J. Math. Anal. Appl. 508 (2022) 125864

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Some identities for the partition function

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ARTICLE INFO

Article history:
Received 30 June 2021
Available online 25 November 2021
Submitted by B.C. Berndt

Keywords:
Partition function
Ramanujan's congruences
Witness identity
Modular forms
Ramanujan's circular summation

ABSTRACT

In his unpublished manuscript on the partition and tau functions, Ramanujan obtained several striking congruences for the partition function $p(n)$, the number of unrestricted partitions of n. The most notable of them are $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$ which holds for all positive integers n. More surprisingly, Ramanujan obtained certain identities between q-series from which the above congruences follow as consequences. In this paper, we adopt Ramanujan's approach and prove an identity which witnesses another famous Ramanujan congruence, namely, $p(11n+6) \equiv 0 \pmod{11}$ and also establish some new identities for the generating functions for $p(17n+5)$, $p(19n+7)$ and $p(23n+1)$. We also find explicit evaluations for $F_p(q)$ in the cases $p = 17, 19, 23$ where F_p is the function appearing in Ramanujan's circular summation formula.

1. Introduction

Euler proved that the generating function for the number of unrestricted partitions of a number n, denoted by $p(n)$ is

$$P(q) := \sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1.1}$$

where $(q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$. Surprisingly, $p(n)$ satisfies amazing divisibility properties modulo $5, 7$ and 11. These properties were empirically observed by Ramanujan from MacMahon's table for values of $p(n)$, and subsequently proved by himself (see [26, Chapter 5, Theorem 5.7]). This stunned G.H. Hardy as they

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https://doi.org/10.1016/j.jmaa.2021.125864

represent a totally unexpected connection between additive number theory (partitions) and multiplicative number theory (divisibility).

Theorem 1.1 *(Ramanujan). For all $n \geq 0$, we have*

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7, \quad p(11n+6) \equiv 0 \pmod{11}.$$

In [2], Alghren and Boylan have showed that congruences similar to the ones in Theorem 1.1 do not exist for primes $p \neq 5, 7$ and 11. However, there exist Ramanujan congruences of the form $p(\ell^k n + \delta_{\ell,k}) \equiv 0 \pmod{\ell^k}$ where $\ell = 5, 7, 11$, $\delta_{\ell,k} \equiv 1/24 \pmod{\ell^k}$, and where for $\ell = 7$, the modulus has to be changed to $7^{\lfloor k/2 \rfloor + 1}$ [26, Theorem 5.7]. Further works of Atkin, Klove, Lovejoy and Ono [5,21,24] have showed that there exists infinitely many subprogressions where Ramanujan congruence modulo 5^k is in fact a congruence modulo 5^{k+1}.

Quite ingeniously, Ramanujan discovered two remarkable identities from which the first two congruences in Theorem 1.1 follow as consequences (See [29, Equations (105.1)-(105.2)]).

Theorem 1.2 *(Ramanujan). We have*

$$\sum_{n \geq 0} p(5n+4)q^n = 5 \cdot \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{1.2}$$

$$\sum_{n \geq 0} p(7n+5)q^n = 7 \cdot \frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49 \cdot q \cdot \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}. \tag{1.3}$$

Hereinafter, we shall refer to such identities like the ones in Theorem 1.2 as *witness* identities since they imply the first two congruences in Theorem 1.1. Ramanujan did not find any witness identity for the third congruence in Theorem 1.1. Such an identity was first obtained by Lehner [23], and subsequently by Atkin [4]. Recently, Paule and Radu [27] obtained a witness identity (different from Atkin and Lehner) using the theory of modular functions.

Unlike the mod 5 and 7 cases, a witness identity in the mod 11 case has always presented greater difficulties. The main reason for this is that there are not enough eta-quotients which generate the corresponding space of modular forms. Another reason from the topological and algebraic standpoint is that the underlying Riemann surface has genus 1 and in view of the Weierstrass gap theorem, there is a pole order such that there does not exist any function with that pole order (see [28] for more details). Inspired by Kolberg's work [18], Radu [30] devised an algorithm which automatically yields identities such as in Theorem 1.2. His algorithm also yields a witness identity for $11 | p(11n+6)$. This fuelled further works in this direction (see [9,15,16,35]).

In this paper, among other things, we obtain a new witness identity for the third congruence in Theorem 1.1. Using the same approach, we also prove an identity of Zuckerman [40] for the generating function of $p(13n+6)$ and obtain some new identities for the generating functions for $p(17n+5)$, $p(19n+7)$ and $p(23n+1)$ (see Section 4). Before stating our results, we note that our approach is motivated from Ramanujan in his unpublished manuscript ([31], see [7] for proofs and commentary). In the manuscript, Ramanujan states (without proof) the following identities between q-series:

$$q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} = \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{(1-q^n)^2}, \tag{1.4}$$

$$q(q;q)_\infty^3 (q^7;q^7)_\infty^3 + 8q^2 \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty} = \sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{q^n(1+q^n)}{(1-q^n)^3}. \tag{1.5}$$

Upon extracting all terms on both sides of (1.4) (resp. (1.5)) where the exponent of q is a multiple of 5 (resp. 7), the first identity (resp. second identity) in Theorem 1.2 follows in a straightforward manner. One also has to use the following identity of Jacobi to prove the second identity in Theorem 1.2:

Theorem 1.3 *(Jacobi). We have*

$$(q;q)_\infty^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{n(n+1)/2}.$$

The proofs of our main results in Section 5 rely mainly on establishing identities similar to the ones in (1.4) and (1.5). As we shall see in Section 3, (1.4) and (1.5) are identities between modular forms (see Section 2 for definition). Establishing these identities amounts to comparing enough terms in the q-expansions of both sides to exceed the dimension of the relevant space of forms. By using a result of Garvan [14], and applying an operator U_p (see Sections 2 and 4 for definition and properties), our results follow.

We emphasize here that our approach of finding a new witness identity modulo 11 for $p(n)$ and other identities is new. The main ingredient in our proofs is the use of Eisenstein series which do not appear in any of the aforementioned papers finding witness identities for $11|p(11n+6)$. The Eisenstein series we consider in our work already appeared in Kolberg's work [20] who obtained several interesting results involving eta-quotients. In fact, his approach leads to special evaluations of certain L-functions which is intrinsic in his work. We, thus, believe that our new idea could be exploited further to obtain several arithmetic properties of coefficients of q-series.

This paper is organized as follows. In Section 2, we introduce some notations and state some basic facts about modular forms. In Section 3, we record some preliminaries. In Section 4, we state our new identities for $p(n)$ modulo $11, 17, 19$ and 23. In Section 5, we obtain proofs of our results in Section 4 and also obtain a new proof of Zuckerman's identity for $p(n)$ modulo 13. In Section 6, we revisit Theorem 1.2, and give a different proof of the theorem. This also leads to some new identities for the generating functions for $p(5n+j)$ and $p(7n+j')$ for $j\in\{0,1,2,3\}$ and $j'\in\{0,1,2,3,4,5\}$. Our proof involves two built-in Mathematica packages, namely, FullSimplify() and GroebnerBasis(). Finally, in Section 7, we briefly discuss Ramanujan's circular summation formula involving his symmetric theta function and obtain explicit evaluations for $F_p(q)$ for $p = 17, 19$ and 23.

2. Basic facts about modular forms

Throughout p will denote a prime ≥ 3. Let $\tau\in\mathbb{H} := \{\tau\in\mathbb{C} : \Im(\tau) > 0\}$ and $q := e^{2\pi i\tau}$. For a positive integer N, we denote by $\Gamma_0(N)$ the congruence subgroup of level N in $SL_2(\mathbb{Z})$, defined by

$$\Gamma_0(N) := \left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in SL_2(\mathbb{Z}) : c\equiv 0 \pmod N\right\}.$$

Let χ be a Dirichlet character modulo N. Then a meromorphic modular form $f:\mathbb{H}\to\mathbb{C}$ of weight k on $\Gamma_0(N)$ and Nebentypus χ is a meromorphic function satisfying the following properties:

(1) For every $\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\Gamma_0(N)$ and $\tau\in\mathbb{H}$ we have

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(d)(c\tau+d)^k f(\tau).$$

(2) f is meromorphic at all cusps of $\Gamma_0(N)$.

If $k = 0$ and $\chi = 1$, we call f a modular function. Moreover, if f is holomorphic in $\mathbb{H}$ and at the cusps of $\Gamma_0(N)$, we call f a holomorphic modular form (or simply, a modular form). If a modular form f vanishes at all the cusps of $\Gamma_0(N)$, f is called a cusp form (see [22,26]).

We shall be concerned with $N = p$, $k = \frac{1}{2}(p-1)$ and $\chi = \chi_p := \left(\frac{\cdot}{p}\right)$ where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. By $\mathcal{M}_{\frac{1}{2}(p-1)}(p, \chi_p)$, we denote the $\mathbb{C}$-vector space of modular forms on $\Gamma_0(p)$ of weight $\frac{p-1}{2}$ and Nebentypus χ_p, and $\mathcal{S}_{\frac{1}{2}(p-1)}(p, \chi_p)$ denotes the subspace of cusp forms in $\mathcal{M}_{\frac{1}{2}(p-1)}(p, \chi_p)$. Given $f(\tau) = \sum_{n\geq 0} a_n q^n \in \mathcal{M}_{\frac{1}{2}(p-1)}(p, \chi_p)$, we define the U_p-operator which is a linear operator (see [3, Chapter 10], [22, Chapter 3, Sec. 5], [26, Chapter 2] for further properties) by

$$U_p(f(\tau)) := \sum_{n\geq 0} a_{np}\, q^n = \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{\tau + j}{p}\right).$$

Define the weight $\frac{p-1}{2}$ Eisenstein series on $\Gamma_0(p)$ with Nebentypus χ_p as follows (see [12, Chapter 2, Sec. 2.7], [37, Chapter 5, Sec. 5.3]):

$$E_{\frac{1}{2}(p-1)}(\tau) := \sum_{n\geq 0}\left(\sum_{\ell | n} \chi_p(n/\ell)\cdot \ell^{\frac{p-1}{2}-1}\right) q^n. \tag{2.1}$$

Finally, we define the Dedekind eta function $\eta(\tau) := q^{\frac{1}{24}}(q;q)_\infty$, which is a modular form of weight 1/2 on $SL_2(\mathbb{Z})$.

