Hindawi Publishing Corporation
Advances in Fuzzy Systems
Volume 2012, Article ID 604396, 7 pages
doi:10.1155/2012/604396

Research Article

Separation Axioms in Intuitionistic Fuzzy Topological Spaces

Amit Kumar Singh[1] and Rekha Srivastava[2]

[1] *Electronics and Communication Sciences Unit, Indian Statistical Institute, Kolkata 700108, India*
[2] *Department of Applied Mathematics, Indian Institute of Technology (BHU), Varanasi 221005, India*

Correspondence should be addressed to Amit Kumar Singh, amitkitbhu@gmail.com

Received 30 April 2012; Accepted 8 November 2012

Academic Editor: Mehmet Bodur

In this paper we have studied separation axioms T_i, $i = 0, 1, 2$ in an intuitionistic fuzzy topological space introduced by Coker. We also show the existence of functors $\mathcal{B}$: IF-Top $\rightarrow$ BF-Top and $\mathcal{D}$: BF-Top $\rightarrow$ IF-Top and observe that $\mathcal{D}$ is left adjoint to $\mathcal{B}$.

1. Introduction

Fuzzy sets were introduced by Zadeh [1] in 1965 as follows: a fuzzy set A in a nonempty set X is a mapping from X to the unit interval $[0, 1]$, and $A(x)$ is interpreted as the degree of membership of x in A. Atanassov [2] generalized this concept and introduced intuitionistic fuzzy sets which take into account both the degrees of membership and of nonmembership subject to the condition that their sum does not exceed 1. Çoker [3] subsequently initiated a study of intuitionistic fuzzy topological spaces.

In this paper we have searched for appropriate definitions of the separation axioms T_i, $i = 0, 1, 2$ in intuitionistic fuzzy topological spaces.

Hausdorffness in an intuitionistic fuzzy topological space has been introduced earlier by Çoker [3], Bayhan and Çoker [4], and Lupianez [5]. In [4], the authors have given six possible definitions of Hausdorffness including that given in [3], and a comparative study has been done. In this paper we have introduced another definition which generalizes the corresponding definition in a fuzzy topological space given in [6]. Our definition is more general than those given in [3, 5], and it turns out to be equivalent to $FT_2(vi)$ in [4].

T_1-ness in an intuitionistic fuzzy topological space has been defined earlier in [4] in six possible ways. Out of those, we have chosen $FT_1(ii)$ as it generalizes the most appropriate definition of T_1-ness in a fuzzy topological space (cf. definition 5.1, [7]). We have also introduced a suitable definition of T_0-ness in an intuitionistic topological space.

The appropriateness of the definitions has been established by proving several basic desirable results; for example, they satisfy hereditary, productive, and projective properties. We have also shown that the functor $\mathcal{B}$: IF-Top $\rightarrow$ BF-Top preserves these separation properties.

2. Preliminaries

Throughout X denotes a nonempty set, I denotes the unit interval $[0, 1]$, and I_0 and I_1 denote the intervals $(0, 1]$ and $[0, 1)$, respectively. A fuzzy set in X is a function from X to I. The collection of all fuzzy sets in X is denoted by I^X. For any $A \in I^X$, A' denotes the fuzzy complement of A, and the constant fuzzy set in X, taking value $\alpha \in I$, is denoted by $\underline{\alpha}$. A crisp subset of X will be identified with its characteristic function. If $Y \subseteq X$, then $A \in I^Y$ will be identified with the fuzzy set in X which takes the same value as A if $x \in Y$ and zero if $x \notin Y$.

Definition 1 (Atanassov [2]). Let X be a nonempty set. An intuitionistic fuzzy set (IFS, in short) A is an ordered pair (μ_A, ν_A) of fuzzy sets in X. Here $(\mu_A, \nu_A)(x) = (\mu_A(x), \nu_A(x))$ and $\mu_A(x)$, $\nu_A(x)$, respectively, denote the degree of membership and the degree of nonmembership of $x \in X$ to the set A and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

We identify an ordinary fuzzy set $A \in I^X$ with the intuitionistic fuzzy set (A, A').

Definition 2 (Atanassov [2]). Let X be a nonempty set and A, B be given by (μ_A, ν_A) and (μ_B, ν_B), respectively,

(a) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$,

(b) $A = B$ if $A \subseteq B$ and $B \subseteq A$,

(c) $\overline{A} = (\nu_A, \mu_A)$,

(d) $A \cap B = (\mu_A \cap \mu_B, \nu_A \cup \nu_B)$,

(e) $A \cup B = (\mu_A \cup \mu_B, \nu_A \cap \nu_B)$.

