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I, ***Pankaj Gautam***, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of ***Prof. Tanmoy Som*** from ***July, 2016*** to ***May, 2021*** at the ***Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi***. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports, dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

Date: 31st May 2021

Place: Varanasi

(Pankaj Gautam)

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To

My Beloved Parents

Mrs. Gulabkali Gautam

&

Mr. Hari Parasd Gautam

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Pankaj Gautam

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Symbols

$\mathcal{D}(T)$	Domain of operator T
$\mathcal{G}(T)$	Graph of operator T
$\mathrm{ran}(T)$	Range of operator T
$\mathrm{Zer}(T)$	Zero of operator T
$\mathcal{F}(T)$	Set of fixed points of T
$\rightarrow$	Strong convergence
$\rightharpoonup$	Weak convergence
2^X	Power set of set X
$L^1_{loc}(\mathbb{R}_+;\mathbb{R})$	Set of locally integrable function on $\mathbb{R}_+$
$L^2([0,T])$	Measurable functions $x:[0,T]\rightarrow\mathbb{R}$ such that $\|x\|^2$ is Lebesgue integrable
$L^2([0,T];H)$	Measurable functions $x:[0,T]\rightarrow H$ such that $\|x\|_H^2$ is Lebesgue integrable

PREFACE

In the consideration of optimization problems and differential equations, the monotone inclusion arises routinely and ordinarily, thus solution techniques for them can be applied to handle many real-world problems.

In this thesis, the author discusses iterative methods and continuous dynamical systems for monotone inclusion problems. This thesis contains six chapters. Chapter 1 introduces monotone inclusion problems along with the historical background of the problem. This chapter includes definitions and some basic properties of monotone and maximal monotone operators, projection operators and fundamental descriptions of the resolvent and Yosida approximations. Some splittings methods and their related dynamical systems to solve the monotone inclusion problems are also demonstrated in this chapter.

Chapter 2 describes the generalized problem of split equality variational inclusion problem. For this purpose, we introduce the problem of finding the zeros of a non-negative lower semicontinuous function over the common solution set of fixed point problem and monotone inclusion problem. We propose and study the convergence behavior of different iterative techniques to solve the generalized problem. Furthermore, we study an inertial form of the proposed algorithm and compare the convergence speed. Moreover, as an application, we analyze split equality equilibrium fixed point problem. Further with the help of numerical computations experimentally, we compare the convergence speed of the proposed algorithm with it's inertial form and already existing algorithms to solve the generalized problem.

In chapter 3, we explore the first-order variable metric backward-forward dynamical systems associated with monotone inclusion and convex minimization problems in real Hilbert space. The operators are chosen so that the backward-forward dynamical system is closely related to the forward-backward dynamical system and has the same computational complexity. We show existence, uniqueness, and weak asymptotic convergence of the generated trajectories and strong convergence if one of the operators is uniformly monotone. We also establish that an equilibrium point of the trajectory is globally exponentially stable and monotone attractor. As a particular case, we explore similar perspectives of the trajectories generated by a dynamical system related to the minimization of the sum of a nonsmooth convex and a smooth convex function. In this sequel, we also study that weak convergence behavior of the orbits associated with minimization problem without the restriction on the choice of the step size. Numerical examples are given to illustrate the convergence of trajectories.

In chapter 4, the first-order forward-backward-half forward dynamical systems associated with the inclusion problem consisting of three monotone operators are analyzed. The framework modifies the forward-backward-forward dynamical system by adding a cocoercive operator to the inclusion. The existence, uniqueness, and weak asymptotic convergence of the generated trajectories are discussed. A variable metric forward-backward-half forward dynamical system with the essence of non-self-adjoint linear operators is introduced. The proposed notion, in turn, extends the forward-backward-forward dynamical system and forward-backward dynamical system in the framework of variable metric by relaxing some conditions on the metrics. By using Lyapunov analysis and a continuous variant of the Opial lemma for the class of operators $\mathfrak{T}$, one can prove the convergence of generated trajectories in variable metric sense. The distributed dynamical system is further explored to compute a generalized Nash equilibrium in a monotone game as an application. A numerical example is provided to illustrate the convergence of the trajectories.

Chapter 5 explores a generalized concept, called warped Yosida approximation, which is formulated with the help of an auxiliary operator. Yosida approximation of set-valued operators performs an extensive role in non-linear operator theory; in particular, in dynamical system governed by maximal monotone operator. The properties of warped Yosida approximation are analyzed and connections are made with existing notions.