + \|s_n - J_\lambda^T z_n\| \\ &= \|z_n - s_n\| + \| J_\lambda^T(w_n - \mu_n d_n) - J_\lambda^T z_n \| \\ &\leq \|z_n - w_n\| + \|w_n - s_n\| + \|\alpha_n(z_n - z_{n-1})\| + \|\mu_n d_n\| \\ &= \|z_n - w_n\| + \|w_n - s_n\| + \|\alpha_n(z_n - z_{n-1})\| + \left\|\frac{\beta_n F(z_n)}{\|d_n\|^2} d_n\right\| \\ &\leq \|z_n - w_n\| + \|w_n - s_n\| + \|\alpha_n(z_n - z_{n-1})\| + \left\|\frac{\beta_n F(z_n)}{\|d_n\|}\right\| \\ &\to 0, \text{ as } n \to \infty.\end{aligned}$$

So, we have $z^* = J_\lambda^T z^*$. As in the Theorem 2.3.1, we can see that $z^* \in \mathcal{F}(R_i)$, for each $i = 1, 2, \ldots, m$. Therefore, we conclude that $z^* \in \Omega$ and $z_n \to z^*$. □

Now with $\alpha_n = 0$, we obtain the following result by Theorem 2.3.2.

Theorem 2.3.3. Let H be a real Hilbert space, $F : H \to \mathbb{R}$ be a nonnegative lower semicontinuous function and $T : H \to 2^H$ be a maximal monotone operator. Suppose that for each $i \in \{1, 2, ..., m\}, R_i : H \to H$ is a quasi-nonexpansive mapping with $I - R_i$ demi-closed at zero and $\Omega \neq \emptyset$. Assume that (A0)-(A2) hold. Let $\{z_n\}$ be the sequence generated by Algorithm 2.3.3. Then the sequence $\{z_n\}$ converges strongly to some point $z^* \in \Omega$.

Remark 2.3.3. The value of $\|z_n - z_{n-1}\|$ is known before the value of α_n. Indeed, the parameters α_n can be chosen such that $0 \leq \alpha_n \leq \alpha'_n$, where

$$\alpha'_n = \begin{cases} \min\left\{\frac{\omega_n}{\|z_n - z_{n-1}\|}, \alpha\right\} & if\ z_n \neq z_{n-1}, \\ \alpha & otherwise, \end{cases} \tag{2.34}$$

where $\{\omega_n\}$ is a positive sequence such that $\sum_{n=1}^{\infty} \omega_n < \infty$.

2.4 Applications

2.4.1 Split Equality Variational Inclusion Fixed Point Problem

In this section first we investigate the split equality variational inclusion fixed point problems as an application.

Let H_1, H_2 and H_3 be Hilbert spaces. In particular, take $H = H_1 \times H_2$ and for any

$(x, y) \in H_1 \times H_2$, define the operators T, F and R_i as

$$\begin{cases} T(x,y) := T_1(x) \times T_2(y), \\ F(x,y) := \frac{1}{2}\|Ax - By\|^2, \\ R_i(x,y) := M_i(x) \times N_i(y), \text{ for each } i = 1, 2, \ldots, m, \end{cases} \tag{2.35}$$

where $T_i : H_i \to 2^{H_i}$, for $i = 1, 2$ are maximal monotone operators and $A : H_1 \to H_3$, $B : H_2 \to H_3$ are bounded linear operators. For integers $1 \leq i \leq m$, $M_i : H_1 \to H_1$ and $N_i : H_2 \to H_2$ are two finite families of set-valued quasi-nonexpansive operators such that

$$\bigcap_{i=1}^{m} \mathcal{F}(M_i) \neq \emptyset \quad \textit{and} \quad \bigcap_{i=1}^{m} \mathcal{F}(N_i) \neq \emptyset.$$

With the above setting, problem (**P**) becomes

$$\text{(SEVIFP)} \qquad \text{find } x \in \bigcap_{i=1}^{m} \mathcal{F}(M_i) \bigcap \, T_1^{-1}(0) \text{ and } y \in \bigcap_{i=1}^{m} \mathcal{F}(N_i) \bigcap T_2^{-1}(0) \\ \text{such that } Ax = By.$$

We assume that the search direction d_n coincides with the gradient $\nabla F(z_n)$ of the function F. So, we have the following result:

Theorem 2.4.1. Let H_1, H_2 and H_3 be real Hilbert spaces, $T_i : H_i \to 2^{H_i}$, for $i = 1, 2$ be maximal monotone operators, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators and for positive integers $1 \leq i \leq m$, $M_i : H_1 \to H_1$ and $N_i : H_2 \to H_2$ be two finite families of quasi-nonexpansive operators with $I - M_i$ and $I - N_i$ to be demi-close at zero. Let A^*, B^* be the adjoint of A, B, respectively. Denote $C_1 = H_1, Q_1 = H_2$. For a given $x_1 \in C_1$ and $y_1 \in Q_1$, let the iterative sequences

