Chapter 1

Introduction

The topic of monotone operators has a significant history and its roots seems to be primarily in functional analysis, rather than mathematical programming. A complete literature review would be prohibitively long, but important early contribution and survey may be found in the works of Browder [1, 2, 3], Minty [4, 5, 6], Kachurovskii [7], and Rockafeller [8, 9]. The original definition of a monotone operator appears to be formulated by Kachurovskii [10].

A monotone operator can be perceived as a two-way generalization: the first one is a non-linear abstraction of linear endomorphisms whose matrices are positive semidefinite, and the second one is a multidimensional abstraction of non-decreasing functions in $\mathbb{R}^n$; i.e., the derivatives of convex and differentiable functions. Hence, not amazingly, a most well-known example of this type of operators in a Banach space is the Fréchet derivative of a convex smooth mapping, or, in the point-to-set notion, the sub-differential of a lower-semi-continuous convex function.

Definition 1.0.1. [11] A set-valued operator $T : H \to 2^H$ is said to be

(i) monotone if

$$\langle x - y, u - v \rangle \geq 0 \ \forall (x, u), (y, v) \in \mathcal{G}(T);$$

(ii) maximal monotone if there exist no monotone operator $S : H \to 2^H$ such that $\mathcal{G}(T)$ is properly contained in $\mathcal{G}(S)$,

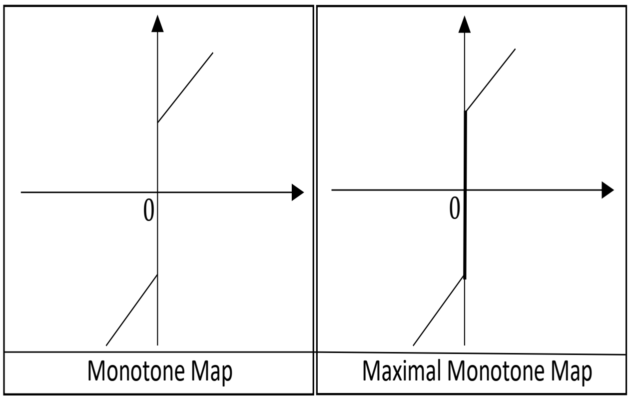

FIGURE 1.1: Monotone and maximal monotone operator

where $\mathcal{G}(T) := \{(x, y) \in H \times H : y \in Tx\}$ is graph of an operator T.

Example 1.0.1. (i) The operator $T : \mathbb{R} \to 2^{\mathbb{R}}$ defined by

$$T(x) = \begin{cases} x - 1, & x < 0, \\ \{-1, 1\}, & x = 0, \\ x + 1, & x > 0 \end{cases}$$

is monotone and

$$T(x) = \begin{cases} x - 1, & x < 0, \\ [-1, 1], & x = 0, \\ x + 1, & x > 0 \end{cases}$$

is maximally monotone.

(ii) Let $\alpha \in [-1, 1]$ and $T : H \to H$ be a non-expansive operator. Then $I + \alpha T$ is maximal monotone operator.

Now we look at some of the features of maximal monotone operators.

Proposition 1.0.1. [11] Let $T : H \to 2^H$ be a maximally monotone operator, then we have the following:

(i) for any sequence $\{(x_n, u_n)\} \in \mathcal{G}(T)$ and $(x, u) \in H \times H$, if $x_n \rightharpoonup x$ and $u_n \to u$, then $(x, u) \in \mathcal{G}(T)(T)$, i.e., $\mathcal{G}(T)$ is sequentially closed in $H^{weak} \times H^{strong}$,

(ii) for any sequence $\{(x_n, u_n)\} \in \mathcal{G}(T)$ and $(x, u) \in H \times H$, if $x_n \to x$ and $u_n \rightharpoonup u$, then $(x, u) \in \mathcal{G}(T)$, i.e., $\mathcal{G}(T)$ is sequentially closed in $H^{strong} \times H^{weak}$,

(iii) $\mathcal{G}(T)$ is closed in $H^{strong} \times H^{strong}$.

