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Solution of the nonlinear fractional diffusion equation with absorbent term and external force

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ABSTRACT

The article presents the approximate analytical solutions of general nonlinear diffusion equation with fractional time derivative in the presence of an absorbent term and a linear external force obtained with the help of powerful mathematical tool like Homotopy Perturbation Method. By using initial value, the approximate analytical solutions of the equation are derived. The fractional derivatives are described in the Caputo sense. Numerical results for different particular cases are presented graphically. The anomalous behavior of nonlinear diffusivity in the presence or absence of external force and reaction term are calculated numerically and presented graphically.

1. Introduction

In this article, a sincere attempt has been taken to solve the nonlinear fractional diffusion equation with external force and reaction term as

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{\partial}{\partial x}\left((u(x,t))^{n}\frac{\partial u(x,t)}{\partial x}\right)-\frac{\partial}{\partial x}(F(x)u(x,t))-\int_{0}^{t}a(t-\xi)u(x,\xi)d\xi,\quad 0<\alpha\leq 1,\quad x>0,\quad t>0, \tag{1}$$

with the initial condition

$$u(x,0)=x, \tag{2}$$

where $F(x)$ is an external force, $a(t)$ is a time-dependent absorbent term which may be related to a reaction diffusion process. The diffusion equation have been widely studied due to its various applications in science and engineering but the study assumes a different dimension when it is nonlinear and also if in the classical diffusion equation the time derivative is replaced by a fractional derivative of order $\alpha(0<\alpha\leqslant 1)$. Nonlinearity is suitable subject which predicts a large extent as a detailed knowledge of the corresponding equations. A complete and extensive study of nonlinear PDE which can be related to the selection mechanics is indispensible. Nonlinear diffusion equations are important class of parabolic equations appear in many physical problems like phase transition in mechanical, electrical and electronic engineering, biochemistry and dynamics of biological sciences and in many methods for image processing and computer vision. The problems become challenging and difficult when singularity appears in it. The fractional differential equations have gained much attention recently due to the fact that fractional order system response ultimately converges to the integer order system response. An important

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outcome of these evolution equations is that it generates fractional Brownian motion which is a generalization of Brownian motion. The above type of anomalous diffusion is a ubiquitous phenomenon in nature and appears in different branches of science and engineering. Eq. (1) represents a model of fractional plasma diffusion for $n = -1/2$, thermal limit approximation of Carle man's model of the Boltzmann equation for $n = -1$, diffusive in higher polymer systems for $n = -2$, isothermal percolations of perfect gas through a micro-porous medium for $n = 1$ and process of melting and evaporation of metals for $n = 2$ (Wazwaz [1]). As one of the particular cases, Eq. (1) can also be related to the Fractional Schrodinger equation or fractional quantum mechanics (FQM), which is fractional differential equation in accordance with modern technology. FQM (Laskin [2]) is obtained by expanding the Feynman path integral over Brownian like quantum mechanical paths to the Levy like quantum mechanical paths, a generalization of quantum mechanics.

Most of the nonlinear problems do not have a precise analytical solution; especially it is hard to get it for the fractional nonlinear equations. So these types of equations should be solved by approximate analytical methods. Schot et al. [3] have given an approximate solution of the Eq. (1) for the linear case i.e., for $n = 0$ in terms of Fox H-function. Zahran [4] has offered a closed form solution in Fox H-function of the generalized linear fractional reaction-diffusion equation subjected to an external linear force field, one that is used to describe the transport processes in disorder systems.

The diffusion equations have been widely studied due to their various applications in Physics and Engineering, but the study related to diffusion equations with nonlinear terms and fractional time derivatives are few in numbers (Bologna et al. [5], Lenzi et al. [6] etc.). Lenzi et al. [7] presented some classes of solutions of a general nonlinear fractional diffusion equation with absorbations. The similar study was made by Assis et al. [8]. A generalized diffusion equation which contains spatial fractional derivatives and nonlinear terms can be found in Silva et al. [9] and Lenzi et al. [10]. Das [11] has used Variational Iteration Method to find the analytical solution of a fractional diffusion equation of order $\alpha(0 < \alpha \leqslant 1)$ only in the presence of external force. Later, Das and Gupta [12] have solved linear diffusion equation with fractional time derivative in presence of external force and a reaction term by using Homotopy perturbation method (HPM). But to the best of authors' knowledge the nonlinear time fractional diffusion equation in the presence of the same type of external force and reaction term has not yet been solved by any researcher.