3. Preliminaries

Here we recall some known results on modular forms. We start with the following special case of a general result of Gordon, Hughes and Newman when dealing with eta-products and quotients (see [26, Theorem 1.64, pp. 18] and [26, Theorem 1.65, pp. 18]).

Theorem 3.1. *Let r_p be an odd integer. Then the eta-quotient $f(\tau) := \eta(\tau)^{r_1}\eta(p\tau)^{r_p} \in \mathcal{M}_{\frac{1}{2}(p-1)}(p, \chi_p)$ if it satisfies the following properties:*

(1) $r_1 + r_p = p - 1$,
(2) $r_1 + p \cdot r_p \equiv 0 \pmod{24}$,
(3) $p \cdot r_1 + r_p \equiv 0 \pmod{24}$, *and*
(4) $r_1 + \frac{r_p}{p} \geq 0$.

If the inequality in (4) is strict and $f(\tau)$ vanishes at $\tau = i\infty$, then it is a cusp form.

Remark 3.2. Note that in Theorem 3.1, the corresponding character should have been $\tilde{\chi}_p = \left(\frac{(-1)^{\frac{p-1}{2}}p}{\cdot}\right)$ in view of [26, Theorem 1.64, pp. 18]. However, it follows using properties of Legendre symbol that $\tilde{\chi}_p = \chi_p$ (see [13, pp. 252, Eq. 2]).

Next, we state a general result which gives a sufficient criterion for the equality of two modular forms (see [37, Cor. 9.20, pp. 174]).

Theorem 3.3. *Let $f(\tau) = \sum_{n\geq 0} a_n q^n$, $g(\tau) = \sum_{n\geq 0} b_n q^n$ be elements in $\mathcal{M}_{\frac{1}{2}(p-1)}(p, \chi_p)$. If*

$$a_n = b_n, \quad \text{for all } n \leq \frac{p^2-1}{24},$$

then $a_n = b_n$ for all $n \geq 0$. In other words, $f \equiv g$.

The next result is crucial and it gives the action of the U_p-operator on elements of $\mathcal{M}_{\frac{1}{2}(p-1)}(p, \chi_p)$ (see [26, Prop. 2.22, pp. 28]).

Proposition 3.4. *Let $f(\tau) \in \mathcal{M}_{\frac{1}{2}(p-1)}(p, \chi_p)$. Then $U_p(f(\tau)) \in \mathcal{M}_{\frac{1}{2}(p-1)}(p, \chi_p)$. If $f(\tau) \in \mathcal{S}_{\frac{1}{2}(p-1)}(p, \chi_p)$, then $U_p(f(\tau)) \in \mathcal{S}_{\frac{1}{2}(p-1)}(p, \chi_p)$.*

The next lemma gives the action of U_p-operator on the Eisenstein series $E_{\frac{1}{2}(p-1)}(\tau)$.

Lemma 3.5. *We have*

$$U_p\left(E_{\frac{1}{2}(p-1)}(\tau)\right) = p^{\frac{p-1}{2}-1} \cdot E_{\frac{1}{2}(p-1)}(\tau).$$

Proof. From (2.1), we have

$$E_{\frac{1}{2}(p-1)}(\tau) := \sum_{n\geq 0}\left(\sum_{\ell \mid n} \chi_p(\ell) \cdot \left(\frac{n}{\ell}\right)^{\frac{p-1}{2}-1}\right) q^n. \tag{3.1}$$

Thus applying the U_p-operator on (3.1) we get

$$U_p\left(E_{\frac{1}{2}(p-1)}(\tau)\right) = \sum_{n\geq 0}\left(\sum_{\ell \mid p\cdot n} \chi_p(\ell) \cdot \left(\frac{p\cdot n}{\ell}\right)^{\frac{p-1}{2}-1}\right) q^n. \tag{3.2}$$

Noting that $\chi_p(\ell) = 0$ when $p \mid \ell$, the result now follows from (3.2). □

We now state an important result of Garvan [14] which would be very useful in our proofs later. In order to state this result, we need to define the genus, g of $\Gamma_0(p)$ for $p > 3$ as follows:

$$g := \begin{cases} \lfloor p/12 \rfloor - 1 & \text{if } p \equiv 1 \pmod{12}, \\ \lfloor p/12 \rfloor & \text{if } p \equiv 5, 7 \pmod{12}, \\ \lfloor p/12 \rfloor + 1 & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

Let F_p denote the set of modular functions f in $\Gamma_0(p)$ satisfying the condition that f is holomorphic on $\mathbb{H}$ and has a pole only at the cusp $\tau = i\infty$ (which means that f is holomorphic at the other cusp $\tau = 0$). For $p > 3$, define

$$W_k = W_{k,p}(q) := q^{\frac{6k^2}{p}-k} \prod_{m\geq 1} \frac{(1-q^{pm-4k})(1-q^{pm+4k-p})}{(1-q^{pm-2k})(1-q^{pm+2k-p})}, \quad k \not\equiv 0 \pmod{p}. \tag{3.3}$$

Garvan constructed polynomial bases out of η-quotients and Fine's W_k-functions following Kolberg's work [19]. Let $\mathcal{B}$ denotes such a basis. Then $\mathcal{B}$ is said to be *good* if it satisfies the following two conditions: For each $f_j \in \mathcal{B}$,

(1) $f_j = q^{-j} + \cdots \in \mathbb{Z}[[q]]$, and
(2) $f_j = 0$ at $\tau = 0$.

The good polynomial basis above then leads to a *good* (linear) basis $\mathscr{B} := \{B_1, B_2, \cdots, B_d\}$ for $S_{\frac{1}{2}(p-1)}(p, \chi_p)$ with $d := \dim(\mathcal{S}_{\frac{1}{2}(p-1)}(p, \chi_p))$ wherein we require that each basis element B_j satisfies $B_j = q^j + \cdots \in \mathbb{Z}[[q]]$.

We are now in a position to state the following result of Garvan [14, Prop. 2.4.13, pp. 20]:

Theorem 3.6. *Let $5 < p \leq 23$. Then we have*

(1) $d := \dim(\mathcal{S}_{\frac{1}{2}(p-1)}(p, \chi_p)) = r_p - g - 1, \; r_p := \frac{p^2-1}{24}$.
(2) *The set $\{B_1(q,p), B_2(q,p), \cdots, B_d(q,p)\}$ forms a good linear basis for $\mathcal{S}_{\frac{1}{2}(p-1)}(p, \chi_p)$ where*

$$B_j(q,p) := \begin{cases} \dfrac{\eta(p\tau)^p}{\eta(\tau)} f_{r_p - j}(q) & \text{if } d - g \leq j \leq d, \\ \dfrac{\eta(p\tau)^p}{\eta(\tau)} f_{g+1}(q)^{t-1} f_{s+g+1}(q) & \text{if } j < d - g, \end{cases}$$

and $t := \lfloor (r_p - j)/(g+1) \rfloor$, $s := r_p - t(g+1) - j$ where the set $\{f_{g+1}, \cdots, f_{2g+1}\}$ forms a good polynomial basis for F_p.

Remark 3.7. We note that the right-hand sides of the identities in (1.4) and (1.5) are, respectively, the Eisenstein series in (2.1) with $p = 5$ and $p = 7$. The left-hand sides of the identities in (1.4) and (1.5) are, respectively, the eta quotient $\frac{\eta(5\tau)^5}{\eta(\tau)}$ and a linear combination of the eta quotients $\eta(\tau)^3\eta(7\tau)^3$ and $\frac{\eta(7\tau)^7}{\eta(\tau)}$. Using Theorem 3.1, it follows that the left-hand sides of (1.4) and (1.5) are modular forms in $\mathcal{M}_2(5, \chi_5)$ and $\mathcal{M}_3(7, \chi_7)$ respectively. By comparing the first few terms, it immediately follows using Theorem 3.3 that the two sides in (1.4) and (1.5) are identical.

4. Some new identities for $p(n)$ modulo 11, 17, 19 and 23

Theorem 4.1. *We have*

$$\sum_{n \geq 0} p(11n+6)\, q^n = 11 \cdot q^4 \cdot \frac{(q^{11}; q^{11})_\infty^{11}}{(q;q)_\infty^{12}} \cdot \left(f_2^2 + 8 \cdot 11 \cdot f_2 + 11 \cdot f_3 + 11^3\right),$$

where f_2 and f_3 are defined by

$$f_2 := W_1^3 W_4 W_5 + W_2^3 W_3 W_1 + W_3^3 W_1 W_4 + W_4^3 W_5 W_2 + W_5^3 W_2 W_3 - 17,$$
$$f_3 := 2 - (W_1^6 W_4 + W_2^6 W_3 + W_3^6 W_1 + W_4^6 W_5 + W_5^6 W_2).$$

Theorem 4.2. *We have*

$$\sum_{n \geq 0} p(17n+5)\, q^n = q^{11} \cdot \frac{(q^{17}; q^{17})_\infty^{17}}{(q;q)_\infty^{18}} \cdot \big(7 \cdot f_2^4 \cdot f_3 + 827 \cdot f_2^5 + 2^2 \cdot 5 \cdot 499 \cdot f_2^3 \cdot f_3$$
$$+ 2^4 \cdot 5 \cdot 14779 \cdot f_2^2 \cdot f_3 - 2 \cdot 191 \cdot 17939 \cdot f_2^3 + 2^2 \cdot 877 \cdot 8893 \cdot f_2 \cdot f_3 - 7 \cdot 19 \cdot 79 \cdot 179 \cdot 563 \cdot f_3$$
$$+ 6028921001 \cdot f_2 + 17^7\big) + 23 \cdot 2381 \cdot q^3 \cdot \frac{(q^{17}; q^{17})_\infty^5}{(q;q)_\infty^6} - 148618367 \cdot q^7 \cdot \frac{(q^{17}; q^{17})_\infty^{11}}{(q;q)_\infty^{12}},$$

where f_2 and f_3 are defined by

$$f_2 := W_1 W_2^2 W_5 + W_2 W_4^2 W_7 + W_3 W_6^2 W_2 + W_4 W_8^2 W_3 + W_5 W_7^2 W_8 + W_6 W_5^2 W_4 + W_7 W_3^2 W_1$$
$$+ W_8 W_1^2 W_6 - 9,$$

$$f_3 := W_1^3W_2W_5W_6 + W_2^3W_4W_7W_5 + W_3^3W_6W_2W_1 + W_4^3W_8W_3W_7 + W_5^3W_7W_8W_4 + W_6^3W_5W_4W_2 + W_7^3W_3W_1W_8 + W_8^3W_1W_6W_3 - 3.$$