Remark 1.0.1. The graph of a maximum monotone operator does not need to be sequentially closed in $H^{weak} \times H^{weak}$. Consider that $H = l^2(N)$ and $C = B(0; 1)$. So, $I - P_C$ is firmly nonexpansive and hence maximally monotone. Take a sequence $\{x_n\} = \{e_1 + e_{2n}\}$, where $\{e_n\}$ is the sequence of unit vectors in $l^2(N)$. Then the sequence $(x_n, (1 - 1/\sqrt{2})x_n) \in \mathcal{G}(T)(I - P_C)$ and $x_n \rightharpoonup e_1$, $(x_n, (1 - 1/\sqrt{2})x_n) \rightharpoonup (1 - 1/\sqrt{2})e_1$. Although, the weak limit $(e_1, (1 - 1/\sqrt{2})e_1) \notin \mathcal{G}(I - P_C)$.

Proposition 1.0.2. [11] Let $T_1 : H_1 \to 2^{H_1}$ and $T_2 : H_2 \to 2^{H_2}$ be two maximal monotone operators, where H_1 and H_2 are real Hilbert spaces. Set $H := H_1 \times H_2$ and $T : H \to 2^H : (x, y) \mapsto T_1x \times T_2y$. Then T is a maximal monotone operator.

To find a zero of an operator is a decrepit and vastly influential problem, and a number of physical and mathematical problems get converted to this problem. When the function is a point-to-set operator, the inclusion problem is a generalized version of this classical problem. Given the operator $T : \mathcal{D}(T) \subset H \to 2^H$, the inclusion problem is:

$$\text{to find } x \in H \text{ such that } 0 \in T(x). \tag{1.1}$$

Recently, this problem has attracted much attention as many nonlinear problems, emanating within applied fields, are mathematically represented as nonlinear operator equations and/or inclusions. If T is maximally monotone, then the set of its zeros has the following characterization.

$$x \in \mathrm{Zer}(T) \Longleftrightarrow \langle u - x, z \rangle \geq 0 \; \forall (u, z) \in \mathcal{G}(T). \tag{1.2}$$

The resolvent and Yosida approximation of T are two single-valued Lipschitz continuous operators that can be associated with a monotone operator T.

Definition 1.0.2. Let $T : H \to 2^H$ be a set-valued operator. Then resolvent and Yosida approximation of T of index $\gamma(> 0)$ are defined by:

$$J_{\gamma A} := (I + \gamma A)^{-1}, A_\gamma := \frac{1}{\gamma}(I - J_{\gamma A}),$$

respectively.

The proof that the resolvent of maximal monotone operator is single-valued everywhere defined may be tracked back to Minty [4]. To evaluate the resolvent by a general iterative method has been proposed by Bruck [12]. The relation between monotone and firmly nonexpansive operators is not distinguished in the literature but Browder and Petryshyn [13] have considered, in discussing, the single-valued case. Some properties of resolvent and Yosida approximations are given in the next results.

Proposition 1.0.3. Let $T : H \to 2^H$ be a set-valued operator. Then, for $x, y \in H$, we have the followings:

(i) $\mathcal{D}(J_{\gamma T}) = \mathcal{D}(T_\gamma) = \mathrm{ran}(I + \gamma T)$ and $\mathrm{ran}(J_{\gamma T}) = \mathcal{D}(T)$;

(ii) $y \in J_{\gamma T}x \Leftrightarrow (y, \gamma^{-1}(x - y)) \in \mathcal{G}(T)$;

(iii) $y \in T_\gamma x \Leftrightarrow (x - \gamma y, y) \in \mathcal{G}(T)$.

Proposition 1.0.4. [14] Let $T : H \to 2^H$ be a maximal monotone operator. Then, we have the followings:

(i) The resolvent $J_{\gamma T} : H \to H$ and $I - J_{\gamma T} : H \to H$ are firmly non-expansive and maximal monotone;

(ii) The reflected resolvent

$$R_{\gamma T} : H \to H : x \mapsto 2J_{\gamma T}x - x$$

is non-expansive;

(iii) The Yosida approximation $T_\gamma : H \to H$ is maximal monotone, γ-cocoercive and hence $\frac{1}{\gamma}$-Lipschitz continuous.