$\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} u_n = J_\lambda^{T_1}(x_n - \mu_n A^*(Ax_n - By_n)), \\ p_n = \delta_{0,n}u_n + \sum\limits_{i=1}^{m} \delta_{i,n}M_i(u_n), \\ v_n = J_\lambda^{T_2}(y_n + \mu_n B^*(Ax_n - By_n)), \\ q_n = \delta_{0,n}v_n + \sum\limits_{i=1}^{m} \delta_{i,n}N_i(v_n), \\ C_{n+1} \times Q_{n+1} = \{(x,y) \in C_n \times Q_n :\| p_n - x \|^2 + \| q_n - y \|^2 \\ \qquad\qquad\qquad\qquad \leq \| x_n - x \|^2 + \| y_n - y \|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_n, \\ y_{n+1} = P_{Q_{n+1}}y_n, \end{cases} \tag{2.36}$$

for all $n \in \mathbb{N}$, where $\{\delta_{i,n}\}$ is a sequence such that $\delta_{i,n} \in (0,1)$, $\sum\limits_{i=0}^{m} \delta_{i,n} = 1$. The step size μ_n is chosen in such a way that

$$\mu_n = \begin{cases} \dfrac{\beta_n F(x_n, y_n)}{\|\nabla F(x_n, y_n)\|^2}, & if\ \nabla F(x_n, y_n) \neq 0 \\ 0, & otherwise, \end{cases}$$

where $\beta_n \in (0,2)$ and $\inf_{n\in\mathbb{N}}[\beta_n(2-\beta_n)] > 0$.

If the solution set $\Omega_1 := \{(p,q) \in H_1 \times H_2 : p \in \bigcap_{i=1}^m \mathcal{F}(M_i) \bigcap T_1^{-1}(0), q \in \bigcap_{i=1}^m \mathcal{F}(N_i) \bigcap T_2^{-1}(0)\ and\ Ap = Bq\}$ is nonempty, then there exists $(x^*, y^*) \in \Omega_1$ such that $x_n \to x^*$ and $y_n \to y^*$.

Proof. Let the operators T, F and R_i be defined by (2.35). From Lemma 1.0.2, T is a maximal monotone operator. Here, function F is of class C^1 and for every $(x,y) \in H_1 \times H_2$, we have $\nabla F(x,y) = (A^*(Ax - By), -B^*(Ax - By))$. Here, R_i is a quasi-nonexpansive mapping such that $I - R_i$ is demiclosed at 0, for each $i = 1, 2, \ldots, m$.

Condition (A0) and (A1) follow from Definition 1.2.2 and the fact that $d_n = \nabla F(x,y) =$

$(A^*(Ax - By), -B^*(Ax - By))$. Hence, from Theorem 2.3.3, we conclude the proof. □

2.4.2 Split Equality Equilibrium Fixed Point Problem

Let C be a nonempty closed and convex subset of a real Hilbert space H and $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for f is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \qquad \forall y \in C. \tag{2.37}$$

The solution set of the equilibrium problem is denoted by EP(f).

Recently, many authors (see, e.g.[62, 63, 64]) have studied strong convergence of iterative schemes for finding a common solution of an equilibrium problem and fixed point problem for a nonlinear mapping .

Let us assume that the bifunction f satisfies the following conditions:

$(B1)$ $f(x, x) = 0, \quad \forall x \in C$,

$(B2)$ f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$, $\forall x, y \in C$,

$(B3)$ $\lim_{t\to 0} f(tz + (1-t)x, y) \leq f(x, y)$, for each $x, y, z \in C$,

$(B4)$ for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Further, we quote the following lemma:

Lemma 2.4.1. [65, Theorem 4.2] Let C be a nonempty closed and convex subset of a Hilbert space H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying $(B1) - (B4)$. Let Φ_f be a set-valued mapping of H into itself defined by

$$\Phi_f(x) = \begin{cases} \{z \in C : f(z, y) + \frac{1}{\lambda}\langle\, y - z, z - x\rangle \;\geq 0\}, & \forall x \in C \\ \emptyset, & \forall x \notin C. \end{cases} \tag{2.38}$$

Then $EP(f) = \Phi_f^{-1}(0)$ and Φ_f is a maximal monotone operator with $\mathcal{D}\Phi_f \subset C$. Furthermore, for any $x \in H$ and $\lambda > 0$, the resolvent G_λ^f of f coincides with the resolvent of Φ_f, where

$$G_\lambda^f x = \{z \in C : f(z,y) + \frac{1}{\lambda}\langle\, y - z, z - x\rangle \ \geq 0 \ \ \forall y \in C\}.$$

The so called *Split equality equilibrium fixed point problem* with respect to bi-function f and g is to find $x \in C$ and $y \in Q$ such that

$$\text{(SEEFP)} \qquad \text{find } x \in \bigcap_{i=1}^m \mathcal{F}(M_i) \bigcap EP(f) \text{ and } y \in \bigcap_{i=1}^m \mathcal{F}(N_i) \bigcap EP(g)$$
$$\text{such that } Ax = By.$$

Using Lemma 2.4.1 and Theorem 2.4.1, we have the following result.