Theorem 4.3. *We have*

$$\sum_{n\geq 0} p(19n+4)\, q^n = q^{14}\cdot\frac{(q^{19};q^{19})_\infty^{19}}{(q;q)_\infty^{20}}\cdot\big(5\cdot f_2^7 + 3\cdot 5\cdot 7\cdot 11\cdot f_2^5\cdot f_3 + 2\cdot 3\cdot 7\cdot 19697\cdot f_2^4\cdot f_3 + 2\cdot 13\cdot 44501\cdot f_2^5 1171\cdot 258991\cdot f_2^4 - 3\cdot 5\cdot 7\cdot 349\cdot 42281\cdot f_2^2\cdot f_3 + 3^2\cdot 7\cdot 19\cdot 4304389\cdot f_2\cdot f_3 + 7\cdot 19^7\cdot f_2^2 + 19^8\cdot f_2 + 19^8\big) + 2^2\cdot 3\cdot 7\cdot 401\cdot q^2\cdot\frac{(q^{19};q^{19})_\infty^{3}}{(q;q)_\infty^{4}} + 3\cdot 37\cdot 127\cdot 1973\cdot q^5\cdot\frac{(q^{19};q^{19})_\infty^{7}}{(q;q)_\infty^{8}} + 17\cdot 19\cdot 4951997\cdot q^8\cdot\frac{(q^{19};q^{19})_\infty^{11}}{(q;q)_\infty^{12}} + 2^3\cdot 3\cdot 19^7\cdot q^{11}\cdot\frac{(q^{19};q^{19})_\infty^{15}}{(q;q)_\infty^{16}},$$

where f_2 and f_3 are defined by

$$f_2 := W_1W_7W_8 + W_2W_5W_3 + W_4W_9W_6 - 5, \quad f_3 := q^{-3}\cdot\frac{(q;q)_\infty^4}{(q^{19};q^{19})_\infty^4}.$$

Theorem 4.4. *We have*

$$\sum_{n\geq 0} p(23n+1)\, q^n = q^{21}\cdot\frac{(q^{23};q^{23})_\infty^{23}}{(q;q)_\infty^{24}}\cdot\big(f_3^7 + 29\cdot 53\cdot f_3^5\cdot f_5 + 2^6\cdot 1097\cdot f_3^5\cdot f_4 + 2^2\cdot 43\cdot 4483\cdot f_3^6 - 3^2\cdot 11\cdot 23\cdot 1140975611\cdot f_3^2 + 3\cdot 23^2\cdot 6673419971\cdot f_5 + 23^2\cdot 1129\cdot 594379\cdot f_4 - 11\cdot 23^2\cdot 5534043863\cdot f_3 + 2^2\cdot 3^2\cdot 5^2\cdot 13\cdot 103\cdot f_3^4\cdot f_5 + 5\cdot 11\cdot 235099\cdot f_3^4\cdot f_4 + 2\cdot 17\cdot 37\cdot 43\cdot 1811\cdot f_3^5 + 6361\cdot 11059\cdot f_3^3\cdot f_5 + 23\cdot 101\cdot 1657\cdot 8089\cdot f_3^3 + 2\cdot 3^2\cdot 23\cdot 17911\cdot 89317\cdot f_3\cdot f_5 + 3\cdot 7^2\cdot 23\cdot 29\cdot 2462023\cdot f_3\cdot f_4 + 2^4\cdot 3\cdot 5\cdot 2629547\cdot f_3^3\cdot f_4 + 3\cdot 12161\cdot 136361\cdot f_3^4 + 2^3\cdot 11\cdot 13\cdot 23\cdot 79\cdot 15313\cdot f_3^2\cdot f_4 + 23^{10}\big) + 23\cdot 461\cdot 99017\cdot q^{10}\cdot\frac{(q^{23};q^{23})_\infty^{11}}{(q;q)_\infty^{12}},$$

where f_2, f_3 and f_4 are defined by

$$f_3 := -8 - \big(W_1^2W_6W_{10} + W_2^2W_{11}W_3 + W_3^2W_5W_7 + W_4^2W_1W_6 + W_5^2W_7W_4 + W_6^2W_{10}W_9 + W_7^2W_4W_1 + W_8^2W_2W_{11} + W_9^2W_8W_2 + W_{10}^2W_9W_8 + W_{11}^2W_3W_5\big),$$

$$f_4 := 7 - \big(W_1^3W_5W_8 + W_2^3W_{10}W_7 + W_3^3W_8W_1 + W_4^3W_3W_9 + W_5^3W_2W_6 + W_6^3W_7W_2 + W_7^3W_{11}W_{10} + W_8^3W_6W_5 + W_9^3W_1W_3 + W_{10}^3W_4W_{11} + W_{11}^3W_9W_4\big),$$

$$f_5 := -32 - \big(W_1^4W_6W_{11} + W_2^4W_{11}W_1 + W_3^4W_5W_{10} + W_4^4W_1W_2 + W_5^4W_7W_9 + W_6^4W_{10}W_3 + W_7^4W_4W_8 + W_8^4W_2W_4 + W_9^4W_8W_7 + W_{10}^4W_9W_5 + W_{11}^4W_3W_6\big).$$

Remark 4.5. We note that the right-hand sides of each identity in Theorems 4.1–4.4 have positive integral powers of q. Also, the definition of W_ks implies that the f_is $(i = 2, 3, 4)$ have integral coefficients. Thus, Theorem 4.1 is a witness identity for $11 \mid p(11n + 6)$.

Remark 4.6. In the proofs of Theorems 4.1–4.4, we use the basis as in Theorem 3.6. By changing one of these basis elements for the space $S_5(11,\chi_{11})$, a slightly modified form of Theorem 4.1 can be obtained as follows:

$$\sum_{n\geq 0} p(11n+6)\, q^n = 11\cdot \frac{B(q)}{q\cdot (q;q)_\infty^{11}} + 11\cdot q^4 \cdot \frac{(q^{11};q^{11})_\infty^{11}}{(q;q)_\infty^{12}}\cdot \left(6\cdot 11\cdot f_2 + 2\cdot 31\cdot f_3 + 11^3\right), \tag{4.1}$$

where $B(q)$ is the following binary theta series considered in [25]:

$$B(q) := \sum_{\substack{x,y\in\mathbb{Z}\\ x^2+11y^2\equiv 0 \pmod 4}} \frac{2x^4-132x^2y^2+242y^4}{64}\cdot q^{\frac{x^2+11y^2}{4}} = q+7q^3+16q^4-49q^5+\cdots,$$

which is a weight 5 form with complex multiplication by $\mathbb{Q}(\sqrt{-11})$ and trivial character. Thus, (4.1) is another witness identity for $11 \mid p(11n+6)$ since $B(q)$ has integral coefficients which can be readily seen.

5. Proofs of results in Section 4 and a new proof of Zuckerman's identity

Here we only give the proof of Theorem 4.1 and a new proof of an identity of Zuckerman. The proofs of Theorems 4.2-4.4 follow in a similar vein and we only outline them here.

5.1. Proof of Theorem 4.1

By choosing $p = 11$ in (2.1), we obtain

$$E_5(\tau) = \sum_{n\geq 1}\left(\sum_{\ell\mid n}\chi_{11}(n/\ell)\,\ell^4\right) q^n = \sum_{n\geq 1}\chi_{11}(n)\cdot \frac{q^n(1+11q^n+11q^{2n}+q^{3n})}{(1-q^n)^5}, \tag{5.1}$$

which is an Eisenstein series in $\mathcal{M}_5(11,\chi_{11})$. Here the last equality in (5.1) follows upon using the generating function for Eulerian polynomials.

Next, we use Theorem 3.6 to obtain the following basis for $\mathcal{S}_5(11,\chi_{11})$:

$$B_1(q,11) := \frac{\eta(11\tau)^{11}}{\eta(\tau)}\cdot f_2^2,\quad B_2(q,11) := \frac{\eta(11\tau)^{11}}{\eta(\tau)}\cdot f_3,\quad B_3(q,11) := \frac{\eta(11\tau)^{11}}{\eta(\tau)}\cdot f_2, \tag{5.2}$$

where f_2 and f_3 are as in [14, pp. 19] with $p = 1$ and $g = 1$. We next claim that

$$E_5(\tau) = B_1(q,11) + 10\cdot B_2(q,11) + 86\cdot B_3(q,11) + 1275\cdot \frac{\eta(11\tau)^{11}}{\eta(\tau)}. \tag{5.3}$$

To prove this claim, we first note using Theorem 3.1 that both sides of (5.3) are elements of $\mathcal{M}_5(11,\chi_{11})$. Using SageMath, we have verified that the first 5 coefficients in the q-expansions of both sides agree. It now follows using Theorem 3.3 that they are identically same. This proves our claim.

We now apply the U_{11}-operator on both sides of (5.3). From Lemma 3.5, we have

$$U_{11}(E_5(\tau)) = 11^4\cdot E_5(\tau). \tag{5.4}$$

To find the action of the U_{11}-operator in the right-hand side of (5.3), we note that each term in this side belongs to $\mathcal{S}_5(11,\chi_{11})$. In view of Proposition 3.4, the U_{11}-operator applied on each term of the right-hand side of (5.3) (except the last term) again belongs to $\mathcal{S}_5(11,\chi_{11})$. Since $\{B_1(q,11), B_2(q,11), B_3(q,11)\}$ forms a basis for $\mathcal{S}_5(11,\chi_{11})$, we obtain using SageMath and Theorem 3.3 the following:

$$\begin{aligned} U_{11}(B_1(q,11)) &= -220\cdot B_1(q,11) + 2541\cdot B_2(q,11) - 6292\cdot B_3(q,11),\\ U_{11}(B_2(q,11)) &= -11\cdot B_1(q,11),\\ U_{11}(B_3(q,11)) &= 11\cdot B_1(q,11) - 121\cdot B_2(q,11) + 363\cdot B_3(q,11). \end{aligned} \tag{5.5}$$

For the last term in the right-hand side of (5.3), it is easy to see that

$$U_{11}\left(\frac{\eta(11\tau)^{11}}{\eta(\tau)}\right) = (q;q)_\infty^{11}\cdot\sum_{n\geq 0} p(11n+6)\,q^{n+1}. \tag{5.6}$$

It now follows from (5.3)–(5.6) that

$$\begin{aligned} 11^4\cdot E_5(\tau) &= (-220\cdot B_1(q,11) + 2541\cdot B_2(q,11) - 6292\cdot B_3(q,11)) + 10\cdot(-11\cdot B_1(q,11))\\ &\quad +\ 86\cdot(11\cdot B_1(q,11) -\ 121\cdot B_2(q,11) + 363\cdot B_3(q,11))\\ &\quad + 1275\cdot(q;q)_\infty^{11}\cdot\sum_{n\geq 0} p(11n+6)\,q^{n+1}. \end{aligned} \tag{5.7}$$

Using (5.3) in (5.7) and rearranging yields

$$\begin{aligned} 1275\cdot(q;q)_\infty^{11}\cdot\sum_{n\geq 0} p(11n+6)\,q^{n+1} &= 14025\cdot B_1(q,11) + 154275\cdot B_2(q,11)\\ &\quad + 1234200\cdot B_3(q,11) + 11^4\cdot 1275\cdot\frac{\eta(11\tau)^{11}}{\eta(\tau)}. \end{aligned} \tag{5.8}$$

The result now follows from (5.8).