The proximal point algorithm [15, 16] is a very prominent method to tackle the inclusion problem (1.1), which is given by

$$x_{n+1} = (Id + \lambda_n T)^{-1}(x_n) \text{ for all } n \in \mathbb{N},$$

where $\lambda_n > 0$. In many cases, computing the resolvent of an operator is also as difficult as solving the original inclusion problem.

1.1 Splitting Algorithms

The idea is clear and old: *divide et impera* (divide and conquer). Splitting is one of the most crucial and standardized approaches for exploring structured algorithms

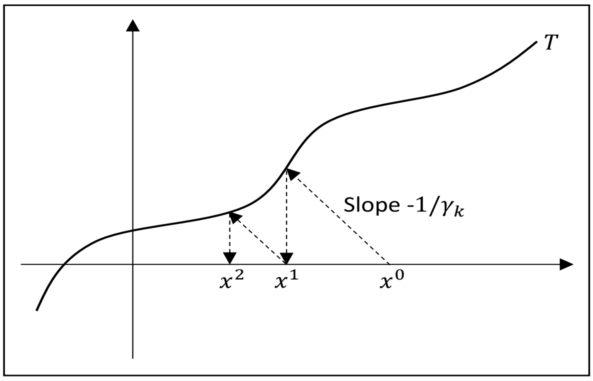

Figure 1.2: The Proximal point algorithm.

to solve complex and structured problems. The problem of accessing the set of zeros of maximally monotone operators employing splitting iterative methods, where the involved operators are appraised independently, either through its resolvent for the case of set-valued or through the operator itself for the case of single-valued, remains to be a very attractive research area. This is due to its usefulness in solving real-world problems that can be modeled as non-differentiable convex optimization problems, like those arising in image processing, signal recovery, support vector machines classification, location theory, clustering, network communications, etc. Difficult models in signal and image processing, optimization, environment science, meteorology, differential equations and variational inequalities have given rise to various operator splitting methods. The overarching concept has always been to simplify, i.e., to improve efficiency in computational work by solving simpler subproblems. As a result, researchers concentrate their efforts on the so-called splitting method, when the operator $T = A + B$. Therefore, monotone inclusion problem is:

$$\text{to find } x \in H \text{ such that } 0 \in (A + B)(x), \tag{1.3}$$

where either both the operators A and B are set-valued or one is set-valued and other one is single-valued. Based on splitting techniques, many iterative methods have been designed to solve problem (1.3). Some well-known techniques are Peaceman–Rachford splitting algorithm [17], Douglas–Rachford splitting algorithm [18], and forward–backward algorithm [19] and forward-backward-forward splitting algorithm [20].

If both the operators A and B are maximal monotone, Douglas–Rachford splitting algorithm finds the solution of problem (1.3), which is as follows: choose $x_0 \in H$ for all $n \geq 0$

$$\begin{cases} y_n = J_{\gamma B} x_n \\ z_n = J_{\gamma A}(2y_n - x_n) \\ x_{n+1} = x_n + \lambda_n (y_n - z_n), \end{cases} \tag{1.4}$$

where $\lambda_n \in [0, 2]$. The following describes the convergence behavior of the Douglas–Rachford algorithm:

Theorem 1.1.1. Let $A, B : H \to 2^H$ be maximally monotone operators with $\mathrm{Zer}(A + B) \neq \emptyset$. Let $\gamma > 0$, $x_0 \in H$, and $\{\lambda_n\}$ be a sequence in $[0, 2]$ with $\sum_{n=1}^{\infty} \lambda_n (2 - \lambda_n) = \infty$. Consider $\{x_n\}$ is a sequence defined by the algorithm (1.4). Then, $\exists x \in \mathcal{F}(R_{\gamma A} \circ R_{\gamma B})$ such that the following statements hold:

(i) $J_{\gamma B} x \in \mathrm{Zer}(A + B)$;

(ii) $y_n - z_n \to 0$;

(iii) $x_n \rightharpoonup x$;

(iv) $y_n \rightharpoonup x$;

(v) $\{z_n\} \rightharpoonup J_{\gamma B} x$;

(vi) Assume that A is a normal cone operator of a closed affine set $C \subseteq H$. Then, $P_C x_n \rightharpoonup J_{\gamma B} x$;

(vii) Assume that either A or B is uniformly monotone on every nonempty bounded subset of $\mathcal{D}(A)$ or $\mathcal{D}(B)$. Then, sequences $\{y_n\}$ and $\{z_n\}$ converge weakly to the unique point in $\mathrm{Zer}(A + B)$.

