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An approximate solution of nonlinear fractional reaction–diffusion equation

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ABSTRACT

The article presents a mathematical model of nonlinear reaction diffusion equation with fractional time derivative $\alpha\,(0 < \alpha \leqslant 1)$ in the form of a rapidly convergent series with easily computable components. Fractional reaction diffusion equation is used for modeling of merging travel solutions in nonlinear system for popular dynamics. The fractional derivatives are described in the Caputo sense. The anomalous behaviors of the nonlinear problems in the form of sub- and super-diffusion due to the presence of reaction term are shown graphically for different particular cases.

1. Introduction

The reaction–diffusion equation has an important role in dissipative dynamical systems as addressed by many scientists, engineers and biologists. Travelling waves appearing in chemical concentration is one of the key areas of research for last few decades. It is seen that travelling waves of chemical concentration have very good effect in biochemical change for the reaction–diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u), \tag{1}$$

where $u(x,t)$ is the chemical concentration, D is diffusion coefficient and $f(u)$ represents the kinetics. The equation can easily be solved when $f(u)$ is linear but it becomes complicated when $f(u)$ is nonlinear. The simplest nonlinear reaction–diffusion equation is the Fisher equation for $f(u) = u(1-u)$. The generalization of Fisher equation which is used as a density dependent diffusion was considered by Feng [1], taking $f(u) = u(\mu + \beta u - \gamma u^2)$, where μ, β, γ are real constants. The detailed study of travelling wave solution of Eq. (1) can be obtained in Volpert [2].

But the situation becomes challenging when both diffusion and kinetic terms become nonlinear. Pablo and Vazquez ([3,4]) have solved the following strong reaction-slow diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{m-1}\frac{\partial u}{\partial x}\right) + u^p, \quad p < 1 < m. \tag{2}$$

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Later, Pablo and Sanchez [5] have studied the large time behaviour of Fisher equation in porous medium of the following type

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{m-1}\frac{\partial u}{\partial x}\right) + u^p(1-u). \tag{3}$$

In 2007, Witelski [6] consider the model as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{\eta}\frac{\partial u}{\partial x}\right) + u(1-u^{\eta}), \quad \eta > 0, \quad -\infty < x < \infty,$$

where he used perturbation method to solve the inner problem for the merging dynamics. He observed that at the neighbourhood of the position where the two populations first meet, the reaction terms do not affect the solution to leading terms. This motivated the authors to solve the fractional reaction–diffusion equation with fractional time derivative

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial}{\partial x}\left(u^{n}\frac{\partial u}{\partial x}\right) + u(1-u^{n}), \quad 0 < \alpha \leqslant 1. \tag{4}$$

The fractional differential equations have gained much attention recently due to the fact that fractional order system response ultimately converges to the integer order system response.

For the study of dynamics of merging travelling waves, the perturbation theory and matched asymptotic expansions are needed. Let two populations be moving towards each other and merge at (x_0, t_0). But for $t > t_0$, let us consider a short time after merger at $t = t_0$ through the time scale $t = t_0 + \varepsilon\tau$, $\varepsilon \ll 1$ with another rescale of dependent variable