Theorem 2.4.2. Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be two nonempty closed convex subset of H_1 and H_2, respectively, and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let $f : C \times C \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying $(B1) - (B4)$. Suppose that for each $i \in \{1, 2, ..., m\}, M_i : H_1 \to H_1$ and $N_i : H_2 \to H_2$ be quasi-nonexpansive operators with $I - M_i$ and $I - N_i$ are demi-close at zero. For a given $x_1 \in C_1$ and $y_1 \in Q_1$, let the iterative sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} u_n = G_\lambda^f(x_n - \mu_n A^*(Ax_n - By_n)), \\ p_n = \delta_{0,n} u_n + \sum\limits_{i=1}^{m} \delta_{i,n} M_i(u_n), \\ v_n = G_\lambda^g(y_n + \mu_n B^*(Ax_n - By_n)), \\ q_n = \delta_{0,n} v_n + \sum\limits_{i=1}^{m} \delta_{i,n} N_i(v_n), \\ C_{n+1} \times Q_{n+1} = \{(x, y) \in C_n \times Q_n :\| p_n - x \|^2 + \| q_n - y \|^2 \\ \qquad\qquad\qquad\qquad \leq \| x_n - x \|^2 + \| y_n - y \|^2 \}, \\ x_{n+1} = P_{C_{n+1}} x_n, \\ y_{n+1} = P_{Q_{n+1}} y_n, \end{cases} \tag{2.39}$$

for all $n \in \mathbb{N}$. Let the sequences $\{\delta_{i,n}\}$ and $\{\mu_n\}$ satisfy the condition of Theorem 2.4.1. If the solution set $\Omega_2 := \{(p, q) \in H_1 \times H_2 : p \in \bigcap_{i=1}^m \mathcal{F}(M_i) \bigcap EP(f), q \in \bigcap_{i=1}^m \mathcal{F}(N_i) \bigcap EP(g)$ *and* $Ap = Bq\}$ is nonempty, then there exists $(x^*, y^*) \in \Omega_2$ such that $x_n \to x^*$ and $y_n \to y^*$.

2.5 Numerical Experiments

In this section, we discuss some examples in support of Theorems 2.3.1, 2.3.2, 2.3.3, 2.4.1 and 2.4.2. We have implemented our code in Python 2.7 (Anaconda) on a personal Dell computer with Intel(R)Core(TM) i5-7200U CPU 2.50GHz and RAM 8.00 GB.

2.5.1 Test Problem for Problem (P)

Example 2.5.1. Let $H = \mathbb{R}^N$, $N \in \mathbb{N}$, be a real Hilbert space. Let $z = (x_1, x_2, \cdots, x_N)$ and $F : H \to \mathbb{R}$ be a function defined by, $F(z) = \|z\|^2$. Let $L : H \to H$ be an operator defined by

$$L[x_1, \cdots, x_N] = \begin{bmatrix} \frac{1}{2N} & 0 & \cdots \\ \vdots & \ddots & 0 \\ 0 & 0 & \frac{1}{2N} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}.$$

Note that L is a nonexpansive operator. Hence, by Example 1.0.1, $T = (I + \frac{1}{2}L)$ is a maximal monotone operator.

For $i = 1, 2, \ldots, m$, $R_i : H \to H$ is defined by

$$R_i(x_1, x_2, \cdots, x_N) = (R_{i_1}(x_1), R_{i_2}(x_2), \cdots, R_{i_N}(x_N)),$$

where

$$R_{i_j}(x_j) = \begin{cases} 0, & if\ x_j = 0, \\ \frac{x_j}{i+1} \sin \frac{1}{x_j}, & if\ x_j \neq 0, \end{cases} \tag{2.40}$$

for $j = 1, 2, \ldots, N$. Here, each R_{i_j} is quasi-nonexpansive operator with $\mathcal{F}(R_{i_j}) = \{0\}$. Also suppose that $\lambda = 2.5$, $\alpha = 0.3$, $\omega_n = \frac{1}{n^2}$, $\beta_n = \frac{n}{n+1}$, $\delta_{i,n} = \frac{1}{m+1}, \forall i = 0, 1, 2, \ldots, m$ and search direction $d_n = \nabla F(z_n)$. Observe that all the assumptions of Theorems 2.3.1, 2.3.2 and 2.3.3 are satisfied. Consequently, we conclude that sequence $\{z_n\}$ converges strongly to $z^* = (0, 0) \in \Omega$.

For stopping criteria $\|z_{n+1} - z_n\| < \epsilon = 10^{-4}$, Figures 2.1, 2.2, 2.3 and 2.6, 2.7, 2.8 show the convergence of sequence $\{\|z_{n+1} - z_n\|\}$ for different values of $z_1 \in \mathbb{R}^7$ and $z_1 \in \mathbb{R}^{25}$ using Algorithms 2.3.1, 2.3.3, 2.3.2, respectively. Figure 2.4 shows

	Alg 2.3.1 $z_1 = u_1$	Alg 2.3.2 $z_0 = z_1 = u_1$	Alg 2.3.3 $z_1 = u_1$	Alg 2.3.1 $z_1 = v_1$	Alg 2.3.2 $z_0 = z_1 = v_1$	Alg 2.3.3 $z_1 = v_1$
CPU time (in second)	11.254	0.565	0.798	11.093	0.372	1.227
number of iterations	181	19	35	180	17	47

TABLE 2.1: CPU time and number of iterations for Algorithms 2.3.1, 2.3.2, 2.3.3 using Example 2.5.1 for $z_1 = u_1 \in \mathbb{R}^7$ and $z_1 = v_1 \in \mathbb{R}^7$.