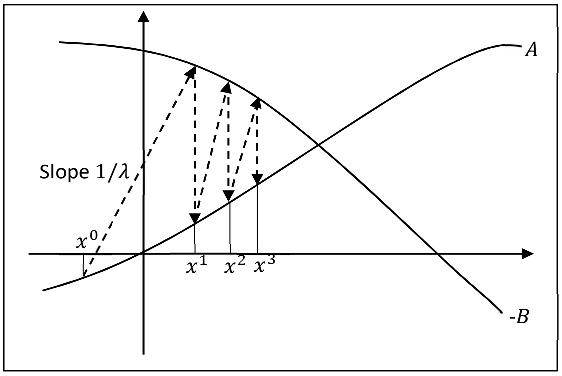

FIGURE 1.3: Douglas–Rachford algorithm.

With a product space approach, the Douglas–Rachford algorithm can easily be adapted to solve monotone inclusions for the sum of more than two operators, see [14, Proposition 25.7]. Currently, the Douglas–Rachford algorithm is a very active field of research, since numerical experiments show that it works exceptionally well even for nonconvex problems, and this behavior remains somewhat miraculous, see e.g. [21].

The next algorithm is applicable for particular instances of problem (1.3) where one of the operators is additionally β-cocoercive for some β. The advantage of this so-called forward-backward algorithm is that it is not necessary to calculate the proximal point of the well-behaved operator B, which might be inaccessible. The forward-backward algorithm and its convergence is given in the following theorem:

Theorem 1.1.2. Let $A : H \to 2^H$ be a maximal monotone operator, and $B : H \to H$ be β-cocoercive operator for $\beta > 0$. Consider that $\gamma \in (0, 2\beta)$ and set $\delta = 2 - \frac{\gamma}{2\beta}$. Let $\lambda_n \in [0, \delta]$ such that $\sum_{n=1}^{\infty} \lambda_n(\delta - \lambda_n) = \infty$ and $\operatorname{Zer}(A + B) \neq \emptyset$. For $x_0 \in H$, forward-backward algorithm is given by

$$x_{n+1} = x_n + \lambda_n(J_{\gamma A}(I - \gamma B)x_n - x_n). \tag{1.5}$$

Then the following statements hold:

(i) $\{x_n\} \rightharpoonup x^* \in \operatorname{Zer}(A + B)$.

(ii) Let $x \in \operatorname{Zer}(A + B)$. Then $\{Bx_n\}$ converges strongly to the unique dual solution Bx.

(iii) Assume that either A or B is uniformly monotone on every nonempty bounded subset of $\mathcal{D}(A)$ or H, respectively. Then $\{x_n\}$ converges strongly to the unique point in $\operatorname{Zer}(A + B)$.

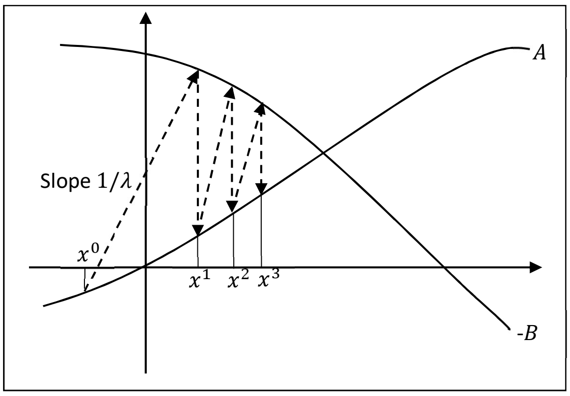

FIGURE 1.4: Forward-backward algorithm.

To relax the cocoercivity condition in forward-backward (FB) algorithm, Tseng [20] has modified the FB algorithm and introduced a new algorithm, called forward-backward-forward algorithm, which is summarized in next result.