5.2. A new proof of Zuckerman's identity

Here we obtain the proof of the following identity of Zuckerman [40] for the generating function of $p(13n+6)$:

$$\begin{aligned} \sum_{n\geq 0} p(13n+6)\,q^n &= 11\cdot\frac{(q^{13};q^{13})_\infty}{(q;q)_\infty^2} + 36\cdot 13\cdot q\cdot\frac{(q^{13};q^{13})_\infty^3}{(q;q)_\infty^4} + 38\cdot 13^2\cdot q^2\cdot\frac{(q^{13};q^{13})_\infty^5}{(q;q)_\infty^6}\\ &\quad + 20\cdot 13^3\cdot q^3\cdot\frac{(q^{13};q^{13})_\infty^7}{(q;q)_\infty^8} + 6\cdot 13^4\cdot q^4\cdot\frac{(q^{13};q^{13})_\infty^9}{(q;q)_\infty^{10}}\\ &\quad + 13^5\cdot q^5\cdot\frac{(q^{13};q^{13})_\infty^{11}}{(q;q)_\infty^{12}} + 13^5\cdot q^6\cdot\frac{(q^{13};q^{13})_\infty^{13}}{(q;q)_\infty^{14}}. \end{aligned} \tag{5.9}$$

By choosing $p = 13$ in (2.1), we obtain

$$E_6(\tau) = \sum_{n\geq 1}\left(\sum_{\ell\mid n}\chi_{13}(n/\ell)\,\ell^5\right) q^n, \tag{5.10}$$

which is an Eisenstein series in $\mathcal{M}_6(13,\chi_{13})$.

Using SageMath, we have verified that the following eta-quotients form a basis for $\mathcal{S}_6(13, \chi_{13})$:

$$e_1(q,13) := \eta(\tau)^{11} \cdot \eta(13\tau), \quad e_2(q,13) := \eta(\tau)^9 \cdot \eta(13\tau)^3, \quad e_3(q,13) := \eta(\tau)^7 \cdot \eta(13\tau)^5,$$
$$e_4(q,13) := \eta(\tau)^5 \cdot \eta(13\tau)^7, \quad e_5(q,13) := \eta(\tau)^3 \cdot \eta(13\tau)^9, \quad e_5(q,13) := \eta(\tau) \cdot \eta(13\tau)^{11}.$$

Using SageMath and Theorem 3.3, we obtain

$$E_6(\tau) = e_1(q,13) + 42 \cdot e_2(q,13) + 578 \cdot e_3(q,13) + 3960 \cdot e_4(q,13) + 15446 \cdot e_5(q,13) + 33462 \cdot e_6(q,13) + 33463 \cdot \frac{\eta(13\tau)^{13}}{\eta(\tau)}. \tag{5.11}$$

Further, using Theorem 3.3 and Lemma 3.5 we find that

$$\begin{aligned} 13^5 \cdot E_6(\tau) = 3200 \cdot e_1(q,13) - 66378 \cdot e_2(q,13) - 292032 \cdot e_3(q,13) - 43940 \cdot e_4(q,13) \\ + 571220 \cdot e_5(q,13) - 371293 \cdot e_6(q,13) \\ + 33463 \cdot (q;q)_\infty^{13} \cdot \sum_{n\geq 0} p(13n+6)\, q^{n+1}. \end{aligned} \tag{5.12}$$

Identity (5.9) now follows upon using (5.11) in (5.12) and rearranging terms.

5.3. Proofs of Theorems 4.2-4.4

As in the proof of Theorem 4.1, we consider Eisenstein series $E_{\frac{p-1}{2}}(\tau)$ where $p = 17, 19$ and 23. Next, we show that $E_{\frac{p-1}{2}}(\tau)$ is a linear combination of Garvan's basis (Theorem 3.6) and $\dfrac{\eta(p\tau)^p}{\eta(\tau)}$ as in (5.3) in the case $p = 11$. We then apply U_p-operator on this identity. The action of the U_p-operator on $E_{\frac{p-1}{2}}(\tau)$ is given by Lemma 3.5 and the actions of U_p-operator on the corresponding basis elements can be obtained using SageMath and Theorem 3.3 as in (5.5) in the case $p = 11$. Finally, the result follows as in the case $p = 11$.

6. Some new identities for $p(n)$ modulo 5 and 7

Here we discuss another proof of Theorem 1.2 using a dissection formula from Ramanujan's Notebook [6, Theorem 12.4, pp. 274]. As bonuses, we obtain new identities for the generating functions of $p(5n + j)$ and $p(7n + j')$ for all $j \in \{0, 1, 2, 3\}$ and $j' \in \{0, 1, 2, 3, 4, 5\}$. To this end, we denote by $f_{j,n}(q)$ the following:

$$f_{j,n}(q) := \frac{(q^{2j}; q^n)_\infty (q^{n-2j}; q^n)_\infty}{(q^j; q^n)_\infty (q^{n-j}; q^n)_\infty}. \tag{6.1}$$

We start with the first identity in Theorem 1.2. Let ζ_5 denote a fifth root of unity. For ease of notation, let $E(q) = (q;q)_\infty := \prod_{n=1}^{\infty}(1-q^n)$. Then we have

$$P(q) = \frac{\prod_{k=1}^{4} E(\zeta_5^k q)}{\prod_{k=0}^{4} E(\zeta_5^k q)} =: \frac{N_5(q)}{D_5(q)}. \tag{6.2}$$

It easily follows via use of the identity, $(1-x^5) = \prod_{k=0}^{4}(1-\zeta_5^k \cdot x)$ that

$$D_5(q) = \frac{E(q^5)^6}{E(q^{25})}. \tag{6.3}$$

Let us assume that

$$N_5(q) := \sum_{n\geq 0} c_n \cdot q^n =: \sum_{j=0}^{4} N_{j,5}(q), \tag{6.4}$$

for some $c_n \in \mathbb{Z}$, and where for $0 \leq j \leq 4$, we denote by $N_{j,5}(q)$ the following:

$$N_{j,5}(q) := \sum_{n\geq 0} c_{5n+j} \cdot q^{5n+j}. \tag{6.5}$$

Clearly, we have

$$D_5(q) = E(q) \cdot N_5(q) = E(q) \cdot \left(\sum_{j=0}^{4} N_{j,5}(q) \right). \tag{6.6}$$

Using [6, Theorem 12.4, pp. 274] with $n = 5$ in the right-hand side of (6.6) and multiplying, we find

$$\begin{aligned} \frac{D_5(q)}{E(q^{25})} &= \left(f_{1,5}(q^5) - q - f_{2,5}(q^5) \cdot q^2\right) \cdot \left(N_{0,5}(q) + N_{1,5}(q) + N_{2,5}(q) + N_{3,5}(q) + N_{4,5}(q)\right) \\ &= \left(N_{0,5}(q) \cdot f_{1,5}(q^5) - N_{3,5}(q) \cdot f_{2,5}(q^5) \cdot q^2 - N_{4,5}(q) \cdot q\right) \\ &+ \left(-N_{0,5}(q) \cdot q + N_{1,5}(q) \cdot f_{1,5}(q^5) - N_{4,5}(q^5) \cdot f_{2,5}(q^5) \cdot q^2\right) \\ &+ \left(-N_{0,5}(q^5) \cdot f_{2,5}(q^5) \cdot q^2 - N_{1,5}(q^5) \cdot q + N_{2,5}(q^5) \cdot f_1\right) \\ &+ \left(-N_{1,5}(q^5) \cdot f_{2,5}(q^5) \cdot q^2 - N_{2,5}(q^5) \cdot q + N_{3,5}(q^5) \cdot f_{1,5}(q^5)\right) \\ &+ \left(-N_{2,5}(q) \cdot f_{2,5}(q^5) \cdot q^2 - N_{3,5}(q) \cdot q + N_{4,5}(q) \cdot f_{1,5}(q^5)\right), \end{aligned} \tag{6.7}$$

where we have grouped the expressions in parentheses above based on residue classes of exponents of q congruent to $0, 1, 2, 3$ and 4 modulo 5 in order. For ease of notation, let us put $f_{j,5}(q^5) = f_j(q^5)$ and $D_5(q)/E(q^{25}) = R_5(q)$. Since the exponents of q in the q-expansion of $R_5(q)$ are all congruent to 0 modulo 5, (6.7) is equivalent to the following matrix equation:

$$A_5(q) := \begin{pmatrix} f_1(q^5) & 0 & 0 & -q^2 \cdot f_2(q^5) & -q \\ -q & f_1(q^5) & 0 & 0 & -q^2 \cdot f_2(q^5) \\ -q^2 \cdot f_2(q^5) & -q & f_1(q^5) & 0 & 0 \\ 0 & -q^2 \cdot f_2(q^5) & -q & f_1(q^5) & 0 \\ 0 & 0 & -q^2 \cdot f_2(q^5) & -q & f_1(q^5) \end{pmatrix} \begin{pmatrix} N_{0,5}(q) \\ N_{1,5}(q) \\ N_{2,5}(q) \\ N_{3,5}(q) \\ N_{4,5}(q) \end{pmatrix} = \begin{pmatrix} R_5(q) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6.8}$$

On evaluating the determinant of the matrix $A_5(q)$, we find

$$\det(A_5(q)) = f_1(q^5)^5 - q^{10} \cdot f_2(q^5)^5 - 5 \cdot q^5 \cdot f_1(q^5) \cdot f_2(q^5) \cdot \left(1 + f_1(q^5) \cdot f_2(q^5)\right) - q^5. \tag{6.9}$$

It is clear that $\det(A_5(q))$ is a continuous function for $|q| < 1$, since $f_1(q^5)$ and $f_2(q^5)$ are continuous. Also, $\det(A_5(0)) = 1 \neq 0$. This implies that there is a neighbourhood around zero, $|q| < r < 1$ where $\det(A_5(q)) \neq 0$. Indeed, it turns out that $\det(A_5(q)) = D_5(q)/E(q^{25})^5$. And so, $\det(A_5(q)) \neq 0$ for all $|q| < 1$.