	Alg 2.3.1 $z_1 = u_1'$	Alg 2.3.2 $z_0 = z_1 = u_1'$	Alg 2.3.3 $z_1 = u_1'$	Alg 2.3.1 $z_1 = v_1'$	Alg 2.3.2 $z_0 = z_1 = v_1'$	Alg 2.3.3 $z_1 = v_1'$
CPU time (second)	3011.928	4.731	36.224	2873.379	3.186	46.752
number of iterations	1071	22	85	1021	27	106

TABLE 2.2: CPU time and number of iterations for Algorithms 2.3.1, 2.3.2, 2.3.3 using Example 2.5.1 for $z_1 \in \mathbb{R}^{25}$ and $z_1 \in \mathbb{R}^{25}$.

the convergence of sequence $\{\|z_{n+1} - z_n\|\}$ for different values of $\alpha \in [0, 1)$ and $z_0 = z_1 = (.23, .4, .6, .52, .7, .8, .7) \in \mathbb{R}^7$ using Algorithm 2.3.2.

Note that for Table 2.2 and Figure 2.9, $z_1 = u_1' = (1.2, .8, .6, .9, .7, 1, .8, .4, .8, .6, .2, .3, .4, .33, .6, 1.2, .35, .47, .8, .6, .5, .8, .4, .7, .3) \in \mathbb{R}^{25}$ and $z_1 = v_1' = (1.2, .5, .8, .7, .8, .3, .6, .2, .7, .3, .1, .2, .3, .23, .2, .1, .15, .17, .5, .4, .3, .6, .7, .1, .4) \in \mathbb{R}^{25}$.

Remark 2.5.1. (i) We observe from Example 2.5.1 that Algorithm 2.3.2 has better performance than Algorithms 2.3.1 and 2.3.3.

(ii) From Figures 2.5 and 2.9, we observe that, when we increase the dimension of the Euclidean space, Algorithm 2.3.2 is stable (approximate the solution after same number of iterations), but Algorithms 2.3.1 and 2.3.3 are not stable.