Theorem 1.1.3. Let $A : H \to 2^H$ be a maximal monotone operator, and $B : H \to H$ be a monotone and β-Lipschitz continuous operator for $\beta > 0$ with $\mathcal{D}(B) = H$. For $\gamma \in (0, \beta)$ and $x_0 \in H$, we have the following algorithm:

$$\begin{cases} y_n = (I - \gamma B)x_n \\ z_n = J_{\gamma A} y_n \\ x_{n+1} = x_n - y_n + z_n - \gamma B z_n. \end{cases} \tag{1.6}$$

Then, we have the following:

(i) $\{x_n - z_n\}$ converges strongly to 0;

(ii) $\{x_n\}$ and $\{z_n\}$ converge weakly to a point in $\text{Zer}(A + B)$;

(iii) Consider that A or B is uniformly monotone on every nonempty bounded subset of $\mathcal{D}(A)$. Then $\{x_n\}$ and $\{z_n\}$ converge strongly to an unique point in $Zer(A + B)$.

1.2 Continuous Dynamical system for inclusion problems

Since the seventies of the 19th century, much attention has been paid to the monotone inclusion and optimization problems through the dynamical systems approach (Baillon, Brezis and Bruck, see ([22, 23, 24]), not only because of their natural importance in the fields such as applied functional analysis and differential equation but also since there are admitted as important tools for the discovery and study of numerical algorithms for minimization problems acquired by the time discretization of continuous dynamics. The dynamic approach to iterative optimization techniques can provide deeper intuitions into the conventional nature of the methods, and the ideas used in the continuous case can be attuned to produce results for discrete algorithms. We refer the reader to [25] for more information on the relationships between continuous and discrete dynamics.

In order to solve minimization problem

$$\min_{x\in H} f(x),$$

where $f : H \to \mathbb{R}$ is a smooth function, one uses the discretization $\dot{x}(t) = \frac{1}{\gamma}(x_{n+1} - x_n)$ $(\gamma > 0)$ of the gradient flow

$$\dot{x}(t) = -\nabla f(x(t)),$$

which produces steepest descent method as

$$x_{n+1} = x_n - \gamma\nabla f(x_n).$$

The discretization of the differential inclusion

$$\dot{x}(t) \in -\partial f(x(t)),$$

generates the subgradient scheme $x_{n+1} \in x_n - \gamma\partial f(x_n)$, or, by the discretization $x_{n+1} - x_n \in -\gamma\partial f(x_n)$, we get the proximal point algorithm [26]

$$x_{n+1} = (I + \gamma\partial f(x_n))^{-1}.$$

Now we go ahead and discuss about continuous implicit-type dynamical systems related with optimization and monotone inclusion problems, which are the initial-valued differential equation. With a view to obtain solution of optimization problem

$$\inf_{x \in C} f(x), \tag{1.7}$$

where $f : H \to \mathbb{R}$ is a smooth function, Antipin [27] studied convergence of the orbits of following dynamical system

$$\begin{cases} \dot{x}(t) + x(t) = P_C(x(t) - \gamma \nabla f(x(t))), \\ x(0) = x_0 \in H, \quad \gamma > 0, \end{cases} \tag{1.8}$$

in finite dimensional space. He has also explored the exponential convergence of the generated trajectories. In the convex setting, Bolte [28] has established that trajectory of (1.8) converges weakly to a minimizer of problem (1.7), in a general real Hilbert space. He has also shown that the orbits can be forced to converge strongly toward a well-specified minimizer. The operator P_C used in (1.8), is called projection operator, which is defined as follows:

Definition 1.2.1. [11] Let C be a subset of H, and $x \in H$, $p \in C$. Then p is a projection of x on to C (or a best approximation to x from C) if $\|x - p\| = d_C(x)$. If every point in H has at least one projection onto C or exactly one projection onto C, then C is proximinal or Chebyshev set, respectively. If every point in H has exactly one projection onto C, then C is a Chebyshev set. In this case, the projector onto C is the operator, denoted by P_C, that maps every point in H to its unique projection onto C.

It is remarkable that the projection mapping P_C is a non-expansive mapping from H to C (for more information on projection mappings, see Agarwal, O'Regan, and

Sahu [29]). Now, we discuss the important properties of the projection operator.