Thus, for $|q| < 1$, we can solve the matrix equation (6.8) for $N_{0,5}(q)$, $N_{1,5}(q)$, $N_{2,5}(q)$, $N_{3,5}(q)$, $N_{4,5}(q)$, and this yields

$$N_{0,5}(q) = \frac{R_5(q) \cdot \left[f_{1,5}(q^5)^4 - 2 \cdot f_{1,5}(q^5) \cdot f_{2,5}(q^5)^2 \cdot q^5 + f_{2,5}(q^5) \cdot q^5\right]}{\det(A_5(q))},$$
$$N_{1,5}(q) = \frac{R_5(q) \cdot \left[f_{1,5}(q^5) \cdot f_{2,5}(q^5)^3 \cdot q^6 + f_{1,5}(q^5)^3 \cdot q + f_{2,5}(q^5)^2 \cdot q^6\right]}{\det(A_5(q))},$$
$$N_{2,5}(q) = \frac{R_5(q) \cdot \left[-f_{1,5}(q^5)^3 \cdot f_{2,5}(q^5) \cdot q^2 + f_{2,5}(q^5)^3 \cdot q^7 - f_{1,5}(q^5)^2 \cdot q^2\right]}{\det(A_5(q))},$$
$$N_{3,5}(q) = \frac{R_5(q) \cdot \left[f_{2,5}(q^5)^4 \cdot q^8 - 2 \cdot f_{1,5}(q^5)^2 \cdot f_{2,5}(q^5) \cdot q^3 + f_{1,5}(q^5) \cdot q^3\right]}{\det(A_5(q))},$$
$$N_{4,5}(q) = \frac{R_5(q) \cdot \left[f_{1,5}(q^5)^2 \cdot f_{2,5}(q^5)^2 \cdot q^4 + 3 \cdot f_{1,5}(q^5) \cdot f_{2,5}(q^5) \cdot q^4 + q^4\right]}{\det\,(A_5(q))}. \tag{6.10}$$

We have

Theorem 6.1. *For $j \in \{0, \cdots, 4\}$ and $|q| < 1$, we have*

$$\sum_{n\geq 0} p(5n+j) \cdot q^n = \frac{E(q^5)^5}{E(q)^6} \cdot G_{j,5}(q),$$

where

$$G_{0,5}(q) = f_{1,5}(q)^4 - f_{2,5}(q) \cdot q, \;\; G_{1,5}(q) = f_{1,5}(q)^3 + 2 \cdot f_{2,5}(q)^2 \cdot q,$$
$$G_{2,5}(q) = f_{2,5}(q)^3 \cdot q - 2 \cdot f_{1,5}(q)^2, \;\; G_{3,5}(q) = f_{2,5}(q)^4 \cdot q - f_{1,5}(q), \;\; G_{4,5}(q) = 5.$$

Proof. Using [6, Theorem 12.4, pp. 274] with $n = 5$ and expanding, we have

$$\begin{aligned} D_5(q) = \prod_{k=0}^{4} E(\zeta_5^k q) &= E(q^{25})^5 \prod_{k=0}^{4} (f_{1,5}(q^5) - \zeta_5^k \cdot q - \zeta_5^{2k} \cdot f_{2,5}(q^5) \cdot q^2) \\ &= E(q^{25})^5 \cdot \det(A_5(q)). \end{aligned} \tag{6.11}$$

Thus, (6.3) and (6.11) yield

$$D_5(q) \cdot \det(A_5(q)) = \frac{D_5(q)^2}{E(q^{25})^5} = \frac{E(q^5)^{12}}{E(q^{25})^7}. \tag{6.12}$$

We prove the result for $j = 4$ since all the other cases are exactly similar. Since $D_5(q)$ is a power series in q^5 with integer coefficients, it follows from (6.2), (6.4) and (6.10) that

$$\begin{aligned} \sum_{n\geq 0} p(5n+4) \cdot q^{5n+4} &= \frac{N_{4,5}(q)}{D_5(q)} \\ &= \frac{R_5(q) \cdot \left(f_{1,5}(q^5)^2 \cdot f_{2,5}(q^5)^2 \cdot q^4 + 3 \cdot f_{1,5}(q^5) \cdot f_{2,5}(q^5)^2 \cdot q^4 + q^4\right)}{D_5(q) \cdot \det(A_5(q))}. \end{aligned} \tag{6.13}$$

Using (6.12), $R_5(q) = D_5(q)/E(q^{25})$ and the fact that $f_{1,5}(q) \cdot f_{2,5}(q) = 1$, we obtain from (6.13) that

$$\sum_{n\geq 0} p(5n+4) \cdot q^{5n+4} = 5 \cdot q^4 \cdot \frac{E(q^{25})^4}{D_5(q)}. \tag{6.14}$$

The result now follows by using (6.3) in (6.14) and $q \to q^{1/5}$. □

Next, we move towards the second identity in Theorem 1.2. Let ζ_7 denote a seventh root of unity. We have

$$P(q) = \frac{\prod_{k=1}^{6} E(\zeta_7^k q)}{\prod_{k=0}^{6} E(\zeta_7^k q)} =: \frac{N_7(q)}{D_7(q)}. \tag{6.15}$$

It easily follows via use of the identity, $(1 - x^7) = \prod_{k=0}^{6}(1 - \zeta_7^k \cdot x)$ that

$$D_7(q) = \frac{E(q^7)^8}{E(q^{49})}. \tag{6.16}$$

Let us assume that

$$N_7(q) := \sum_{n\geq 0} c_n \cdot q^n =: \sum_{j=0}^{6} N_{j,7}(q), \tag{6.17}$$

for some $c_n \in \mathbb{Z}$, and where for $0 \leq j \leq 6$, we denote by $N_{j,7}(q)$ the following:

$$N_{j,7}(q) := \sum_{n\geq 0} c_{7n+j} \cdot q^{7n+j}. \tag{6.18}$$

Clearly, we have

$$D_7(q) = E(q) \cdot N_7(q) = E(q) \cdot \left(\sum_{j=0}^{6} N_{j,7}(q)\right). \tag{6.19}$$

Using [6, Theorem 12.4, pp. 274] with $n = 7$ in the right-hand side of (6.19) and multiplying, we find

$$\begin{aligned}
\frac{D_7(q)}{E(q^{49})} &= \left(f_{1,7}(q^7) - f_{2,7}(q^7) \cdot q - q^2 + f_{3,7}(q^7) \cdot q^5\right) \\
&\quad \times \ (N_{0,7}(q) + N_{1,7}(q) + N_{2,7}(q) + N_{3,7}(q) + N_{4,7}(q) + N_{5,7}(q) + N_{6,7}(q)) \\
&= \left(N_{0,7}(q) \cdot f_{1,7}(q^7) + N_{2,7}(q) \cdot f_{3,7}(q^7) \cdot q^5 - N_{5,7}(q) \cdot q^2 - N_{6,7}(q) \cdot f_{2,7}(q^7) \cdot q\right) \\
&+ \ \left(-N_{0,7}(q) \cdot f_{2,7}(q^7) \cdot q + N_{1,7}(q) \cdot f_{1,7}(q^7) + N_{3,7}(q^7) \cdot f_{3,7}(q^7) \cdot q^5 - N_{6,7}(q) \cdot q^2\right) \\
&+ \ \left(-N_{0,7}(q^7) \cdot q^2 - N_{1,7}(q^7) \cdot f_{2,7}(q^7) \cdot q + N_{2,7}(q^7) \cdot f_{1,7}(q^7) + N_{4,7}(q) \cdot f_{3,7}(q^7) \cdot q^5\right) \\
&+ \ \left(-N_{1,7}(q^7) \cdot q^2 - N_{2,7}(q^7) \cdot f_{2,7}(q^7) \cdot q + N_{3,7}(q^7) \cdot f_{1,7}(q^7) + N_{5,7}(q) \cdot f_{3,7}(q^7) \cdot q^5\right) \\
&+ \ \left(-N_{2,7}(q) \cdot q^2 - N_{3,7}(q) \cdot f_{2,7}(q^7) \cdot q + N_{4,7}(q) \cdot f_{1,7}(q^7) + N_{6,7}(q) \cdot f_{3,7}(q^7) \cdot q^5\right) \\
&+ \ \left(N_{0,7}(q) \cdot f_{3,7}(q^7) \cdot q^5 - N_{3,7}(q) \cdot q^2 - N_{4,7}(q) \cdot f_{2,7}(q^7) \cdot q + N_{5,7}(q) \cdot f_{1,7}(q^7)\right) \\
&+ \ \left(N_{1,7}(q) \cdot f_{3,7}(q^7) \cdot q^5 - N_{4,7}(q) \cdot q^2 - N_{5,7}(q) \cdot f_{2,7}(q^7) \cdot q + N_{6,7}(q) \cdot f_{1,7}(q^7)\right),
\end{aligned} \tag{6.20}$$

where we have grouped the expressions in parentheses above based on residue classes of exponents of q congruent to $0, 1, 2, 3, 4, 5$ and 6 modulo 7 in order. For ease of notation, let us put $f_{j,7}(q^7) = f_j(q^7)$, $D_7(q)/E(q^{49}) = R_7(q)$ and $N_{j,7}(q) = N_j(q)$. Since the exponents of q in the q-series expansion of $R_7(q)$ are all congruent to 0 modulo 7, (6.20) is equivalent to the following matrix equation:

$$A_7(q) := \begin{pmatrix} f_1(q^7) & 0 & q^5 \cdot f_3(q^7) & 0 & 0 & -q^2 & -q \cdot f_2(q^7) \\ -q \cdot f_2(q^7) & f_1(q^7) & 0 & q^5 \cdot f_3(q^7) & 0 & 0 & -q^2 \\ -q^2 & -q \cdot f_2(q^7) & f_1(q^7) & 0 & q^5 \cdot f_3(q^7) & 0 & 0 \\ 0 & -q^2 & -q \cdot f_2(q^7) & f_1(q^7) & 0 & q^5 \cdot f_3(q^7) & 0 \\ 0 & 0 & -q^2 & -q \cdot f_2(q^7) & f_1(q^7) & 0 & q^5 \cdot f_3(q^7) \\ q^5 \cdot f_3(q^7) & 0 & 0 & -q^2 & -q \cdot f_2(q^7) & f_1(q^7) & 0 \\ 0 & q^5 \cdot f_3(q^7) & 0 & 0 & -q^2 & -q \cdot f_2(q^7) & f_1(q^7) \end{pmatrix} \begin{pmatrix} N_0(q) \\ N_1(q) \\ N_2(q) \\ N_3(q) \\ N_4(q) \\ N_5(q) \\ N_6(q) \end{pmatrix} = \begin{pmatrix} R_7(q) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6.21}$$