It is very difficult to get the exact analytical solutions of fractional order problems especially for nonlinear cases. The authors have made a sincere effort to solve the above specified model taken the full advantage of powerful and efficient mathematical tool HPM. The HPM is the new approach for finding the approximate analytical solution of linear and nonlinear problems. The method was first proposed by He ([13–15]) and was successfully applied to solve nonlinear wave equation by He ([16–19]), fractional Lotka–Volterra equation by Das and Gupta [20], boundary value problems by He [21], fractional predator prey model by Das et al. [22], linear fractional Schrodinger equation by Das et al. [23] and fractional logistic equation by Das et al. [24] etc. The basic difference of this method from the other perturbation techniques is that it does not require small parameters in the equation which overcomes the limitations of traditional perturbation techniques. He [25] applied the method successfully to solve a duffing equation with high order of nonlinearity. The result shows that its first order of approximation is valid uniformly for very large parameter with accuracy better than the perturbation solutions. He [14] claimed that the approximations obtained by this method are valid not only for small parameters but for very large parameters. Also in the study of Monami and Odibat [26], it is seen that HPM is very effective, convenient, supplies quantitatively reliable results. The main advantage of the HPM is that it reduces both nonlinear differential equation and fractional nonlinear differential equation to a series of ordinary differential equations, which are easy to solve for any order of approximations, as and when required. In 2008, Odibat and Momani [27] presented a modification of HPM and the algorithm is applied to solve the quadratic Riccati differential equation of fractional order. In their earlier article [28], they successfully applied the method to solve the nonlinear time fractional advection equation and nonlinear time fractional hyperbolic equation and made a statement that HPM is a universal one by which various kinds of nonlinear equation can easily be solved. Wang [29] has made a sincere effort to find the approximate solution of the nonlinear fractional order KdV-Burgers equation with time and space fractional derivatives with high accuracy.

In this article, the Homotopy Perturbation Method is used to solve the nonlinear fractional diffusion Eq. (1). Using the initial condition, the approximate analytical expressions of $u(x, t)$ for different fractional Brownian motions and also for standard motion are obtained. The effects of external force and absorbent term in the solution are obtained numerically for different particular cases, which are depicted graphically. The elegance of this article can be attributed to the simplistic approach in seeking the approximate analytical solution of the problem and also in the demonstration of the effect of damping for the stability of the nonlinear system of fractional order in presence of reaction term.

2. Basic ideas of fractional calculus

In this section, we give some definitions and properties of the fractional calculus [30] which are used further in this paper.

Definition 1. A real function $f(t)$, $t > 0$, is said to be in the space $C_\mu, \mu \in \Re$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if $f^{(n)} \in C_\mu$, $n \in N$.

Definition 2. The Riemann–Liouville fractional integral operator (J_t^α) of order $\alpha \geqslant 0$, of a function $f \in C_\mu$, $\mu \geqslant -1$, is defined as

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0,\ t > 0, \tag{3}$$

$$J_t^0 f(t) = f(t).$$

where $\Gamma(\alpha)$ is the well-known gamma function. Some of the properties of the operator J_t^α, which we will need here, are as follows:For $f \in C_\mu$, $\mu \geqslant -1$, α, $\beta \geqslant 0$ and $\gamma \geqslant -1$

(1) $J_t^\alpha J_t^\beta f(t) = J_t^{\alpha+\beta} f(t)$.
(2) $J_t^\alpha J_t^\beta f(t) = J_t^\beta J_t^\alpha f(t)$.
(3) $J_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 3. The fractional derivative (D_t^α) of $f(t)$, in the Caputo sense is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, \tag{4}$$

for $n-1 < \alpha < n, n \in N, t > 0, f \in C_{-1}^n$.

The following are two basic properties of the Caputo fractional derivative [31]:

(1) Let $f \in C_{-1}^n, n \in N$, then $D_t^\alpha f, 0 \le \alpha \le n$ is well defined and $D_t^\alpha f \in C_{-1}$.
(2) Let $n-1 \le \alpha \le n, n \in N$ and $f \in C_\mu^n, \mu \ge -1$. Then

$$(J_t^\alpha D_t^\alpha) f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}. \tag{5}$$

3. Basic idea of homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following non-linear functional equation

$$A(u) - f(x) = 0, \quad x \in \Omega, \tag{6}$$

with the following boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad x \in \Gamma, \tag{7}$$

where A is a general functional operator, B a boundary operator, $f(x)$ is a known analytical function and Γ is the boundary of the domain Ω. The operator A can be decomposed into two operators L and N, where L is linear, and N is nonlinear operator. Eq. (6) can be, therefore, written as follows

$$L(u) + N(u) - f(x) = 0. \tag{8}$$

Using the homotopy technique [13–19], we construct a homotopy $v(x,p) : \Omega \times [0,1] \to \Re$, which satisfies

$$\mathscr{H}(v,p) \equiv (1-p)[L(v) - L(u_0)] + p[A(v) - f(x)] = 0, \tag{9}$$

or

$$\mathscr{H}(v,p) \equiv L(v) - L(u_0) + pL(u_0) + p[N(v) - f(x)] = 0, \tag{10}$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation for the solution of Eq. (6), which satisfies the boundary conditions. Obviously, from Eqs. (9) and (10) we will have

$$\mathscr{H}(v,0) \equiv L(v) - L(u_0) = 0, \tag{11}$$

and

$$\mathscr{H}(v,1) \equiv A(v) - f(x) = 0, \tag{12}$$

the changing values of p from zero to unity is just that of $v(x,p)$ from $u_0(x)$ to $u(x)$. In topology, this is called deformation, and also, $L(v) - L(u_0)$, $A(v) - f(x)$ are called homotopic.