Lemma 1.2.1. [29] Let C be a nonempty closed convex subset of a real Hilbert space H, then we have the following:

(i) $P_C(x) \in C$, $\forall x \in H$;

(ii) $\langle x - P_C(x), P_C(x) - y \rangle \geq 0$, $\forall x, y \in C$;

(iii) $\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2$, $\forall x \in H$ and $y \in C$;

(iv) $\langle P_C(x) - P_C(y), x - y \rangle \geq \|P_C(x) - P_C(y)\|^2$, $\forall x, y \in H$.

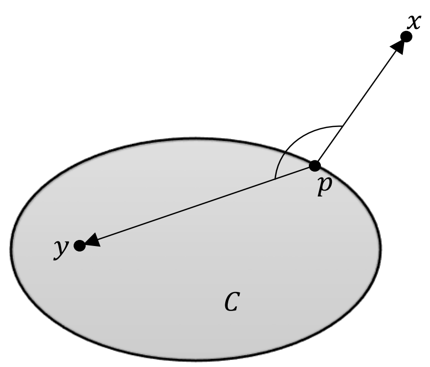

FIGURE 1.5: Projection onto a nonempty closed convex set C in the Euclidean plane.

A Newton-like dynamical approach to solve monotone inclusion problem (1.1) has been delineated by Attouch et al. [30]. By using Lyapunov analysis, authors have proved that the trajectories converges weakly to equilibria. In this context, to study monotone inclusion problem via splitting method, a Newton-like dynamical system has been developed by Abbas et al. [31], where the operator is the sum of the sub-differential of a convex lower semi-continuous function, and the gradient of a convex differentiable function. Abbas et al. [32] studied the following dynamical system to extend this study to a non-potential case and thus broaden its range of applications:

$$\begin{cases} \dot{x}(t) + x(t) = \text{prox}_{\gamma\Phi}(I - \gamma B)x(t) \\ x(0) = x_0, \end{cases}$$

where $\Phi : H \to \mathbb{R}_\infty$ is a proper, lower semicontinuous and convex function, $B : H \to H$ is a cocoercive operator and $\text{prox}_{\gamma\Phi} : H \to H$ represents the proximal point operator of $\gamma\Phi$, which is defined by:

$$\text{prox}_{\gamma\Phi}(x) = \text{argmin}_{y \in H} \left\{ \Phi(y) + \frac{1}{\gamma}\|x - y\|^2 \right\}. \tag{1.9}$$

Note that if $\text{Zer}(\partial\Phi + B) \neq \emptyset$, then weak convergence of orbit of dynamical system (1.9) is assured by selecting the step-size γ in an appropriate domain bounded by cocoercive parameter of the operator B. One can observe that the time discretization of the dynamical system (1.9) corresponds to forward-backward algorithm, that converges weakly to zero of $\text{Zer}(\partial\Phi + B)$. Here $\partial\Phi$ is subdifferential of Φ, which is defined as follows:

Definition 1.2.2. [14, Definition 16.1] Let $\Phi : H \to (-\infty, \infty]$ be proper. The subdifferential of Φ is the set-valued operator

$$\partial\Phi : H \to 2^H : x \mapsto \{u \in H | (\forall y \in H)\langle y - x, u\rangle + \Phi(x) \leq \Phi(y)\}.$$

Let $x \in H$. If $\partial\Phi \neq \emptyset$, then Φ is subdifferentiable at x.

The first order dynamical system, which is linked to forward-backward (FB) algorithm to find a solution of (1.3) has been studied by Bot et al. [33]. For a given