On evaluating the determinant of the matrix $A_7(q)$, we find

$$\begin{aligned} \det(A_7(q)) = {} & f_1(q^7)^7 - \big[7f_1(q^7)^3 \cdot f_2(q^7) + 14 \cdot f_1(q^7)^2 \cdot f_2(q^7)^3 + 7 \cdot f_1(q^7) \cdot f_2(q^7)^5 + f_2(q^7)^7 \\ & -7 \cdot f_1(q^7)^5 \cdot f_3(q^7) - 7 \cdot f_1(q^7)^4 \cdot f_2(q^7)^2 \cdot f_3(q^7)\big] \cdot q^7 + \big[1 + 14 \cdot f_1(q^7) \cdot f_2(q^7) f_3(q^7) \\ & +7 \cdot f_2(q^7)^3 \cdot f_3(q^7) - 14 \cdot f_1(q^7)^3 \cdot f_3(q^7)^2 - 7 \cdot f_1(q^7)^2 \cdot f_2(q^7)^2 \cdot f_3(q^7)^2 \\ & -7 \cdot f_1(q^7) \cdot f_2(q^7)^4 \cdot f_3(q^7)^2\big] \cdot q^{14} + \big[7 \cdot f_1(q^7) \cdot f_3(q^7)^3 + 14 \cdot f_2(q^7)^2 \cdot f_3(q^7)^3 \\ & - 7 \cdot f_1(q^7)^2 f_2(q^7) f_3(q^7)^4\big] \cdot q^{21} - 7 \cdot f_2(q^7) \cdot f_3(q^7)^5 \cdot q^{28} + f_3(q^7)^7 \cdot q^{35}. \end{aligned} \tag{6.22}$$

It is clear that $\det(A_7(q))$ is a continuous function for $|q| < 1$, since $f_1(q^7),\ f_2(q^7)$ and $f_3(q^7)$ are continuous. Also, $\det(A_7(0)) = 1 \neq 0$. This implies that there is a neighbourhood around zero, $|q| < r < 1$ where $\det(A_7(q)) \neq 0$. Indeed, it turns out that $\det(A_7(q)) = D_7(q)/E(q^{49})^8$. And so, $\det(A_7(q)) \neq 0$ for all $|q| < 1$.

Thus, for $|q| < 1$, we can solve the matrix equation (6.21) for $N_{0,7}(q),\ N_{1,7}(q),\ N_{2,7}(q),\ N_{3,7}(q),\ N_{4,7}(q),\ N_{5,7}(q),\ N_{6,7}(q)$, and this yields

$$\begin{gathered} N_{0,7}(q) = \frac{R_7(q) \cdot G_{0,7}(q)}{\det(A_7(q))}, \quad N_{1,7}(q) = \frac{R_7(q) \cdot G_{1,7}(q)}{\det(A_7(q))}, \\ N_{2,7}(q) = \frac{R_7(q) \cdot G_{2,7}(q)}{\det(A_7(q))}, \quad N_{3,7}(q) = \frac{R_7(q) \cdot G_{3,7}(q)}{\det(A_7(q))} \\ N_{4,7}(q) = \frac{R_7(q) \cdot G_{4,7}(q)}{\det(A_7(q))}, \quad N_{5,7}(q) = \frac{R_7(q) \cdot G_{5,7}(q)}{\det(A_7(q))}, \\ N_{6,7}(q) = \frac{R_7(q) \cdot G_{6,7}(q)}{\det(A_7(q))}, \end{gathered} \tag{6.23}$$

where

$$\begin{aligned} G_{0,7}(q^7) = {} & f_{1,7}(q^7)^6 - \big[3 \cdot f_{1,7}(q^7)^2 \cdot f_{2,7}(q^7) + 4 \cdot f_{1,7}(q^7) \cdot f_{2,7}(q^7)^3 + f_{2,7}(q^7)^5 - 5 \cdot f_{1,7}(q^7)^4 \cdot f_{3,7}(q^7) \\ & -4 \cdot f_{1,7}(q^7)^3 \cdot f_{2,7}(q^7)^2 \cdot f_{3,7}(q^7)\big] \cdot q^7 - \big[2 \cdot f_{2,7}(q^7) \cdot f_{3,7}(q^7) - 6 \cdot f_{1,3}(q^7)^2 \cdot f_{3,7}(q^7)^2 \\ & - 2 \cdot f_{1,7}(q^7) \cdot f_{2,7}(q^7)^2 \cdot f_{3,7}(q^7)^2 - f_{2,7}(q^7)^4 \cdot f_{3,7}(q^7)^2\big] \cdot q^{14} + \big[f_{3,7}(q^7)^3 - 2 \cdot f_{1,7}(q^7) \cdot f_{2,7}(q^7) \cdot f_{3,7}(q^7)^4\big] \cdot q^{21}, \\ G_{1,7}(q^7) = {} & f_{1,7}(q^7)^5 \cdot f_{2,7}(q^7) \cdot q + \big[f_{1,7}(q^7)^2 + 3 \cdot f_{1,7}(q^7) \cdot f_{2,7}(q^7)^2 + f_{2,7}(q^7)^4 + f_{1,7}(q^7)^3 \cdot f_{2,7}(q^7) \cdot f_{3,7}(q^7) \\ & + 3 \cdot f_{1,7}(q^7)^2 \cdot f_{2,7}(q^7)^3 \cdot f_{3,7}(q^7)\big] \cdot q^8 + \big[f_{3,7}(q^7) + 2 \cdot f_{1,7}(q^7) \cdot f_{2,7}(q^7) \cdot f_{3,7}(q^7)^2 - 3 \cdot f_{2,7}(q^7)^3 \cdot f_{3,7}(q^7)^2 \\ & -f_{1,7}(q^7)^3 \cdot f_{3,7}(q^7)^3\big] \cdot q^{15} - \big[2 \cdot f_{1,7}(q^7) \cdot f_{3,7}(q^7)^4 - f_{2,7}(q^7)^2 \cdot f_{3,7}(q^7)^4\big] \cdot q^{22}, \\ G_{2,7}(q^7) = {} & \big[f_{1,7}(q^7)^5 + f_{1,7}(q^7)^4 \cdot f_{2,7}(q^7)^2\big] \cdot q^2 - \big[2 \cdot f_{1,7}(q^7) \cdot f_{2,7}(q^7) + f_{2,7}(q^7)^3 - 4 \cdot f_{1,7}(q^7)^3 \cdot f_{3,7}(q^7) \\ & -2 \cdot f_{1,7}(q^7)^2 \cdot f_{2,7}(q^7)^2 \cdot f_{3,7}(q^7) - 2 \cdot f_{1,7}(q^7) \cdot f_{2,7}(q^7)^4 \cdot f_{3,7}(q^7)\big] \cdot q^9 + \big[3 \cdot f_{1,7}(q^7) \cdot f_{3,7}(q^7)^2 \\ & + 6 \cdot f_{2,7}(q^7)^2 \cdot f_{3,7}(q^7)^2 - 4 \cdot f_{1,7}(q^7)^2 \cdot f_{2,7}(q^7) \cdot f_{3,7}(q^7)^3\big] \cdot q^{16} - 5 \cdot f_{2,3}(q^7) \cdot f_{3,7}(q^7)^4 \cdot q^{23} + f_{3,7}(q^7)^6 \cdot q^{30}, \\ G_{3,7}(q^7) = {} & \big[2 \cdot f_{1,7}(q)^4 \cdot f_{2,7}(q^7) + f_{1,7}(q^7)^3 \cdot f_{2,7}(q^7)^3\big] \cdot q^3 + \big[f_{1,7}(q^7)^4 \cdot f_{3,7}(q^7)^2 + 2 \cdot f_{1,7}(q)^2 \cdot f_{2,7}(q^7) \cdot f_{3,7}(q^7) \\ & +f_{1,7}(q^7) \cdot f_{2,7}(q^7)^3 \cdot f_{3,7}(q^7) + f_{2,7}(q^7)^5 \cdot f_{3,7}(q^7) + f_{2,7}(q^7)^2\big] \cdot q^{10} + \big[3 \cdot f_{1,7}(q^7)^2 \cdot f_{3,7}(q^7)^3 \\ & -3 \cdot f_{1,7}(q^7) \cdot f_{2,7}(q^7)^2 \cdot f_{3,7}(q^7)^3 - 3 \cdot f_{2,7}(q^7) \cdot f_{3,7}(q^7)^2 \cdot\big] \cdot q^{17} + f_{3,7}(q^7)^4 \cdot q^{24}, \end{aligned}$$