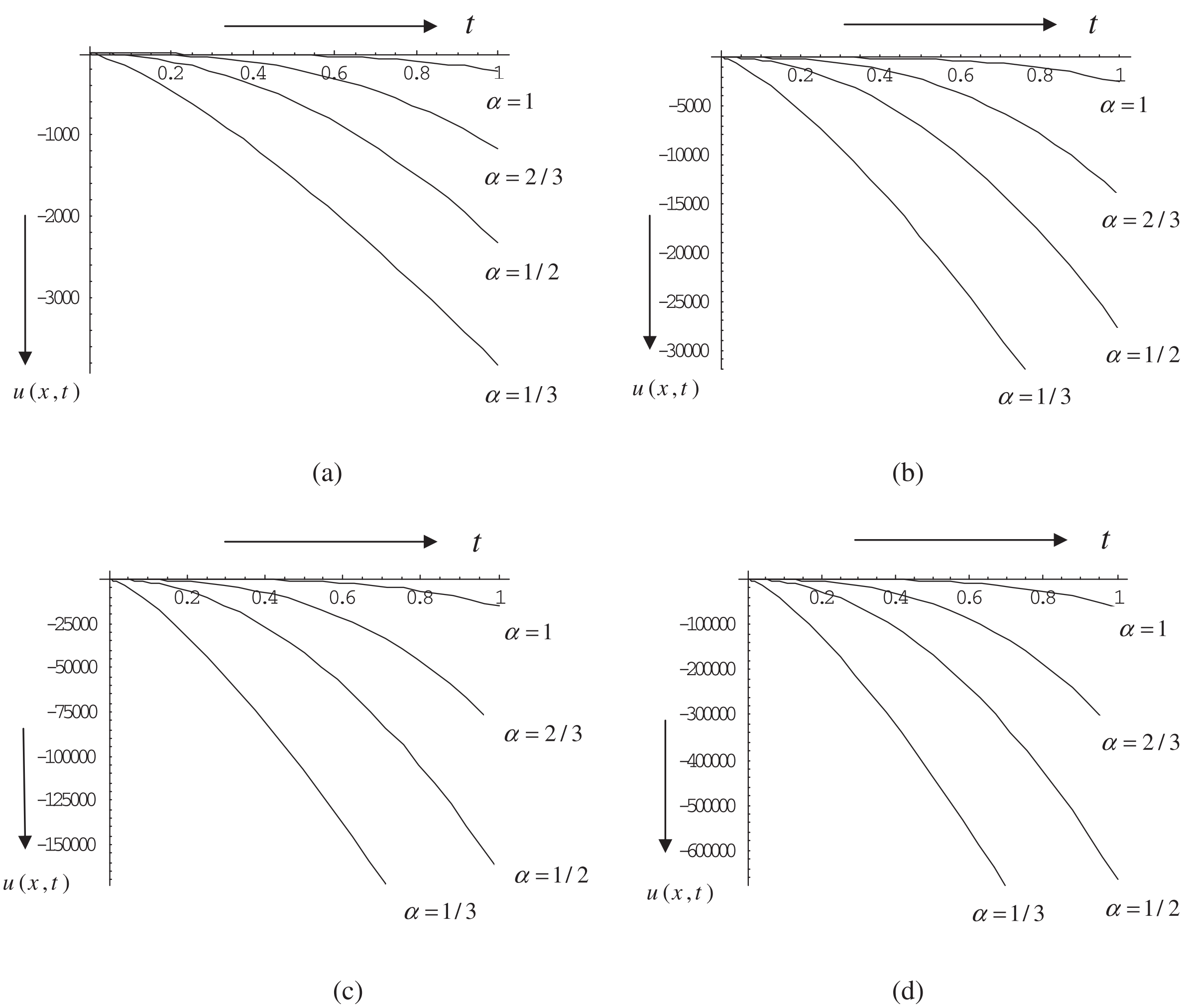

Fig. 1. Plots of $u(x,t)$ vs. t when (a) $n = -\frac{1}{2}$, (b) $n = -1$, (c) $n = -\frac{3}{2}$, and (d) $n = -2$ at $x = 1$ for different values of α.

$$u(x,t) = \varepsilon^{\frac{\alpha}{n}}\bar{u}(x,t) \quad \text{and} \quad x = x_0 + \varepsilon\bar{x}.$$

Then the governing equation reduces to

$$\frac{\partial^\alpha \bar{u}}{\partial \tau^\alpha} = \varepsilon^{2(\alpha-1)}\frac{\partial}{\partial x}\left(\bar{u}^n \frac{\partial \bar{u}}{\partial \bar{x}}\right) + \varepsilon^\alpha \bar{u}(1 - \varepsilon^\alpha \bar{u}). \tag{5}$$

Now applying the perturbation technique

$$\bar{u}(\bar{x},\tau) = \bar{u}_0(\bar{x},\tau) + \varepsilon\bar{u}_1(\bar{x},\tau) + \varepsilon^2\bar{u}_2(\bar{x},\tau) + \cdots$$

the Eq. (5) reduces to

$$\frac{\partial^\alpha \bar{u}_0}{\partial \tau^\alpha} = \varepsilon^{2(\alpha-1)}\frac{\partial}{\partial x}\left(\bar{u}_0^n \frac{\partial \bar{u}_0}{\partial \bar{x}}\right) + O(\varepsilon^\alpha),$$

when α becomes unity i.e., for standard motion the population-merging dynamics are diffusion dominated but for fractional Brownian motion the affect of reaction terms can be observed in diffusion process in the neighbourhood of x_0.