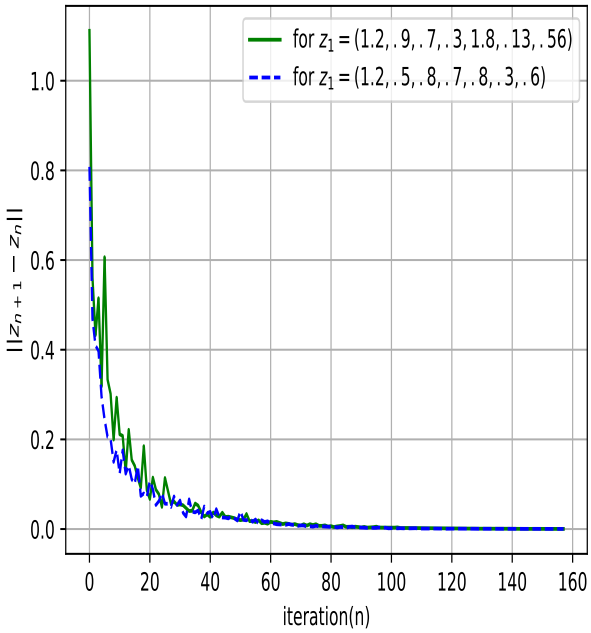

FIGURE 2.1: For Algorithm 2.3.1

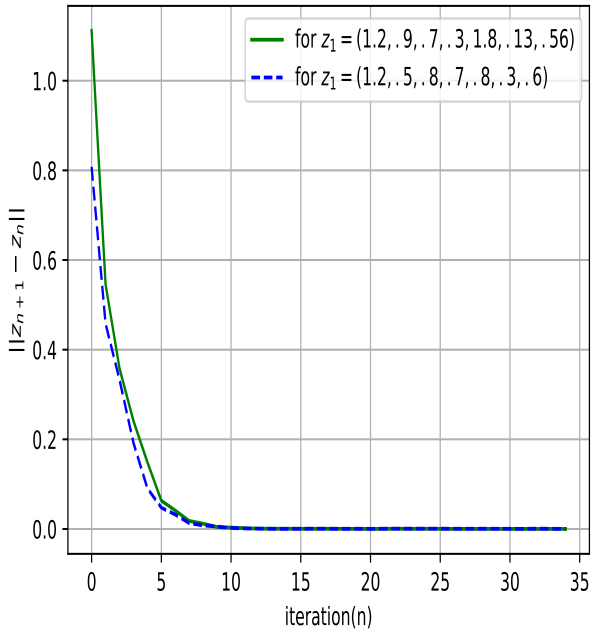

FIGURE 2.2: For Algorithm 2.3.3

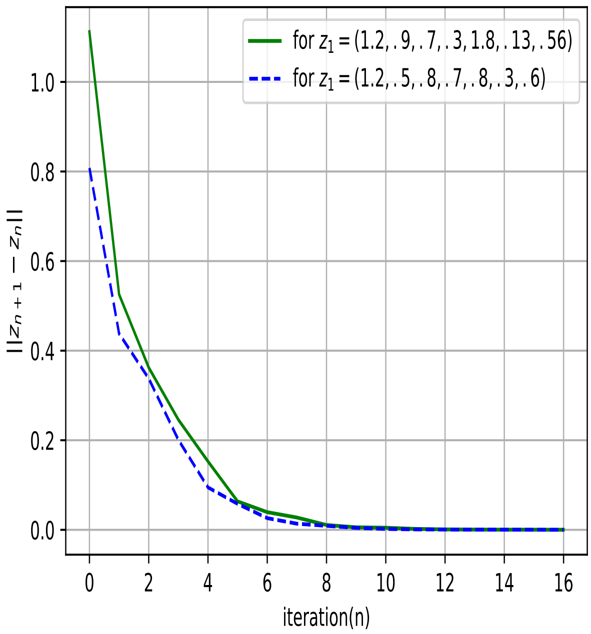

FIGURE 2.3: For Algorithm 2.3.2

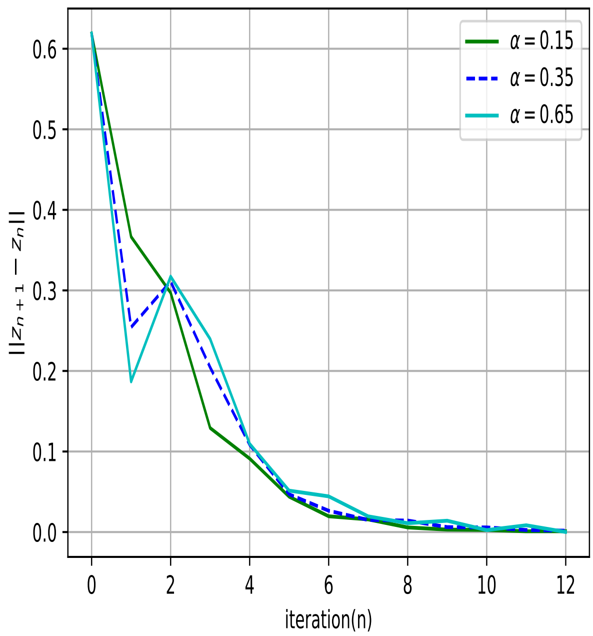

FIGURE 2.4: For Algorithm 2.3.2 for different values of α

FIGURE 2.5: Convergence of sequence $\{\|z_{n+1} - z_n\|\}$ for Example 2.5.1 for $z_1 \in \mathbb{R}^7$