According to HPM, the embedding parameter $p(0 \leqslant p \leqslant 1)$ is used as a "small parameter", and the solution of Eqs. (9) and (10) are written as a power series in p

$$v = v_0 + pv_1 + p^2 v_2 + \cdots. \tag{13}$$

Setting $p = 1$ results in the approximation to the solution of Eq. (6)

$$u(x) = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{14}$$

Finally, we approximate the analytical solution $u(x)$ by truncated series

$$u(x) = \lim_{N\to\infty} \Phi_N(x), \tag{15}$$

where $\Phi_N(x) = \sum_{m=0}^{N-1} v_m(x)$, $N \geqslant 1$.

The above series solutions generally converge very rapidly.

4. Solution of the problem by HPM

Eq. (1) can be written in the operator form as

$$D_t^\alpha u(x,t) = D_x((u(x,t))^n D_x u(x,t)) - D_x(F(x)u(x,t)) - \int_0^t a(t-\xi)u(x,\xi)d\xi, \tag{16}$$

where $D_t^\alpha \equiv \frac{\partial^\alpha}{\partial t^\alpha}$, $D_x \equiv \frac{\partial}{\partial x}$.

Using the homotopy technique, we construct a homotopy $v(x,t,p) : \Omega \times [0,1] \to \Re$, which satisfies

Table 1
Approximate solutions for Case I.

t	x	u_{HPM}				u_{Exact} (for standard motion)
		$\alpha = 1/3$	$\alpha = 1/2$	$\alpha = 2/3$	$\alpha = 1$	
0.00	0.25	0.25000	0.25000	0.25000	0.25000	0.25000
	0.50	0.50000	0.50000	0.50000	0.50000	0.50000
	0.75	0.75000	0.75000	0.75000	0.75000	0.75000
	1.00	1.00000	1.00000	1.00000	1.00000	1.00000
0.25	0.25	0.95546	0.81419	0.68960	0.50000	0.50000
	0.50	1.20546	1.06419	0.93960	0.75000	0.75000
	0.75	1.45546	1.31419	1.18960	1.00000	1.00000
	1.00	1.70546	1.56419	1.43960	1.25000	1.25000
0.50	0.25	1.13882	1.04788	0.94783	0.75000	0.75000
	0.50	1.38882	1.29788	1.19783	1.00000	1.00000
	0.75	1.63882	1.54788	1.44783	1.25000	1.25000
	1.00	1.88882	1.79788	1.69783	1.50000	1.50000
0.75	0.25	1.26745	1.22721	1.16441	1.00000	1.00000
	0.50	1.51745	1.47721	1.41441	1.25000	1.25000
	0.75	1.76745	1.72721	1.66441	1.50000	1.50000
	1.00	2.01745	1.97721	1.91441	1.75000	1.75000
1.00	0.25	1.36985	1.37838	1.35773	1.25000	1.25000
	0.50	1.61985	1.62838	1.60773	1.50000	1.50000
	0.75	1.86985	1.87838	1.85773	1.75000	1.75000
	1.00	2.11985	2.12838	2.10773	2.00000	2.00000

Table 2
Approximate solutions for Case II.

t	x	u_{HPM}				u_{Exact} (for standard motion)
		$\alpha = 1/3$	$\alpha = 1/2$	$\alpha = 2/3$	$\alpha = 1$	
0.00	0.25	0.25000	0.25000	0.250000	0.250000	0.250000
	0.50	0.50000	0.50000	0.500000	0.500000	0.500000
	0.75	0.75000	0.75000	0.750000	0.750000	0.750000
	1.00	1.00000	1.00000	1.000000	1.000000	1.000000
0.05	0.25	7.38567	1.22319	0.443121	0.279547	0.279508
	0.50	14.7713	2.44638	0.886242	0.559094	0.559017
	0.75	22.1570	3.66957	1.329360	0.838641	0.838525
	1.00	29.5427	4.89276	1.772480	1.118190	1.118030
0.10	0.25	16.9059	3.26391	0.861900	0.322750	0.322749
	0.50	33.8118	6.52782	1.723900	0.645500	0.645497
	0.75	50.7177	9.79173	2.585700	0.968250	0.968246
	1.00	67.6236	13.0556	3.447610	1.291000	1.290990
0.15	0.25	27.8713	6.37155	1.621180	0.389547	0.395285
	0.50	55.7425	12.7431	3.242360	0.779094	0.790569
	0.75	83.6138	19.1147	4.863540	1.168640	1.185850
	1.00	111.485	25.4862	6.48472	1.558190	1.581140
0.20	0.25	39.9364	10.5297	2.80473	0.534000	0.559020
	0.50	79.8728	21.0595	5.60945	1.098800	1.118030
	0.75	119.809	31.5892	8.41418	1.658200	1.677050
	1.00	159.746	42.1189	11.2189	2.217600	2.236070