$x_0 \in H$, the dynamical system considered is given by:

$$\begin{cases} \dot{x}(t) = \lambda(t)[J_{\gamma A}(I - \gamma B)x(t) - x(t)], \\ x(0) = x_0, \end{cases} \tag{1.10}$$

where $\lambda : [0, \infty) \to [0, \infty)$ is a Lebesgue measurable function, $A : H \to 2^H$ is maximal monotone operator, $B : H \to H$ is β-cocoercive operator for $\beta > 0$, and $J_{\gamma A}$ is resolvent of operator A for $\gamma > 0$. In this work, the authors have studied the convergence of trajectories for $\gamma \in (0, 2\beta)$ and have shown the convergence of forward-backward dynamical system with the help of convergence of dynamical system generated by non-expansive mapping T:

$$\begin{cases} \dot{x}(t) = \lambda(t)[Tx(t) - x(t)], \\ x(0) = x_0 \in H, \end{cases} \tag{1.11}$$

where $\lambda : [0, \infty) \to [0, 1]$ is a Lebesgue measurable function satisfying the following criteria:

$$\int_0^\infty \lambda(t)(1 - \lambda(t))dt = \infty \ or \ \inf_{t \geq 0} \lambda(t) > 0.$$

Authors have also used the fact that zero of forward-backward operator is the same as the fixed point of a nonexpansive operator. The dynamical system (1.11) is a continuous form of the classical Krasnosel'ski'ı–Mann algorithm to find the fixed points of the nonexpansive operator T [14]. To verify the existance and uniqueness of strong global solution of a dynamical system, researches have verified the Cauchy-Picard-Lipschitz theorem for absolute continuous trajectories, which is as follows:

Theorem 1.2.1. [34, Theorem 58], [35, Proposition 6.2.1] Assume that a function $G : [0, \infty) \times H \to H$ satisfies the following:

(i) $G(\cdot, x) : [0, \infty) \to H$ is measurable $\forall x \in H$;

(ii) $G(t, \cdot) : H \to H$ is continuous $\forall t \geq 0$;

(iii) there is a function $l_1 \in L^1_{loc}(\mathbb{R}_+; \mathbb{R})$ such that

$$\|G(t, x_1) - G(t, x_2)\| \leq l_1(t)\|x_1 - x_2\| \ \forall x_1, x_2 \in H \ \forall t \in [0, b] \ b \in \mathbb{R}_+;$$

(iv) $\exists$ a function $l_2 \in L^1_{loc}(\mathbb{R}_+; \mathbb{R})$ such that

$$\|G(t, x)\| \leq l_2(t) \ \forall x \in H \ \forall t \in [0, b] \ b \in \mathbb{R}_+.$$

Then $\exists$ a unique solution $t \mapsto x(t)$ as $t \to \infty$ of the dynamical system defined by

$$\begin{cases} \dot{x}(t) = G(\gamma(t), x(t)), \\ x(0) = x_0, \end{cases}$$

where $x_0 \in H$.

Moreover, Bot et al. [36] have also studied that the trajectory generated by the dynamical system (1.10) converges strongly with an exponential rate to the solution of the problem (1.3) when the operator $A : H \to 2^H$ is maximal monotone, $B : H \to H$ is monotone and $\frac{1}{\beta}$-Lipschitz for $\beta > 0$ such that sum of both the operators is ρ-strongly monotone for $\rho > 0$. In this work, authors have also derived the convergence rates of the orbits of dynamical system related with minimization of sum of a proper, lower semicontinuous and convex function with a smooth convex such that the objective function satisfies a strong convexity assumption. The dynamical system is as follows:

$$\begin{cases} \dot{u}(t) = \lambda(t)[\text{prox}_{\gamma \mathrm{f}}(I - \gamma \nabla g)u(t) - u(t)] \\ u(0) = u_0, \end{cases}$$

where function $f : H \to \mathbb{R}_\infty$ is proper, lower semicontinuous, convex and function $g : H \to \mathbb{R}$ is convex, $1/\beta$-Lipschitz continuous gradient for $\beta > 0$ and Fréchet differentiable.

In 2020, Csetnek [37] has presented a survey on the first and second-order dynamical system to solve the monotone inclusion problem. In this survey, he has studied dynamical systems to solve non-convex optimization problems.