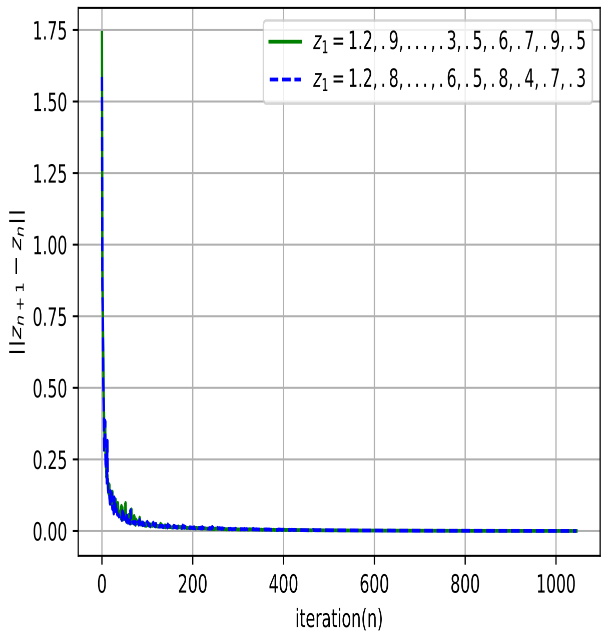

FIGURE 2.6: For Algorithm 2.3.1

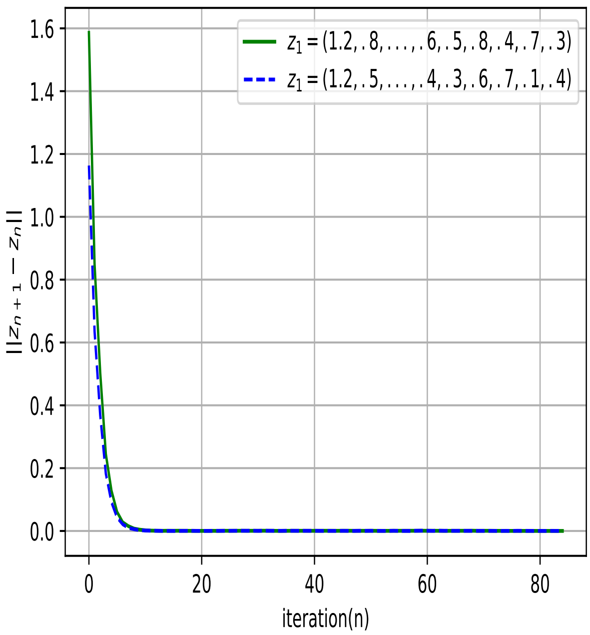

FIGURE 2.7: For Algorithm 2.3.3

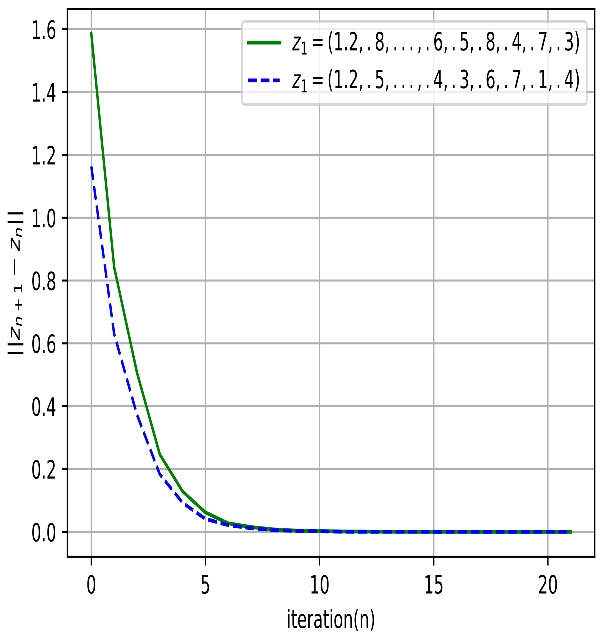

FIGURE 2.8: For Algorithm 2.3.2

FIGURE 2.9: Convergence of sequence $\{\|z_{n+1} - z_n\|\}$ for Example 2.5.1 for $z_1 = u'_1 \in \mathbb{R}^{25}$ and $z_1 = v'_1 \in \mathbb{R}^{25}$

2.5.2 Test Problem for Split Equality Variational Inclusion Fixed Point Problem

Example 2.5.2. In Theorem 2.4.1, set $H_1 = H_2 = H_3 = \mathbb{R}^N$, $N \in \mathbb{N}$. Let $Ax = x, By = 4y$, where $x = (x_1, x_2, \cdots, x_N)$ and $y = (y_1, y_2, \cdots, y_N)$. Let $L_1 : H \to H$ be an operator defined by

$$L_1[x_1, \cdots, x_N] = \begin{bmatrix} \frac{1}{2} & 0 & \cdots \\ \vdots & \ddots & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

and $L_2 : H \to H$ be an operator defined by

$$L_2[x_1, \cdots, x_N] = \begin{bmatrix} \frac{1}{3} & 0 & \cdots \\ \vdots & \ddots & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix},$$

which are nonexpansive operators. Hence, by Example 1.0.1, $T_i = (I + \frac{1}{2}L_i)$, for $i = 1, 2$ are maximal monotone operators. Let $M_i : H \to H$, for $i = 1, 2, \ldots, m$ be defined as

$$M_i(x_1, x_2, \cdots, x_N) = (M_{i_1}(x_1), M_{i_2}(x_2), \cdots, M_{i_N}(x_N)),$$

where

$$M_{i_j}(x_j) = \begin{cases} 0, & if\ x_j = 0, \\ \frac{x_j}{i+1} \sin \frac{1}{x_j}, & if\ x_j \neq 0, \end{cases} \tag{2.41}$$

for $j = 1, 2, \ldots, N$. Also, suppose that $\lambda = 2.5$, $\beta_n = \frac{n}{n+1}$ and $\delta_{i,n} = \frac{1}{m+1}, \forall i = 0, 1, 2, \ldots, m$.

No. of itr n	$\|\|x_{n+1} - x_n\|\|$ for $x_1 = (0.3, 0.4, 0.1, 0.5, 0.3, 0.4, 0.8)$	$\|\|y_{n+1} - y_n\|\|$ for $y_1 = (0.2, 0.5, 0.2, 0.6, 0.9, 0.4, 0.2)$
1	0.441459366509	0.670736200741
2	0.314515785474	0.335249956424
3	0.199682315695	0.159382646293
4	0.122192308631	0.0743614822021
5	0.0596315388329	0.0346232671892
6	0.0295076103278	0.0161793158314
7	0.0138869344931	0.00767955223807
8	0.00744587236442	0.0036488401429
9	0.005087552239	0.00215841680041
10	0.00299508323333	0.000845020212871
11	0.00174528964407	0.000498184762847
12	0.00116160902035	0.000182465498299
13	0.000757512055394	0.000225281371797
14	2.88242408366e-08	1.94312998658e-08
CPU time (second)	0.318000078201	0.318000078201

TABLE 2.3: Numerical values for $\|x_{n+1} - x_n\|$ and $\|y_{n+1} - y_n\|$ using Theorem 2.4.1 and Example 2.5.2

Let $N_i : H \to H$, for $i = 1, 2, \dots, m$ be defined by

$$N_i(x_1, x_2, \cdots, x_N) = (N_{i_1}(x_1), N_{i_2}(x_2), \cdots, N_{i_N}(x_N)),$$

where

$$N_{i_j}(x_j) = \begin{cases} 0, & if \ \| x_j \| \leq 1, \\ (1 - \frac{1}{(i+1)\|x_j\|})x_j, & if \ \| x_j \| > 1, \end{cases} \tag{2.42}$$

for $j = 1, 2, \dots, N$. Here, each M_{i_j} and N_{i_j} are quasi-nonexpansive mappings. Observe that all the assumptions of Theorem 2.4.1 are satisfied. So, we conclude that sequence $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) = (0, 0) \in \Omega_1$.