$$G_{4,7}(q^7) = [f_{1,7}(q^7)^4 + 3 \cdot f_{1,7}(q^7)^3 \cdot f_{2,7}(q^7)^2 + f_{1,7}(q^7)^2 \cdot f_{2,7}(q^7)^4] \cdot q^4 + [-f_{2,7}(q^7) + 3 \cdot f_{1,7}(q^7)^2 \cdot f_{3,7}(q^7)$$
$$-2 \cdot f_{1,7}(q^7) \cdot f_{2,7}(q^7)^2 \cdot f_{3,7}(q^7) - 2 \cdot f_{2,7}(q^7)^4 \cdot f_{3,7}(q^7) + 3 \cdot f_{1,7}(q^7)^3 \cdot f_{2,7}(q^7) \cdot f_{3,7}(q^7)^2] \cdot q^{11}$$
$$+ [f_{3,7}(q^7)^2 + f_{1,7}(q^7) \cdot f_{2,7}(q^7) \cdot f_{3,7}(q^7)^3 + f_{2,7}(q^7)^3 \cdot f_{3,7}(q^7)^3] \cdot q^{18} - f_{1,7}(q^7) \cdot f_{3,7}(q^7)^5 \cdot q^{25},$$
$$G_{5,7}(q^7) = [3 \cdot f_{1,7}(q^7)^3 \cdot f_{2,7}(q^7) + 4 \cdot f_{1,7}(q^7)^2 \cdot f_{2,7}(q^7)^3 + f_{1,7}(q^7) \cdot f_{2,7}(q^7)^5 - f_{1,7}(q^7)^5 \cdot f_{3,7}(q^7)] \cdot q^5$$
$$+ \ [1 + 8 \cdot f_{1,7}(q^7) \cdot f_{2,7}(q^7) \cdot f_{3,7}(q^7) + 3 \cdot f_{2,7}(q^7)^3 \cdot f_{3,7}(q^7) - f_{1,7}(q^7)^2 \cdot f_{2,7}(q^7)^2 \cdot f_{3,7}(q^7)^2] \cdot q^{12}$$
$$- [3 \cdot f_{1,7}(q^7) \cdot f_{3,7}(q^7)^3 + 4 \cdot f_{2,7}(q^7)^2 \cdot f_{3,7}(q^7)^3] \cdot q^{19} + f_{2,7}(q^7) \cdot f_{3,7}(q^7)^5 \cdot q^{26},$$
$$G_{6,7}(q^7) = [f_{1,7}(q^7)^3 + 6 \cdot f_{1,7}(q^7)^2 \cdot f_{2,7}(q^7)^2 + 5 \cdot f_{1,7}(q^7) \cdot f_{2,7}(q^7)^4 + f_{2,7}(q^7)^6 - 2 \cdot f_{1,7}(q^7)^4 \cdot f_{2,7}(q^7)$$
$$\cdot f_{3,7}(q^7)] \cdot q^6 + [2 \cdot f_{1,7}(q^7) \cdot f_{3,7}(q^7) + 3 \cdot f_{2,7}(q^7)^2 \cdot f_{3,7}(q^7) - 2 \cdot f_{1,7}(q^7)^2 \cdot f_{2,7}(q^7) \cdot f_{3,7}(q^7)^2 - 4 \cdot f_{1,7}(q^7)$$
$$\cdot f_{2,7}(q^7)^3 \cdot f_{3,7}(q^7)^2] \cdot q^{13} + [-4 \cdot f_{2,7}(q^7) \cdot f_{3,7}(q^7)^3 + f_{1,7}(q^7)^2 \cdot f_{3,7}(q^7)^4] \cdot q^{20} + f_{3,7}(q^7)^5 \cdot q^{27}. \tag{6.24}$$

Theorem 6.2. *For $j \in \{0, \cdots, 6\}$, we have*

$$\sum_{n \geq 0} p(7n + j)\, q^n = \frac{E(q^7)^7}{E(q)^8} \cdot \tilde{G}_{j,7}(q),$$

where

$$\tilde{G}_{0,7}(q) = f_{1,7}(q)^2 \left(11 \cdot f_{1,7}(q)^4 - 13 \cdot f_{1,7}(q)^3 f_{2,7}(q)^2 - f_{1,7}(q)^2 f_{2,7}(q)^4 + 2 \cdot f_{1,7}(q) f_{2,7}(q)^6 \right.$$
$$\left. +2 \cdot f_{2,7}(q)^8 \right),$$
$$\tilde{G}_{1,7}(q) = f_{1,7}(q)^2 \left(-3 \cdot f_{1,7}(q)^5 f_{3,7}(q) + 15 \cdot f_{1,7}(q)^3 f_{2,7}(q) - 15 \cdot f_{1,7}(q)^2 f_{2,7}(q)^3 \right.$$
$$\left. +9 \cdot f_{1,7}(q) f_{2,7}(q)^5 - \ 5 \cdot f_{2,7}(q)^7 \right),$$
$$\tilde{G}_{2,7}(q) = f_{1,7}(q)^2 \left(-5 \cdot f_{1,7}(q)^3 + 26 \cdot f_{1,7}(q)^2 f_{2,7}(q)^2 - 31 \cdot f_{1,7}(q) f_{2,7}(q)^4 + 11 \cdot f_{2,7}(q)^6 \right.$$
$$\left. +f_{1,7}(q)^6 f_{3,7}(q)^2 \right),$$
$$\tilde{G}_{3,7}(q) = f_{1,7}(q) \left(5 \cdot f_{1,7}(q)^5 f_{3,7}(q) - 12 \cdot f_{1,7}(q)^3 f_{2,7}(q) + 19 \cdot f_{1,7}(q)^2 f_{2,7}(q)^3 - 8 \cdot f_{1,7}(q) f_{2,7}(q)^5 \right.$$
$$\left. -f_{2,7}(q)^7 \right),$$
$$\tilde{G}_{4,7}(q) = f_{1,7}(q) \left(12 \cdot f_{1,7}(q)^3 - 12 \cdot f_{1,7}(q)^2 f_{2,7}(q)^2 + 3 \cdot f_{1,7}(q) f_{2,7}(q)^4 + 3 \cdot f_{2,7}(q)^6 \right.$$
$$\left. -f_{1,7}(q)^6 f_{3,7}(q)^2 \right),$$
$$\tilde{G}_{5,7}(q) = 7 \cdot \frac{E(q)^4}{E(q^7)^4} + 49 \cdot q,$$
$$\tilde{G}_{6,7}(q) = -10 \cdot f_{1,7}(q)^3 + 17 \cdot f_{1,7}(q)^2 f_{2,7}(q)^3 + f_{1,7}(q) f_{2,7}(q)^4 + f_{2,7}(q)^6 + 2 \cdot f_{1,7}(q)^6 f_{3,7}(q)^2.$$

Proof. Using [6, Theorem 12.4, pp. 274] with $n = 7$ and expanding, we have

$$D_7(q) = \prod_{k=0}^{6} E(\zeta_7^k q) = E(q^{49})^7 \prod_{k=0}^{6} (f_{1,7}(q^7) - f_{2,7}(q^7) \cdot \zeta_7^k \cdot q - \zeta_7^k \cdot q^2 + \zeta_7^k \cdot f_{3,7}(q^7) \cdot q^5)$$
$$= E(q^{49})^7 \cdot \det(A_7(q)). \tag{6.25}$$

Thus, (6.16) and (6.25) yields

$$D_7(q) \cdot \det(A_7(q)) = \frac{D_7(q)^2}{E(q^{49})^7} = \frac{E(q^7)^{16}}{E(q^{49})^9}. \tag{6.26}$$

Since $D_7(q)$ is a power series in q^7 with integer coefficients, it follows from (6.15), (6.17) and (6.23) that

$$\sum_{n\geq 0} p(7n+j)\cdot q^{7n+j} = \frac{N_{j,7}(q)}{D_7(q)} = \frac{R_7(q)\cdot G_{j,7}(q^7)}{D_7(q)\cdot \det(A_7(q))}, \tag{6.27}$$

where $G_{j,7}(q^7)$ are as in (6.24). Using (6.16), (6.26) and $R_7(q) = D_7(q)/E(q^{49})$, we obtain from (6.27) that

$$\sum_{n\geq 0} p(7n+j)\cdot q^{7n+j} = \frac{E(q^{49})^7}{E(q^7)^8}\cdot G_{j,7}(q^7). \tag{6.28}$$

Using [6, Theorem 12.4, pp. 274] with $n = 7$ we get

$$E(q)^3 = E(q^{49})^3\cdot (f_{1,7}(q^7) - f_{2,7}(q^7)\cdot q - q^2 + f_{3,7}(q^7)\cdot q^5)^3. \tag{6.29}$$

We now recall Jacobi's identity (Theorem 1.3). From the right-hand side of this identity we see that $n(n+1)/2 \not\equiv 2, 4, 5 \pmod 7$. Also, $n(n+1)/2 \equiv 6 \pmod 7$ if and only if $n \equiv 3 \pmod 7$. Thus (6.29) yields the following relations upon comparing both sides:

$$a^2 - ab^2 - cq^7 = 0,\ \ a - b^2 - bc^2q^7 = 0,\ \ b - a^2c + c^2q^7 = 0,\ \ abc = 1, \tag{6.30}$$

where for notational convenience we have put $a := f_{1,7}(q^7)$, $b := f_{2,7}(q^7)$ and $c := f_{3,7}(q^7)$.

When $j = 5$, we can adopt the same approach as in [17, Section 7.3, pp. 73] to obtain the following expression for $G_{5,7}(q^7)$ under the constraints (6.30):

$$G_{5,7}(q^7) = 7\cdot q^5\cdot \frac{E(q^7)^4}{E(q^{49})^4} + 49\cdot q^{12}. \tag{6.31}$$

Thus, identities (6.28) and (6.31) yield the second identity in Theorem 1.2 with $q \to q^{1/7}$. However, we chose to adopt a computational approach to see if $G_{j,7}(q^7)$ $(j = 0, \cdots, 6)$ simplifies to anything interesting under the relations (6.30). Thus, we use the following command on Mathematica (check our Mathematica supplement here):

$$\text{FullSimplify}(G_{j,7}(q^7), R_1 == 0, R_2 == 0, R_3 == 0, R_4 == 1),$$

where R_1, R_2, R_3, and R_4 are, respectively, the left-hand sides of the relations defined in (6.30). We obtain the following expressions for $G_{j,7}(q^7)$:

$$\begin{aligned}
G_{0,7}(q^7) &= a^2\left(11a^4 - 13a^3b^2 - a^2b^4 + 2ab^6 + 2b^8\right),\\
G_{1,7}(q^7) &= a^2\left(-3a^5c + 15a^3b - 15a^2b^3 + 9ab^5 - 5b^7\right)\cdot q,\\
G_{2,7}(q^7) &= a^2(-5a^3 + 26a^2b^2 - 31ab^4 + 11b^6 + a^6c^2)\cdot q^2,\\
G_{3,7}(q^7) &= a\left(5a^5c - 12a^3b + 19a^2b^3 - 8ab^5 - b^7\right)\cdot q^3,\\
G_{4,7}(q^7) &= a(12a^3 - 12a^2b^2 + 3ab^4 + 3b^6 - a^6c^2)\cdot q^4,\\
G_{5,7}(q^7) &= -7a\left(a^4c - 2a^2b - ab^3 + b^5\right)\cdot q^5,\\
G_{6,7}(q^7) &= (-10a^3 + 17a^2b^2 + ab^4 + b^6 + 2a^6c^2)\cdot q^6.
\end{aligned} \tag{6.32}$$