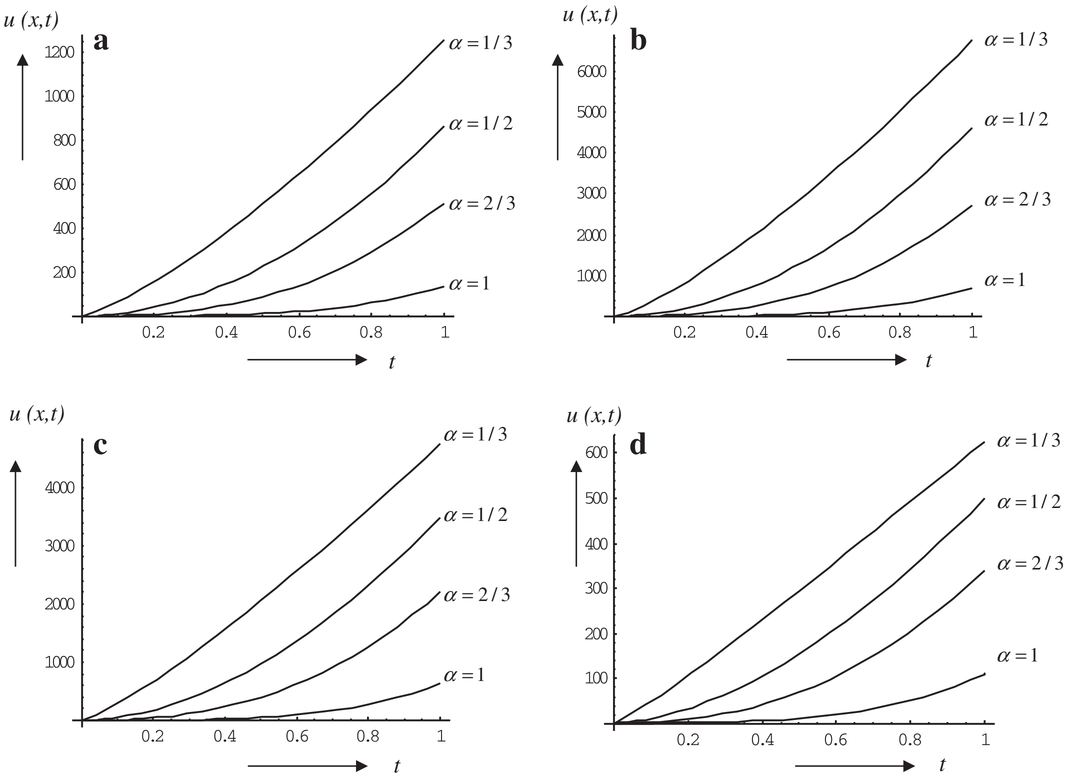

Fig. 1. (a) Plot of $u(x, t)$ vs. t for different values of α at $n = 2$, $x = 1$, $a = 0$ and $k = 0$. (b) Plot of $u(x, t)$ vs. t for different values of α at $n = 2$, $x = 1$, $a = 0$ and $k = 1$. (c) Plot of $u(x, t)$ vs. t for different values of α at $n = 2$, $x = 1$, $a = 1$ and $k = 1$. (d) Plot of $u(x, t)$ vs. t for different values of α at $n = 2$, $x = 1$, $a = 1$ and $k = 0$.

$$\mathscr{H}(v,p) \equiv (1-p)D_t^{\alpha}(v) - D_t^{\alpha}(u_0)] + p\left[D_t^{\alpha}v - D_x(v^nD_xv) + D_x(F(x)v) + \int_0^t a(t-\xi)vd\xi\right],$$

Applying the idea of Caputo derivative $D_t^{\alpha}(u_0) = J_t^{1-\alpha}D_t(u_0) = 0$ [J_t^{α} is the Reimann–Liovillue operator], we get

$$D_t^{\alpha}v = p\left[D_x(v^nD_xv) - D_x(F(x)v) - \int_0^t a(t-\xi)v(x,\xi)d\xi\right], \tag{17}$$

where the homotopy parameter p is considered to be small $0 \leqslant p \leqslant 1$. Now applying the classical perturbation technique, Eq. (7) can be expressed as a power series of p as

$$v(x,t) = v_0(x,t) + pv_1(x,t) + p^2v_2(x,t) + p^3v_3(x,t) + \cdots, \tag{18}$$

When $p \to 1$, Eq. (18) becomes the approximate solution of Eq. (16). Substituting Eq. (18) into Eq. (17) and equating the terms with identical powers of p, we obtain the following set of linear differential equations:

$$p^0 : D_t^{\alpha}v_0 = 0, \tag{19}$$

$$p^1 : D_t^{\alpha}v_1 = D_x(v_0^nD_xv_0) - D_x(F(x)v_0) - \int_0^t a(t-\xi)v_0(x,\xi)d\xi, \tag{20}$$

$$p^2 : D_t^{\alpha}v_2 = D_x(v_0^nD_xv_1 + nv_0^{n-1}v_1D_xv_0) - D_x(F(x)v_1) - \int_0^t a(t-\xi)v_1(x,\xi)d\xi, \tag{21}$$

$$p^3 : D_t^{\alpha}v_3 = D_x(v_0^nD_xv_2 + nv_0^{n-1}v_1D_xv_1 + {}^nC_2v_0^{n-2}v_1^2D_xv_0 + nv_0^{n-1}v_2D_xv_0) - D_x(F(x)v_1) - \int_0^t a(t-\xi)v_2(x,\xi)d\xi, \tag{22}$$

and so on.