There are two types of dynamical systems in the literature to find the roots of the sum of a maximal monotone operator $A : H \to 2^H$ and a monotone and L-Lipschitz continuous operator $B : H \to H$ for $L > 0$ in a real Hilbert space. Firstly, Banart et al. [38] have studied the following forward-backward-forward (FBF) dynamical system:

$$\begin{cases} z(t) = J_{\gamma(t)A}(I - \gamma(t)B)x(t), \\ \dot{x}(t) = z(t) - x(t) + \gamma(t)(Bx(t) - Bz(t)), \\ x(0) = x_0, \end{cases} \tag{1.12}$$

where $x_0 \in H$ and $\gamma : [0, \infty) \to (0, \frac{1}{L})$ is a Lebesgue measurable function. In this consideration, the authors have relaxed the cocoercivity condition of the operator B of the dynamical system (1.10). On the other hand, Csetnek et al. [39] have investigated the following dynamical system:

$$\begin{cases} \dot{x}(t) + x(t) = J_{\gamma A}(x(t) - y(t)) - \dot{y}(t), \\ y(t) = \gamma B x(t), \\ x(0) = x_0, \end{cases}$$

where $x_0 \in H$ and $\gamma \in [\epsilon, \frac{1-3\epsilon}{3L}]$ for $\epsilon > 0$.

1.3 Problem statement and Thesis Objectives

In the second chapter, we look at a problem that is a coalition of the three problems listed below:

(i) to find a root of a nonnegative function F;

(ii) to find a root of a set-valued operator T;

(iii) to find common fixed points of operators $R_1, R_2, \ldots, R_m$;

i.e., consider the following problem

$$(\mathbf{P}) \qquad \text{find } z^* \in T^{-1}(0) \cap (\cap_{i=1}^{m} \mathcal{F}(R_i)) \text{ such that } F(z^*) = 0.$$

Here we introduce and deliberate the convergence behavior of different iterative techniques for solving the generalized problem and compare the convergence speed.

In chapter 3, we study the existence, uniqueness, and weak asymptotic convergence as well as strong convergence of the generated orbits of first-order backward-forward dynamical system to solve the structured monotone inclusion problem of the form:

$$\text{find } x \in H : 0 \in (A + B)x,$$

where $A : H \to 2^H$ is maximal $(\gamma-\alpha)$-cohypomonotone for $\gamma \in \mathbb{R}, \alpha > 0$, $B : H \to H$ is a β-cocoercive for $\beta > 0$. We also establish that an equilibrium point of the trajectory is globally exponentially stable and monotone attractor. As a particular case, we explore similar perspectives of the orbits generated by a dynamical system related to the minimization of the sum of a nonsmooth convex and a smooth convex function.

In chapter 4, we investigate the existence, uniqueness, and weak asymptotic convergence of the orbits of first-order forward-backward-half forward dynamical systems related with the inclusion problem of the form:

$$\text{find } x \in H : 0 \in Ax + B_1x + B_2x,$$

where, $A : H \to 2^H$ is maximally monotone operator, $B_1 : H \to H$ is β-cocoercive for $\beta > 0$, and $B_2 : H \to H$ is monotone and L-Lipschitz continuous and $\mathcal{D}(B_2) = H$. We also explore a variable metric forward-backward-half forward dynamical system in the sense of non-self-adjoint linear operators.

1.4 Outline of the Thesis

The outline of the thesis is as follows:

Chapter 1 presents the introduction of the thesis. Basic definitions that are being used throughout this thesis are provided. It also provides the literature survey and some recent works on monotone inclusion problems. The motivation behind choosing the topic and problem statement of the thesis is also explained.

Chapter 2 describes three iterative methods to find the zero of a nonnegative lower semicontinuous function over the common solution set of the fixed point problem and monotone inclusion problem. The split equality variational inclusion and split equality equilibrium problems with numerical examples are also studied in this chapter.

Chapter 3 provides the existence, uniqueness, weak and strong convergence of the orbits of first-order backward-forward dynamical systems to deal with the monotone inclusion problems. In addition, convergence of trajectories generated by convex minimization problem is also discussed.

Chapter 4 is based on the convergence of the trajectories of forward-backward-half forward dynamical system to tackle monotone inclusion problem consisting of three operators. The same has been studied in non-self-adjoint variable metric sense. As an application, generalized Nash equilibrium problem are studied.

Chapter 5 explore the warped Yosida approximation and its properties.

Chapter 6 concludes the thesis and hints towards possible future work in solving inclusion problem and non-convex optimization problem.

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