In (6.32), we notice that in the expression for $G_{5,7}(q^7)$, the right-hand side has a factor of 7 multiplied outside. In view of (6.28) and the fact that the q-expansions have integral coefficients, we obtain a witness identity for $7 \mid p(7n+5)$ which is a slightly disguised form of the second identity in Theorem 1.2. We now use another function in Mathematica, namely, GroebnerBasis() via the following sequence of commands:

$$\text{gb} := \text{GroebnerBasis}[\{s - G_{5,7}(q^7),\ t - T(q),\ R1,\ R2,\ R3,\ R4 - 1\},\ \{a,\ b,\ c,\ s,\ t,\ q\},\ \{a,\ b,\ c\}],$$
$$\text{Solve}[\text{gb}[[1]] == 0, s], \tag{6.33}$$

where $T(q) := \prod_{k=0}^{6}(f_{1,7}(q^7) - f_{2,7}(q^7)\cdot\zeta_7^k\cdot q - \zeta_7^k\cdot q^2 + \zeta_7^k\cdot f_{3,7}(q^7)\cdot q^5)$. This yields the following relation between $G_{5,7}(q^7)$ and $T(q)$:

$$G_{5,7}(q^7) = 7\cdot(7q^{12} + q^5\sqrt{T(q)}). \tag{6.34}$$

Noting that $T(q) = D_7(q)/E(q^{49})^7$ and using (6.16), we immediately obtain (6.31). This again yields the second identity in Theorem 1.2. □

Using [6, Theorem 12.4, pp. 274], the following dissection formula can be obtained for the partition function $P(q)$. We omit the details.

Theorem 6.3. *Let $p \geq 5$ be a prime, and ζ_p denote a pth root of unity. If $p = 6g+1$, for some $g \geq 1$, then we have the following p-dissections of $P(q)$:*

$$P(q) = \frac{E(q^{p^2})^p}{E(q^p)^{p+1}} \prod_{k=1}^{p-1}\Bigg((-1)^g\zeta_p^{k(p^2-1)/24}q^{(p^2-1)/24} + \sum_{j=1}^{(p-1)/2}(-1)^{j+g}\zeta_p^{k(j-g)(3j-3g-1)/2}q^{(j-g)(3j-3g-1)/2}f_{j,p}(q^p)\Bigg).$$

If $p = 6g-1$, for some $g \geq 1$, then we have

$$P(q) = \frac{E(q^{p^2})^p}{E(q^p)^{p+1}} \prod_{k=1}^{p-1}\Bigg((-1)^g\zeta_p^{k(p^2-1)/24}q^{(p^2-1)/24} + \sum_{j=1}^{(p-1)/2}(-1)^{j+g}\zeta_p^{k(j-g)(3j-3g+1)/2}q^{(j-g)(3j-3g+1)/2}f_{j,p}(q^p)\Bigg),$$

where $f_{j,p}(q)$ are as in (6.1).

7. Circular summation for Ramanujan's theta function

For $|ab| < 1$, let $f(a,b)$ denote Ramanujan's theta function

$$f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2}b^{k(k-1)/2}.$$

In [32, pp. 54], Ramanujan claims (without proof) that if $n \geq 2$, then

$$\sum_{r=0}^{n-1}\Bigg(\sum_{\substack{k=-\infty\\k\equiv r \pmod n}}^{\infty} a^{k(k+1)/2n}b^{k(k-1)/2n}\Bigg)^n = f(a,b)F_n(ab),$$

where

$$F_n(q) := 1 + 2nq^{n(n-1)/2} + \cdots \tag{7.1}$$

Rangachari [33] proved Ramanujan's claim using the theory of modular forms, and Son [36] obtained an elementary proof of the claim. Rangachari also verified Ramanujan's explicit and elegant formula for F_n for $n = 2, 3, 4, 5, 7$ (see [33,36]). There has been much interest in finding further explicit evaluation of F_n (see Ahlgren [1] for $n = 6, 8, 9, 10$, Chua [11] for $n = 9$, Ono [25] for $n = 11$ and Chua [10] for $n = 13$).

Here we obtain explicit evaluations for F_{17}, F_{19} and F_{23}. We also obtain an explicit evaluation for F_{11}, different from [8] and [25]. More precisely, we have the following theorem:

Theorem 7.1. *We have*

$$\begin{aligned}
F_{11}(q) &= 11 \cdot B_1(q, 11) - 99 \cdot B_2(q, 11) + 308 \cdot B_3(q, 11) + 11 \cdot \frac{\eta(11\tau)^{11}}{\eta(\tau)} + \frac{\eta(\tau)^{11}}{\eta(11\tau)},\\
F_{17}(q) &= 17 \cdot B_1(q, 17) - 255 \cdot B_2(q, 17) + 1836 \cdot B_3(q, 17) - 14671 \cdot e_4(q, 17) - 195636 \cdot B_5(q, 17)\\
&\quad + 1981758 \cdot B_6(q, 17) - 7244652 \cdot B_7(q, 17) + 43765021 \cdot e_8(q, 17) + 318766813 \cdot B_9(q, 17)\\
&\quad - 1753217327 \cdot B_{10}(q, 17) + 17 \cdot \frac{\eta(17\tau)^{17}}{\eta(\tau)} + \frac{\eta(\tau)^{17}}{\eta(17\tau)},\\
F_{19}(q) &= 19 \cdot B_1(q, 19) - 171 \cdot B_2(q, 19) + 608 \cdot e_3(q, 19) + 4902 \cdot B_4(q, 19) - 34846 \cdot B_5(q, 19)\\
&\quad + 295279 \cdot e_6(q, 19) + 2187451 \cdot B_7(q, 19) - 16815931 \cdot B_8(q, 19) + 5399819 \cdot e_9(q, 19)\\
&\quad - 41513727 \cdot B_{10}(q, 19) + 1501 \cdot B_{11}(q, 19) + 43776 \cdot e_{12}(q, 19) + 361 \cdot B_{13}(q, 19)\\
&\quad + 19 \cdot \frac{\eta(19\tau)^{19}}{\eta(\tau)} + \frac{\eta(\tau)^{19}}{\eta(19\tau)},\\
F_{23}(q) &= 23 \cdot B_1(q, 23) - 575 \cdot B_2(q, 23) + 5704 \cdot B_3(q, 23) - 30406 \cdot B_4(q, 23) + 243524 \cdot B_5(q, 23)\\
&\quad - 201641 \cdot B_6(q, 23) - 1700804 \cdot B_7(q, 23) + 8375841 \cdot B_8(q, 23) - 20572304 \cdot B_9(q, 23)\\
&\quad - 24390925 \cdot B_{10}(q, 23) + 61521159 \cdot e_{11}(q, 23) + 566887486 \cdot B_{12}(q, 23)\\
&\quad - 5668863199 \cdot B_{13}(q, 23) + 38063023518 \cdot B_{14}(q, 23) + 918978133 \cdot B_{15}(q, 23)\\
&\quad - 255391020775 \cdot B_{16}(q, 23) + 616128851321 \cdot B_{17}(q, 23) - 30602390905 \cdot B_{18}(q, 23)\\
&\quad - 2659690891913 \cdot B_{19}(q, 23) + 23 \cdot \frac{\eta(23\tau)^{23}}{\eta(\tau)} + \frac{\eta(\tau)^{23}}{\eta(23\tau)}
\end{aligned} \tag{7.2}$$

where $B_j(q, p)$ are as in Theorem 3.6 and

$$\begin{aligned}
&e_4(q, 17) := \eta(17\tau)^5 \cdot \eta(\tau)^{11},\ e_8(q, 17) := \eta(\tau)^7 \cdot \eta(17\tau)^{11},\ e_3(q, 19) := \eta(\tau)^{15} \cdot \eta(19\tau)^3,\\
&e_6(q, 19) := \eta(\tau)^{11} \cdot \eta(19\tau)^7,\ e_9(q, 19) := \eta(\tau)^7 \cdot \eta(19\tau)^{11},\ e_{12}(q, 19) := \eta(\tau)^3 \cdot \eta(19\tau)^{15},\\
&e_{11}(q, 23) := \eta(\tau)^{11} \cdot \eta(23\tau)^{11}.
\end{aligned}$$

Proof. We observe that both sides of the equalities in (7.2) are modular forms in $\mathcal{M}_{\frac{1}{2}(p-1)}(p, \chi_p)$, in view of Theorem 3.1 and Theorem 3.6. By comparing coefficients in the q-expansions of both sides using Theorem 3.3, the results follow. □

8. Further comments

As mentioned in [14], Theorem 3.6 is probably true for primes $p > 23$. One interesting problem in this direction would be to explicitly find a *good basis* for $p > 23$. Let $\{B_1(q, p), \cdots, B_d(q, p)\}$ be a good basis for $\mathcal{S}_{\frac{1}{2}(p-1)}(p, \chi_p)$ where $d = \dim(\mathcal{S}_{\frac{1}{2}(p-1)}(p, \chi_p))$. Then we conjecture that the following holds:

$$E_{\frac{1}{2}(p-1)}(\tau) = c_1 \cdot B_1(q,p) + \cdots + c_d \cdot B_d(q,p) + c_{d+1} \cdot \frac{\eta(p\tau)^p}{\eta(\tau)}, \;\; c_1, \cdots, c_{d+1} \in \mathbb{Z}, \;\; p > 23.$$

Another interesting problem would be to obtain witness identities for Ramanujan's congruences modulo powers of $5, 7$ and 11. A modest goal in this direction would be to start with moduli which are powers of 5. In this direction, Watson [38, pp. 110] obtained witness identities involving infinite sums. For moduli 5^2 and 7^2, Zuckerman [40] obtained witness identities involving finite sums, in the spirit of Theorem 1.2.

Acknowledgments

This work was done when the first author was a postdoc at Research Institute for Symbolic Computation (RISC), Austria. He was supported by grant SFB F50-06 of the Austrian Science Fund (FWF). The authors thank the anonymous referee for the comments and feedback. The first author thanks Ralf Hemmecke for explaining how the function GroebnerBasis() in Mathematica works. The authors thank Madeline Locus Dawsey, Atul Dixit, Frank Garvan, Ben Kane, Robert Osburn and Peter Paule for their feedback. Our computation have been done in SageMath [34] and Mathematica [39].

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