Now considering $a(t) = a\frac{t^{\beta-1}}{\Gamma(\beta)}$ $(0 < \beta \leqslant 1)$, $F(x) = -kx$ and applying the operator J_t^{α} (the inverse of Caputo operator D_t^{α}) on the both sides of Eqs. (19)–(22), we obtain

$$v_0(x,t) = x,$$

$$v_1(x,t) = (2kx + nx^{n-1})\frac{t^{\alpha}}{\Gamma(\alpha+1)} - ax\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},$$

$$v_2(x,t) = \left[4k^2x + kn(3n+2)x^{n-1} + 2n(n-1)(2n-1)x^{2n-3}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - a\left(4kx + n(n+2)x^{n-1}\right)\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} + a^2x\frac{t^{2\alpha+2\beta}}{\Gamma(2\alpha+2\beta+1)},$$

$$\begin{aligned} v_3(x,t) = &\Big[8k^3x + k^2n(3n^2+6n+4)x^{n-1} + 10kn^2(n-1)(2n-1)x^{2n-3} + 6n(n-1)^2(2n-1)(3n-4)x^{3n-5} \\ &+ \frac{n^2}{2}\{4k^2(n+1)x^{n-1} + 8k(n-1)(2n-1)x^{2n-3} + 3n(n-1)(3n-4)x^{3n-5}\}\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\Big] \\ &\times \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - a\Big[12k^2x + kn(n^2+9n+6)x^{n-1} + 2n(n-1)(2n-1)(n+3)x^{2n-3} + 2n^2\{k(n+1)x^{n-1} \\ &+ (n-1)(2n-1)x^{2n-3}\}\frac{\Gamma(2\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\Big]\frac{t^{3\alpha+\beta}}{\Gamma(3\alpha+\beta+1)} \\ &+ a^2\left[6kx + n(2n+3)x^{n-1} + \frac{n^2(n+1)}{2}x^{n-1}\frac{\Gamma(2\alpha+2\beta+1)}{(\Gamma(\alpha+\beta+1))^2}\right]\frac{t^{3\alpha+2\beta}}{\Gamma(3\alpha+2\beta+1)} - a^3x\frac{t^{3\alpha+3\beta}}{\Gamma(3\alpha+3\beta+1)}, \end{aligned} \quad (23)$$

Proceeding in this manner, the rest of the components $v_m(x,t)$, $m > 3$ can be completely obtained and the series solutions are thus entirely determined.

Finally, we approximate the analytical solution $u(x,t)$ by truncated series

$$u(x,t) = \lim_{N\to\infty}\Psi_N(x,t), \quad (24)$$

where $\Psi_N(x,t) = \sum_{m=0}^{N-1} v_m(x,t)$, $N \geqslant 1$

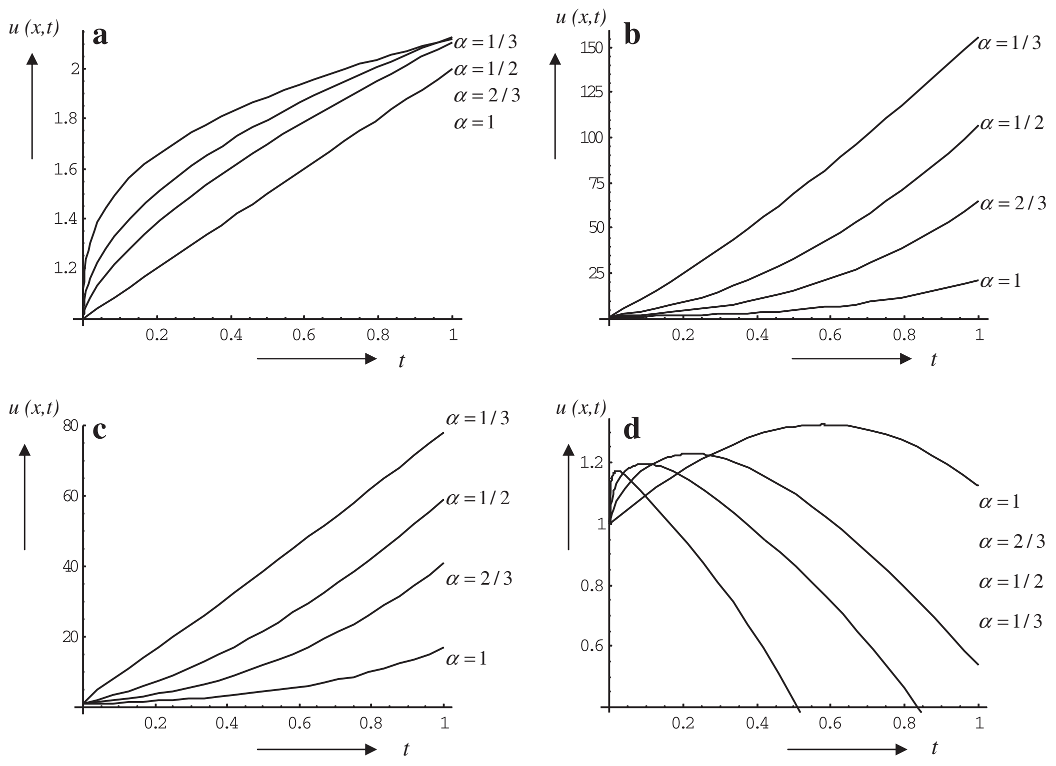

Fig. 2. (a) Plot of $u(x, t)$ vs. t for different values of α at $n = 1$, $x = 1$, $a = 0$ and $k = 0$. (b) Plot of $u(x, t)$ vs. t for different values of α at $n = 1$, $x = 1$, $a = 0$ and $k = 1$. (c) Plot of $u(x, t)$ vs. t for different values of α at $n = 1$, $x = 1$, $a = 1$ and $k = 1$. (d) Plot of $u(x, t)$ vs. t for different values of α at $n = 1$, $x = 1$, $a = 1$ and $k = 0$.