The perturbation methods which are generally used to solve nonlinear problems have some limitations e.g., the approximate solution involves series of small parameters which poses difficulty since majority of nonlinear problems have no small parameters at all. Although appropriate choices of small parameters some time leads to ideal solution but in most of the cases unsuitable choices lead to serious effects in the solutions. The homotopy perturbation method (HPM) proposed by the Chinese scientist He [7] is a new approach for finding the approximate solution of linear and nonlinear partial differential equations, which does not require small parameters in the equation and so overcomes the limitations of the traditional perturbation techniques. The objective of the present article is to solve the time fractional reaction diffusion equation for

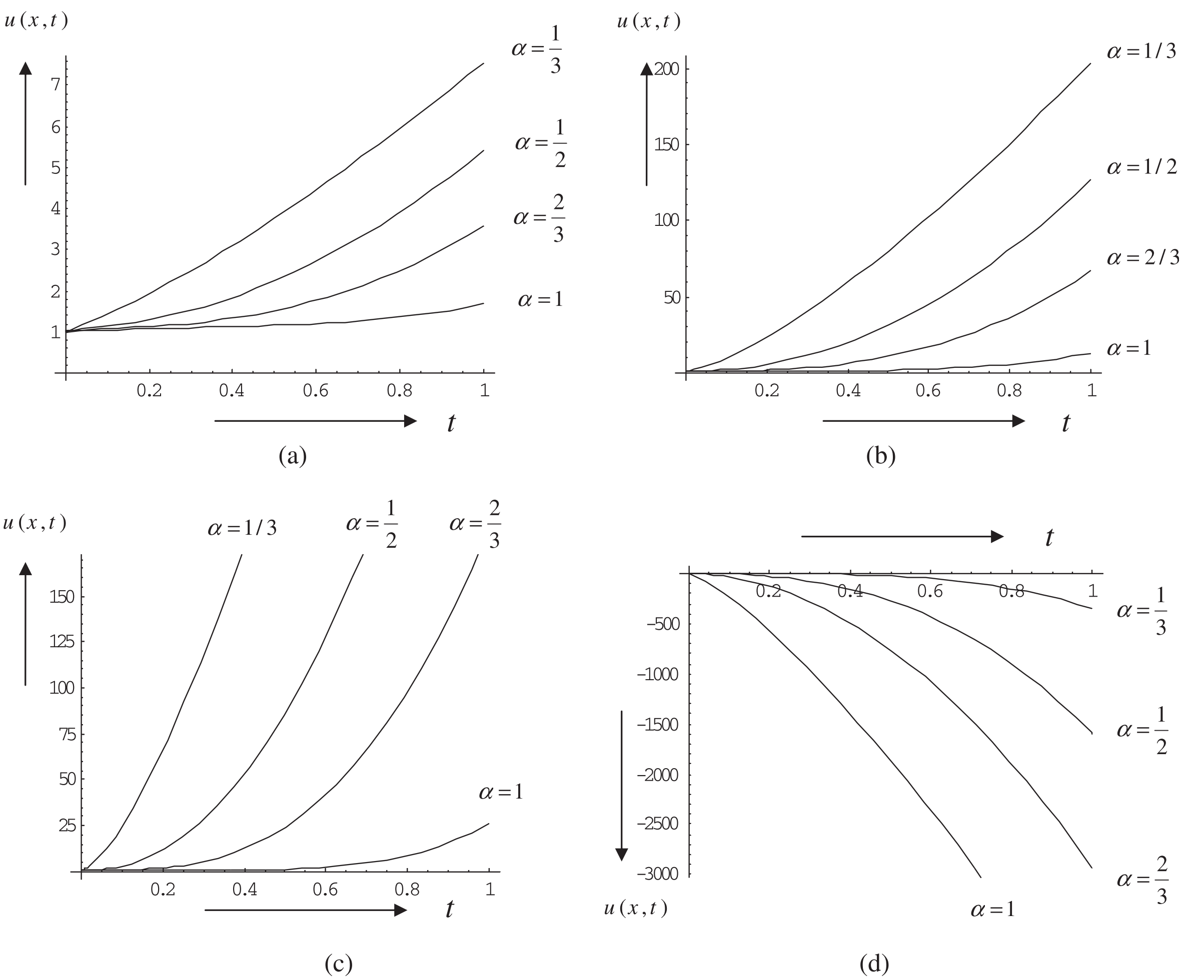

Fig. 2. Plots of $u(x,t)$ vs. t when (a) $n = \frac{1}{2}$, (b) $n = 1$, (c) $n = \frac{3}{2}$, and (d) $n = 2$ at $x = 1$ for different values of α.

different fractional Brownian motions and also for standard motion using the powerful mathematical tool HPM. The numerical studies of anomalous diffusion for different particular cases are presented through graphs.

2. Solution of the problem by HPM

The fractional advection Eq. (4) in an operator form can be written as

$$D_t^{\alpha} u = D_x(u^n D_x u) + u(1 - u^n), \quad 0 < \alpha \leqslant 1, \tag{6}$$

with initial condition

$$u(x, 0) = u_0(x, t) = f(x). \tag{7}$$

Using the homotopy technique ([8–11]), we construct a homotopy $\mathrm{v}(x, t, p) : \Omega \times [0, 1] \to \Re$, which satisfies

$$H(\mathrm{v},p) \equiv (1 - p)\left[D_t^{\alpha}(\mathrm{v}) - D_t^{\alpha}(u_0)\right] + p\left[D_t^{\alpha}(\mathrm{v}) - D_x(\mathrm{v}^n D_x \mathrm{v}) - \mathrm{v}(1 - \mathrm{v}^n)\right] = 0, \quad (x, t) \in \Omega.$$

Applying the idea of Caputo derivative $D_t^{\alpha}(u_0) = J_t^{1-\alpha} D_t(u_0) = 0$ [J_t^{α} is the Riemann–Liouville operator], we get

$$D_t^{\alpha} \mathrm{v} = p[D_x(\mathrm{v}^n D_x \mathrm{v}) + \mathrm{v}(1 - \mathrm{v}^n)], \tag{8}$$