For stopping criteria $\|x_{n+1} - x_n\| < \epsilon = 10^{-4}$ and $\|y_{n+1} - y_n\| < \epsilon = 10^{-4}$, Figure 2.10 and Table 2.3 show the convergence of sequences $\{\|x_{n+1} - x_n\|\}$ and $\{\|y_{n+1} - y_n\|\}$ using Theorem 2.4.1. Table 2.4 and Figure 2.13 show the comparison between the convergence of algorithm of Theorem 2.4.1 and algorithm of Theorem 2.1.1 [52].

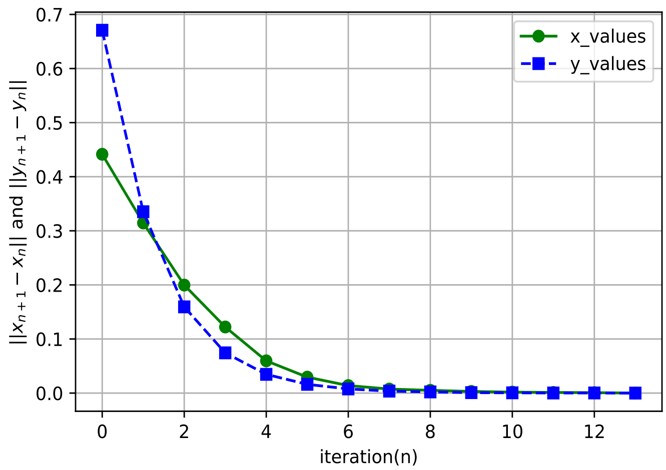

FIGURE 2.10: Convergence of sequences $\{\|x_{n+1} - x_n\|\}$ and $\{\|y_{n+1} - y_n\|\}$ for Example 2.5.2

	Algorithm 2.3.3 $\|\|x_{n+1} - x_n\|\|$	Algorithm 2.3.3 $\|\|y_{n+1} - y_n\|\|$	Theorem 2.1.1 $\|\|x_{n+1} - x_n\|\|$	Theorem 2.1.1 $\|\|y_{n+1} - y_n\|\|$
No. of itr n	13	13	23	23
CPU time (second)	0.631999969482	0.631999969482	8.0529999733	8.0529999733

TABLE 2.4: CPU time and number of iteration for $\|x_{n+1} - x_n\|$ and $\|y_{n+1} - y_n\|$ using Theorem 2.4.1 (with $M_i = N_i = 0$, for each i) and Theorem 2.1.1 based on Example 2.5.2 for $x_1 = (.5, .4, .7, .5, .8, .4, .8, .6, .4, .9) \in \mathbb{R}^{10}$ and $y_1 = (.6, .5, .8, .6, .9, .4, 0.7, .5, .7, .4) \in \mathbb{R}^{10}$

2.5.3 Test Problem for Split Equality Equilibrium Fixed Point Problem

Example 2.5.3. Let $H_1 = H_2 = H_3 = \mathbb{R}$ and $C = Q = [0, \infty)$, and define the bifunctions $f : C \times C \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ by

$$f(x, y) = y^2 + xy - 2x^2, \qquad g(x, y) = x(y - x).$$

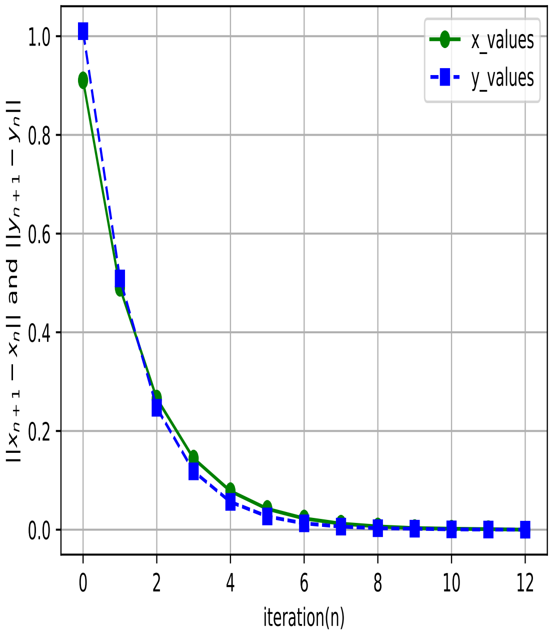

FIGURE 2.11: For Theorem 2.4.1 with $M_i = N_i = 0$, for each i

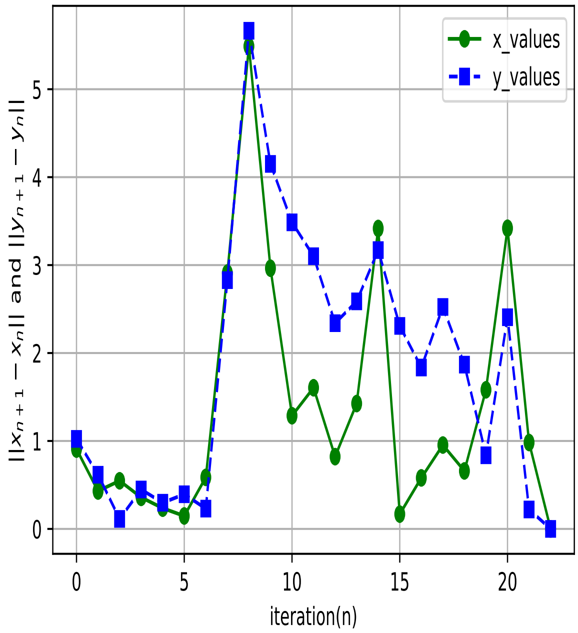

FIGURE 2.12: For Theorem 2.1.1

FIGURE 2.13: Convergence of sequences $\{\|x_{n+1} - x_n\|\}$ and $\{\|y_{n+1} - y_n\|\}$ based on Example 2.5.2