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [32].

5. Particular cases

In this section, we are considering some cases of Eq. (1) to demonstrate the reliability of the method HPM and its wider applicability for solving linear and nonlinear diffusion equations of fractional order.

Case I: Consider $n = 1$, $k = 0$, $a = 0$ i.e., the nonlinear time-fractional diffusion equation in absence of both external force and reaction term

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right). \tag{25}$$

The approximate results of the solutions of Eq. (25) obtained for different α for various values of x and t using HPM is presented in Table 1. The exact solution of $u(x, t)$ for $\alpha = 1$ is $u(x, t) = x + t$.

Case II: Consider $n = 2$, $k = 0$, $a = 0$ i.e., the slow diffusion equation with fractional time derivative

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left((u(x,t))^2\frac{\partial u(x,t)}{\partial x}\right). \tag{26}$$

Table 2 exhibits the approximate solutions of Eq. (26) obtained for different α with variations in x and t. The value $\alpha = 1$ is the only case for which we know the exact solution of Eq. (26) as $u(x,t) = \frac{x}{\sqrt{1-4t}}$.

Case III: Consider $n = 0$, $k = 1$, $a = 1$ the Eq. (1) reduces to the following linear fractional diffusion equation in presence of external force and reaction term

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial}{\partial x}(xu(x,t)) - \int_0^t a(t-\xi)u(x,\xi)d\xi, \tag{27}$$

whose solution is

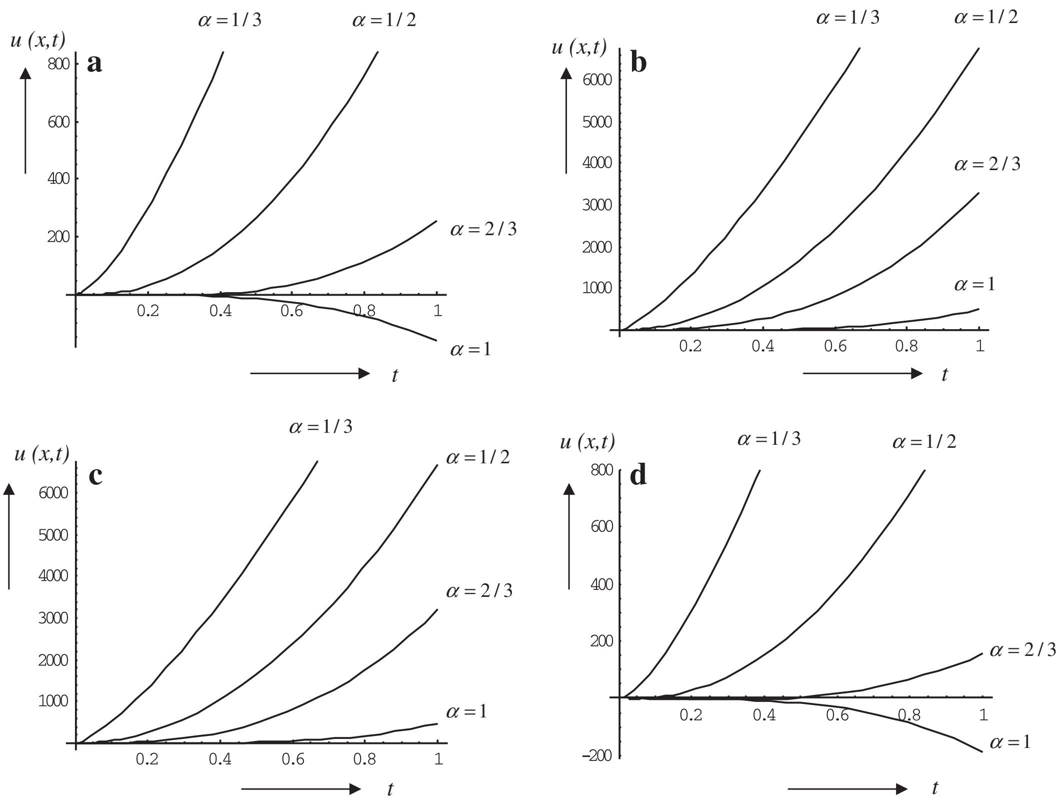

Fig. 3. (a) Plot of $u(x, t)$ vs. t for different values of α at $n = -1$, $x = 1$, $a = 0$ and $k = 0$. (b) Plot of $u(x, t)$ vs. t for different values of α at $n = -1$, $x = 1$, $a = 0$ and $k = 1$. (c) Plot of $u(x, t)$ vs. t for different values of α at $n = -1$, $x = 1$, $a = 1$ and $k = 1$. (d) Plot of $u(x, t)$ vs. t for different values of α at $n = -1$, $x = 1$, $a = 1$ and $k = 0$.

whose solution is

$$u(x,t) = x\left[1 + \frac{2t^{\beta}}{\Gamma(\beta+1)} + \frac{3t^{2\beta}}{\Gamma(2\beta+1)} + \frac{4t^{3\beta}}{\Gamma(3\beta+1)} + \cdots\right] = x\sum_{r=0}^{\infty}\frac{(r+1)t^{r\beta}}{\Gamma(r\beta+1)} = xE_{\beta}(K_2 t^{\beta}),\ K_2^r = r+1,$$

where $E_{\beta}(t) = \sum_{r=0}^{\infty}\frac{t^r}{\Gamma(r\alpha+1)}\,(\alpha > 0)$ is the Mittag–Leffler function of first kind.