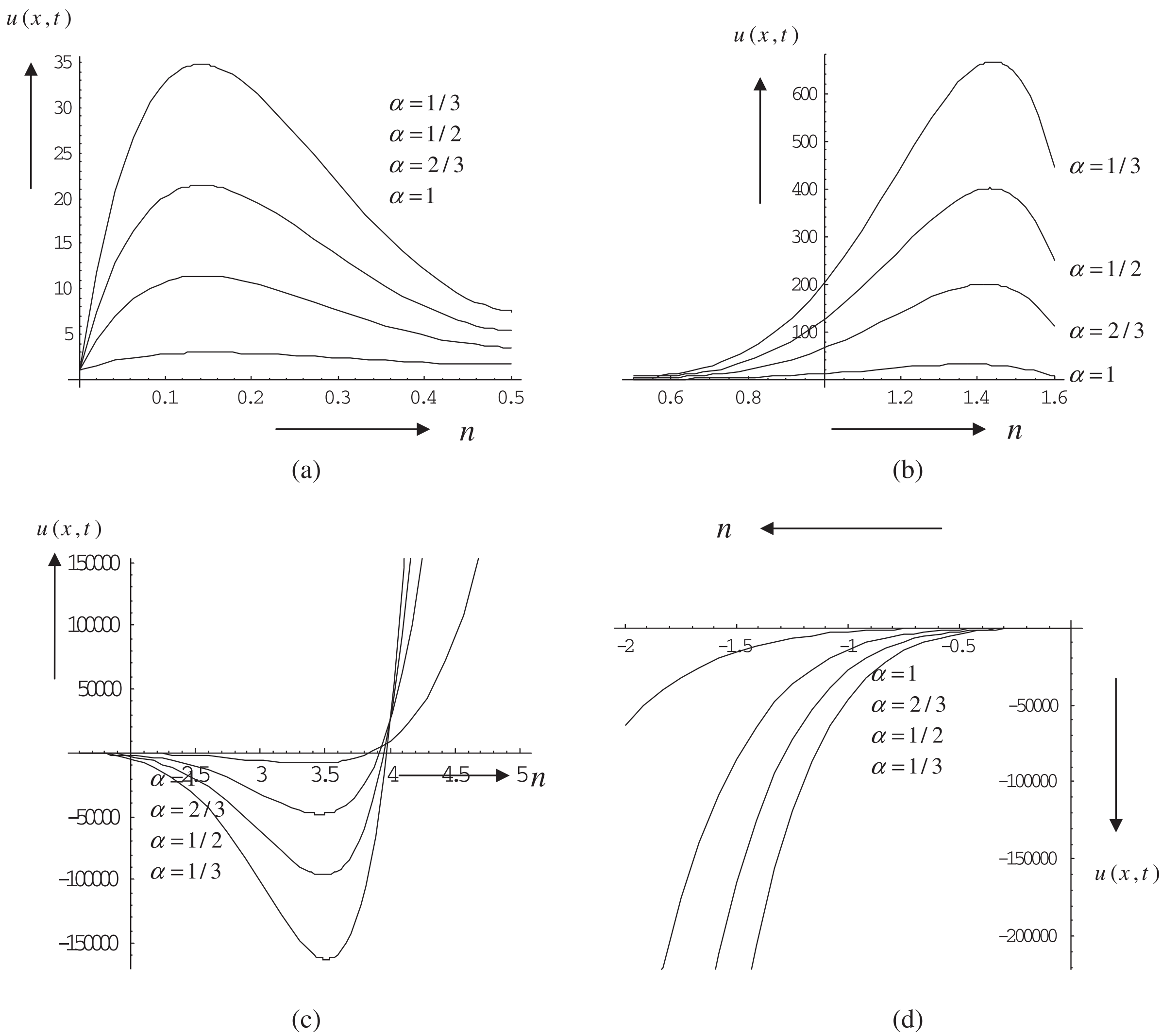

Fig. 3. Plots of $u(x,t)$ vs. different range of n (a) n = 0 to 0.5, (b) n = 0.5 to 1.6, (c) n = 1.6 to 5, and (d) n = −2 to 0, for various values of α at x = 1 and t = 1.

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$). Now applying the classical perturbation technique, we can assume that the solution of Eq. (8) can be expressed as a power series in p as given below

$$v = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots. \tag{9}$$

When $p \to 1$, Eq. (8) corresponds to Eq. (6) and Eq. (9) becomes the approximate solution of Eq. (6), that is, of Eq. (4). Substituting Eq. (9) in Eq. (8) and comparing the like powers of p, we obtain the following set of differential equations

$$p^0 : D_t^\alpha v_0 = 0, \tag{10}$$

$$p^1 : D_t^\alpha v_1 = D_x\left(v_0^n D_x\ v_0\right) + v_0 - v_0^{n+1}, \tag{11}$$

$$p^2 : D_t^\alpha v_2 = D_x\left({}^nC_1 v_0^{n-1} v_1 D_x v_0 + v_0^n D_x v_1\right) + v_1 - {}^{n+1}C_1 v_0^n v_1, \tag{12}$$

$$p^3 : D_t^\alpha v_3 = D_x\left({}^nC_1 v_0^{n-1} v_2 D_x v_0 + {}^nC_2 v_0^{n-2} v_1^2 D_x v_0 + {}^nC_1 v_0^{n-1} v_1 D_x v_1 + v_0^n D_x v_2\right) + v_2 - {}^{n+1}C_1 v_0^n v_2 - {}^{n+1}C_2 v_0^{n-1} v_1^2, \tag{13}$$

and so on. Here, ${}^nC_r = \frac{n!}{r!(n-r)!}$ is the binomial coefficient.