We observe that the functions f and g satisfy the conditions $(B1) - (B4)$. Also, we have $G^f_\lambda x = \frac{x}{3\lambda+1}$ and $G^g_\lambda x = \frac{x}{\lambda+1}$. Let $Ax = x, By = 4y$. Let $M_i : H \to H$, for $i = 1, 2, \ldots, m$ be defined by

$$M_i(x) = \begin{cases} 0, & if\ x = 0, \\ \frac{x}{i+1} \sin \frac{1}{x}, & if\ x \neq 0. \end{cases} \tag{2.43}$$

Also, suppose that $\lambda = 1$, $\beta_n = \frac{n}{n+1}$ and $\delta_{i,n} = \frac{1}{m+1}, \forall i = 0, 1, 2, \ldots, m$. Let $N_i : H \to H$, for $i = 1, 2, \ldots, m$ be defined by

$$N_i(x) = \begin{cases} 0, & if\ \ |x| \leq 1, \\ (1 - \frac{1}{(i+1)|x|})x, & if\ \ |x| > 1. \end{cases} \tag{2.44}$$

Number of iteration n	$\|\|x_{n+1} - x_n\|\|$ for $x_1 = 1.5$	$\|\|y_{n+1} - y_n\|\|$ for $y_1 = 1.3$
1	0.5	0.918007096641
2	0.330035960122	0.161488666253
3	0.20922616607	0.08583619912
4	0.153955532019	0.0510805504004
5	0.138720172321	0.0314529235324
6	0.080385051789	0.0196247393265
7	0.0484058981149	0.0122275565211
8	0.0155829454539	0.00762143353299
9	0.0132753732442	0.00472982657387
10	0.00326142540624	0.00300107406444
11	0.00222545130171	0.00186115299614
12	0.00158178377996	0.00115021857958
13	0.00171855240509	0.000649649786976
14	0.00084490137982	0.000450952075506
15	0.000419623863243	0.000313142969688
16	0.000166003354156	0.000222866067994
17	5.81580335432e-05	0.000129818654887

TABLE 2.5: Numerical values for $\|x_{n+1} - x_n\|$ and $\|y_{n+1} - y_n\|$ using Theorem 2.4.2 and Example 2.5.3

Here, each M_i and N_i are quasi-nonexpansive mappings. Observe that all the assumptions of Theorem 2.4.2 are satisfied. So, we conclude that sequence $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) = (0, 0) \in \Omega_1$.

For stopping criteria $\|x_{n+1}-x_n\| < \epsilon = 10^{-4}$ and $\|y_{n+1}-y_n\| < \epsilon = 10^{-4}$, Figure 2.14 and Table 2.5 show the convergence of sequences $\{\|x_{n+1} - x_n\|\}$ and $\{\|y_{n+1} - y_n\|\}$ using Theorem 2.4.2. The CPU time is 0.0920000076294.

2.6 Conclusion

In this chapter, the minimization of a nonnegative lower semicontinuous function over the intersection of a finite number of fixed point sets and a zero set has been

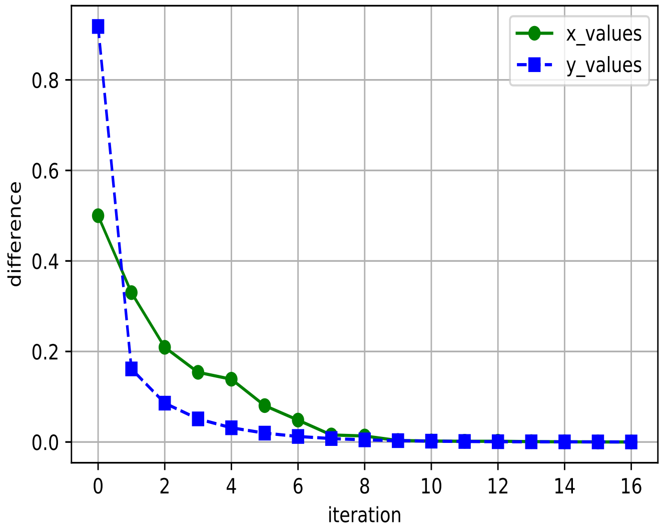

FIGURE 2.14: Convergence of sequences $\{\|x_{n+1} - x_n\|\}$ and $\{\|y_{n+1} - y_n\|\}$ for Example 2.5.3

studied. The generalized version of the algorithm given by Chang et al.[52] is obtained and three new algorithms with some modifications are presented. The comparison through example is made for the three algorithms, which further suggests that the rate of convergence of the third and second algorithms are faster than that of the generalized version. Also, we have obtained a common solution of three problems, so that a single solution can be used for three different purposes. The work to prove the convergence of these algorithms without considering some of the assumptions could be the scope for future research.