The result is in complete agreement with the result of Das and Gupta [12].

6. Numerical results and discussion

In this section, numerical results of the probability density function $u(x,t)$ for different fractional Brownian motions $\alpha = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and also for standard motion $\alpha = 1$ are calculated at $n = 2, 1, -1, -2$ for specific cases. The variation of $u(x,t)$ w.r.to t at $x = 1$ considering $\beta = \alpha$ are depicted through Figs. 1–4. The corresponding results of $u(x,t)$ w.r.to x and t are depicted through Fig. 5 for four different cases with $\alpha = \frac{1}{2}, \beta = \frac{1}{3}$ at $n = 2$. During numerical computation only fourth order term of the series solution is considered. The accuracy of the result can be improved by introducing more terms of the approximate solution.

It is seen from the Fig. 1(a) that $u(x,t)$ increases with the increase in t for all α. However, it is found to decrease with the increase in α. When there exists only external force (i.e., $a = 0$, $k = 1$) which is graphically described by Fig. 1(b), it is seen that the magnitude of $u(x,t)$ rapidly increases. But if in addition, there exists the absorbent term (i.e., $a = 1$, $k = 1$) then it is seen from Fig. 1(c) that magnitude of $u(x,t)$ decreases. Fig. 1(d) which graphically describes the effect of absorbent term only (i.e., $a = 1$, $k = 0$) clearly reveals that the nonlinear diffusion process becomes stable. All the figures of Fig. 1(a)–(d) are calculated for $n = 2$.

For $n = 1$, which is represented by Fig. 2, it is seen the magnitude of $u(x,t)$ decreases than the previous one in all the considered cases. Here the system becomes much stable in presence of only reaction term.

It is observed from Fig. 3 which represents $n = -1$ that the magnitude becomes higher in the presence of external force and the reaction term has negligible effect in comparison to $n = 1$ and $n = 2$. For $n = -2$, Fig. 4 depicts that the external force has a tremendous effect on the given model.

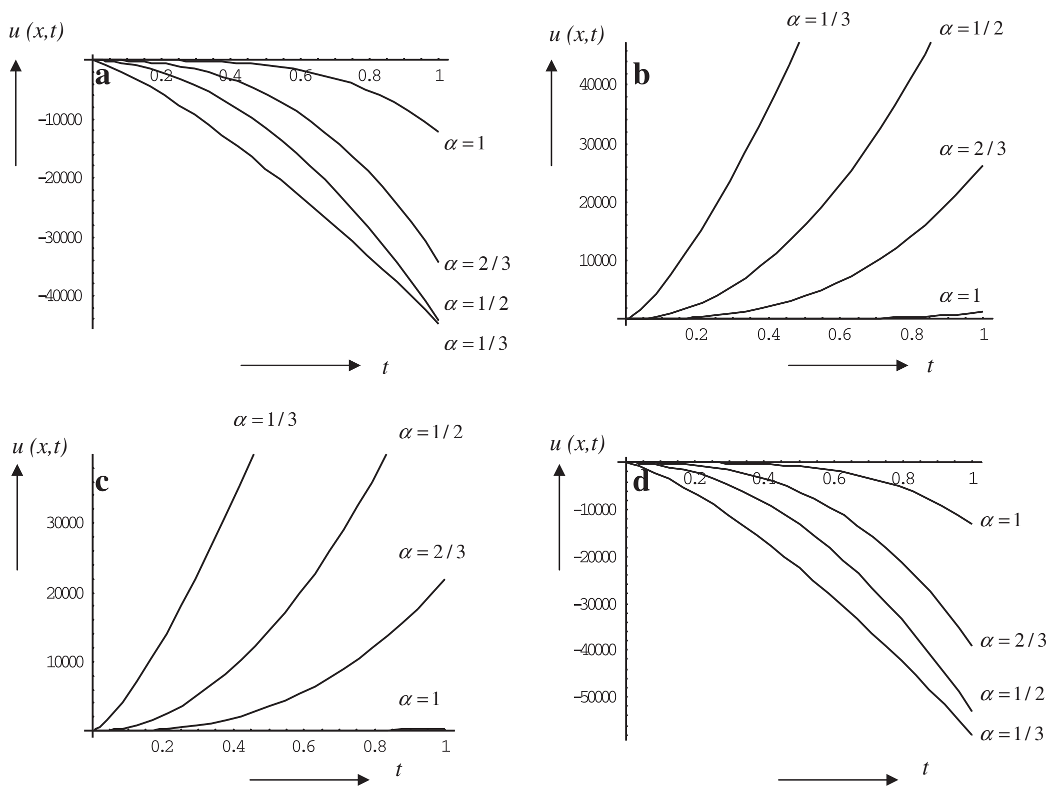

Fig. 4. (a) Plot of $u(x,t)$ vs. t for different values of α at $n = -2$, $x = 1$, $a = 0$ and $k = 0$. (b) Plot of $u(x,t)$ vs. t for different values of α at $n = -2$, $x = 1$, $a = 0$ and $k = 1$. (c) Plot of $u(x,t)$ vs. t for different values of α at $n = -2$, $x = 1$, $a = 1$ and $k = 1$. (d) Plot of $u(x,t)$ vs. t for different values of α at $n = -2$, $x = 1$, $a = 1$ and $k = 0$.