Now, applying the operator J_t^α on both sides of the Eqs. (10)–(13), we obtain

$$v_0(x,t) = u_0(x,t) = f(x), \tag{14}$$

$$v_1(x,t) = \left[f(x) + n(f(x))^{n-1}(f'(x))^2 + (f(x))^n f''(x) - (f(x))^{n+1}\right]\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{15}$$

$$\begin{aligned} v_2(x,t) = \Big[&f(x) + n(n+2)(f(x))^{n-1}(f'(x))^2 + (n+2)(f(x))^n f^{(2)}(x) - (n+2)(f(x))^{n+1} \\ &+(n+1)(f(x))^{2n+1} + (f(x))^{2n} f^{(4)}(x) - (3n+2)(f(x))^{2n} f^{(2)}(x) - n(5n+3)(f(x))^{2n-1}(f'(x))^2 \\ &+4n(f(x))^{2n-1}(f^{(2)}(x))^2 + 6n(f(x))^{2n-1} f'(x) f^{(3)}(x) + 7n(2n-1)(f(x))^{2n-2}(f'(x))^2 f^{(2)}(x) \\ &+2n(n-1)(2n-1)(f(x))^{2n-3}(f'(x))^4\Big]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{16}$$

$$\begin{aligned} v_3(x,t) = \Big[\Big\{&f(x) + n(2n+3)(f(x))^{n-1}(f'(x))^2 + (2n+3)(f(x))^n f^{(2)}(x) - (2n+3)(f(x))^{n+1} \\ &+2n(n+3)(n-1)(2n-1)(f(x))^{2n-3}(f'(x))^4 + 7n(n+3)(2n-1)(f(x))^{2n-2}(f'(x))^2 f^{(2)}(x) \\ &+4n(n+3)(f(x))^{2n-1}(f^{(2)}(x))^2 + 6n(n+3)(f(x))^{2n-1} f'(x) f^{(3)}(x) - n(n+3)(5n+3)(f(x))^{2n-1}(f'(x))^2 \\ &-(n+3)(3n+2)(f(x))^{2n} f^{(2)}(x) + (n+3)(f(x))^{2n} f^{(4)}(x) + (n+1)(n+3)(f(x))^{2n+1} \\ &+6n(n-1)^2(2n-1)(3n-4)(f(x))^{3n-5}(f'(x))^6 + 3n(n-1)(2n-1)(39n-32)(f(x))^{3n-4}(f'(x))^4 f^{(2)}(x) \\ &-n(49n^3 - 20n^2 - 21n + 8)(f(x))^{3n-3}(f'(x))^4 + n(294n^2 - 353n + 102)(f(x))^{3n-3}(f'(x))^2(f^{(2)}(x))^2 \\ &+2n(77n^2 - 88n + 24)(f(x))^{3n-3}(f'(x))^3 f^{(3)}(x) - 4n(29n^2 + 9n - 7)(f(x))^{3n-2}(f'(x))^2 f^{(2)}(x) \\ &+2n(20n-9)(f(x))^{3n-2}(f^{(2)}(x))^3 + 2n(93n-38)(f(x))^{3n-2} f'(x) f^{(2)}(x) f^{(3)}(x) \\ &+n(59n-22)(f(x))^{3n-2}(f'(x))^2 f^{(4)}(x) - n(23n+16)(f(x))^{3n-1}(f^{(2)}(x))^2 \\ &-2n(17n+12)(f(x))^{3n-1} f'(x) f^{(3)}(x) + 12n(f(x))^{3n-1} f'(x) f^{(5)}(x) + 14n(f(x))^{3n-1}(f^{(3)}(x))^2 \\ &+2n(n+1)(7n+3)(f(x))^{3n-1}(f'(x))^2 + 23n(f(x))^{3n-1} f^{(2)}(x) f^{(4)}(x) - (4n+3)(f(x))^{3n} f^{(4)}(x) \\ &+3(n+1)(2n+1)(f(x))^{3n} f^{(2)}(x) + (f(x))^{3n} f^{(6)}(x) - (n+1)^2(f(x))^{3n+1}\Big\} + \frac{n}{2}\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} \\ &\{n(n+1)(f(x))^{n-1}(f'(x))^2 + (n+1)(f(x))^n f^{(2)}(x) - (n+1)(f(x))^{n+1} + 4n(n-1)(2n-1)(f(x))^{2n-3}(f'(x))^4 \\ &+ 14n(2n-1)(f(x))^{2n-2}(f'(x))^2 f^{(2)}(x) + 8n(f(x))^{2n-1}(f^{(2)}(x))^2 + 12n(f(x))^{2n-1} f'(x) f^{(3)}(x) \\ &- 2n(5n+3)(f(x))^{2n-1}(f'(x))^2 - 2(3n+2)(f(x))^{2n} f^{(2)}(x) + 2(f(x))^{2n} f^{(4)}(x) + 2(n+1)(f(x))^{2n+1} \\ &+ 3n^2(n-1)(3n-4)(f(x))^{3n-5}(f'(x))^6 + 3n(n-1)(15n-4)(f(x))^{3n-4}(f'(x))^4 f^{(2)}(x) \\ &+ (51n^2 - 29n + 2)(f(x))^{3n-3}(f'(x))^2(f^{(2)}(x))^2 + 8n(2n-1)(f(x))^{3n-3}(f'(x))^3 f^{(3)}(x) \\ &- n(19n^2 - 17n + 4)(f(x))^{3n-3}(f'(x))^4 - 2n(25n-7)(f(x))^{3n-2}(f'(x))^2 f^{(2)}(x) \\ &+ (7n-1)(f(x))^{3n-2}(f^{(2)}(x))^3 + 4(6n-1)(f(x))^{3n-2} f'(x) f^{(2)}(x) f^{(3)}(x) + 2n(f(x))^{3n-2}(f'(x))^2 f^{(4)}(x) \\ &+ 2(f(x))^{3n-1} f^{(2)}(x) f^{(4)}(x) - (11n+1)(f(x))^{3n-1}(f^{(2)}(x))^2 - 16n(f(x))^{3n-1} f'(x) f^{(3)}(x) \\ &+ 2(f(x))^{3n-1}(f^{(3)}(x))^2 + n(11n+5)(f(x))^{3n-1}(f'(x))^2 + (5n+3)(f(x))^{3n} f^{(2)}(x) \\ &-2(f(x))^{3n} f^{(4)}(x) - (n+1)(f(x))^{3n+1}\}\Big]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{aligned} \tag{17}$$