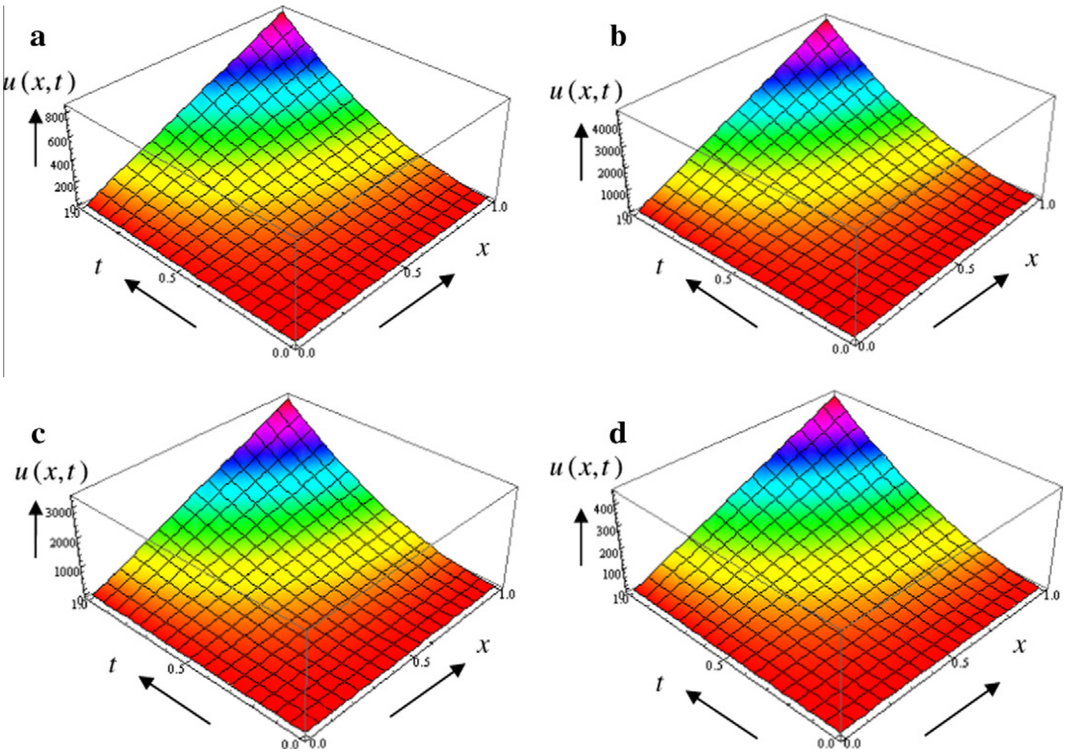

Fig. 5. (a) Plot of $u(x, t)$ w.r.to x and t at $n = 2$, $\alpha = 1/2$ for $a = 0$ and $k = 0$. (b) Plot of $u(x, t)$ w.r.to x and t at $n = 2$, $\alpha = 1/2$ for $a = 0$ and $k = 1$. (c) Plot of $u(x, t)$ w.r.to x and t at $n = 2$, $\alpha = 1/2$ for $a = 1$ and $k = 1$. (d) Plot of $u(x, t)$ w.r.to x and t at $n = 2$, $\alpha = 1/2$ for $a = 1$ and $k = 0$.

It is also seen from the 3-D figures which are described through Fig. 5 that the variations of $u(x, t)$ are linear with x but it becomes exponential with t for different particular cases as stated in the caption of the figures.

Therefore, for positive integral power of n, the absorbent term plays an important role for damping the system even in presence of external force. But for negative power of n, the presence of external force has causes the system unstable which cannot be retrieved even by adding the absorbent term into the system.

Tables 1 and 2, which show the approximate and exact values for Cases I and II, clearly exhibit that even four order terms of the approximation of the solutions are sufficient to get good approximation to the exact solution. It is evident that the accuracy can further be enhanced by computing few more terms of the approximate solutions.

7. Conclusions

Another important study of this article is to show the effect of reaction term on the nonlinear fractional diffusion equation. Here the presence of external force increases the rate of diffusion in the considered nonlinear system for positive integral value of n. But the rate becomes slower in the presence of reaction term. Again in the absence of the external force, the reaction term helps the diffusivity of the system much slower which indicates that the reaction term controls the system stability. Therefore, the dynamic response and stability margin are improved in the presence of absorbent term which provides damping force.

But for negative integral value of n, the rate of diffusion becomes higher which cannot be controlled even in presence of absorbent term. Therefore it can be concluded that the damping term has lesser effect for controlling the system for $n < 0$ but its effect increases for $n \geqslant 0$.

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