Proceeding in this manner, the rest of the components $v_n(x,t)$ can be obtained and the series solutions are thus entirely determined.

Finally, we approximate the analytical solution $u(x,t)$ by the truncated series

$$u(x,t) = \lim_{N\to\infty} \Phi_N(x,t), \tag{18}$$

where $\Phi_N(x,t) = \sum_{n=0}^{N-1} v_n(x,t)$.

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [12].

3. Numerical results and discussion

Numerical results of the probability density function $u(x,t)$ for different fractional Brownian motions $\alpha = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and standard motion $\alpha = 1$ for $f(x) = x$ are calculated at different values of n and x and are depicted in Figs. 1–3.

It has been observed from Fig. 1 that the probability density function rapidly decreases with the increase in t when n becomes more negative i.e., diffusion increases in opposite direction with the increase in time and negative values of n. Again the magnitude of $u(x,t)$ decreases with the increase ofα, which conforms the exponential decay of regular Brownian motion and the result is in complete agreement with results of Das [13], Giona and Roman [14].

It is seen that $u(x,t)$ increases with the increase in t for $n = \frac{1}{2}, 1, \frac{3}{2}$ but decreases with t for $n = 2$ at $x = 1$ for every α as described in Fig. 2. Again the behavior of $u(x,t)$ with α is same as the previous case.

Fig. 3 describes the explicit nature of $u(x,t)$ for various range of n at $x = 1$ and $t = 1$. It is seen from the figures that three consecutive sub-diffusions occur where first two are slow and in the positive direction but the third one is faster and in the opposite direction and after that super-diffusion occurs. Again for the negative values of n, the rapid diffusion occurs in the opposite direction.

4. Conclusion

The important part of the study is the effect of reaction term in the nonlinear fractional diffusion equation in the range $0 < n < 5$ where anomalous diffusion is observed with sub-diffusion in the range $0 < n < 4$ and super-diffusion in the range of $4 < n < 5$. As revealed by Fig. 3 unlike standard nonlinear fractional diffusion equation, there is no threshold between sub-diffusion and super-diffusion. Moreover, slow and fast diffusions have been observed with reverse diffusion in the range of $2.5 < n < 4$. But in the range $-2 < n < 0$ (Fig. 1) no sub-diffusion or super-diffusion occurs, demarcation has been observed and $u(x,t)$ describes the asymptotic behavior with t.

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