Definition 3 (Çoker [3]). Let $\{A_i : i \in J\}$ be an arbitrary family of IFSs in X. Then

(a) $\cap A_i = (\cap \mu_{A_i}, \cup \nu_{A_i})$,

(b) $\cup A_i = (\cup \mu_{A_i}, \cap \nu_{A_i})$,

(c) $0_{\sim} = (\underline{0}, \underline{1})$, $1_{\sim} = (\underline{1}, \underline{0})$.

Definition 4 (Çoker [3]). Let X and Y be two nonempty sets and $f : X \to Y$ be a function. If A and B be IFSs in X and Y, respectively, then

(a) $f(A) = (f(\mu_A), (1 - f(1 - \nu_A)))$,

(b) $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B))$.

It is easy to verify that $f^{-1}(\cap A_i) = \cap f^{-1}(A_i)$ and $f^{-1}(\cup A_i) = \cup f^{-1}(A_i)$.

Definition 5 (Wong [8]). A fuzzy point x_r in X is a fuzzy set in X taking value $r \in (0,1)$ at x and zero elsewhere, and x and r are, respectively, called the support and value of x.

A fuzzy point x_r is said to belong to a fuzzy set A (notation : $x_r \in A$) if $r < A(x)$ (cf. [6]).

Two fuzzy points are said to be distinct if their supports are distinct.

Definition 6. Let X be a nonempty set and $x \in X$ a fixed element in X. If $\alpha \in (0,1)$ and $\beta \in (0,1)$ are two fixed real numbers such that $\alpha + \beta \leq 1$, then the IFS $x_{(\alpha,\beta)} = (x_\alpha, 1 - x_{(1-\beta)})$ is called an intuitionistic fuzzy point (IFP, in short) in X, and x is called its support. Two IFPs are said to be distinct if their supports are distinct.

Let $x_{(\alpha,\beta)}$ be an IFP in X and $A = (\mu_A, \nu_A)$ be an IFS in X. Then $x_{(\alpha,\beta)}$ is said to belong to A (notation : $x_{(\alpha,\beta)} \in A$, in short) if $\alpha < \mu_A(x)$, $\beta > \nu_A(x)$ (cf. [9]).

We identify a fuzzy point x_r in X by the intuitionistic fuzzy point $x_{(r,(1-r))}$ in X.

Proposition 7. *An intuitionistic fuzzy set A in X is the union of all intuitionistic fuzzy points belonging to A.*

The proof is on similar lines as in [10, Theorem 2.4] and hence is omitted.

Replacing fuzzy sets by intuitionistic fuzzy sets in Chang's definition of a fuzzy topological space, we get the following.

Definition 8 (Çoker [3]). An intuitionistic fuzzy topology (IFT, in short) on a nonempty set X is a family τ of IFSs in X satisfying the following axioms:

(1) $0_{\sim}, 1_{\sim} \in \tau$,

(2) $G_1 \cap G_2 \in \tau$, for all $G_i \in \tau$, $i = 1, 2$,

(3) $\cup G_i \in \tau$ for any arbitrary family $\{G_i \in \tau : i \in J\}$.

The pair (X, τ) is called an intuitionistic fuzzy topological space (IFTS, in short), members of τ are called intuitionistic fuzzy open sets (IFOS, in short) in X, and their complements are called intuitionistic fuzzy closed sets (IFCS, in short).

Definition 9. Let (X, τ) be an IFTS. A subfamily $\mathcal{B} \subseteq \tau$ is called a base for τ if every $U \in \tau$ can be written as a union of members of $\mathcal{B}$.

Proposition 10. *Let (X, τ) be an IFTS, and then a subfamily $\mathcal{B} \subseteq \tau$ is a base for τ if and only if for all $U \in \tau$ and intuitionistic fuzzy point $x_{(\alpha,\beta)} \in U$, $\exists B \in \mathcal{B}$ such that $x_{(\alpha,\beta)} \in B \subseteq U$.*

The proof is easy omitted.

Definition 11. Let (X, τ) be an IFTS. Then a subfamily $\mathcal{S} \subseteq \tau$ is called a subbase for τ if the family of finite intersections of members of $\mathcal{S}$ forms a base for τ.

Given any collection $\mathcal{S}$ of IFSs in X, containing $0_{\sim}$ and $1_{\sim}$, the set τ consisting of arbitrary unions of finite intersections of members of $\mathcal{S}$ forms an IFT on X. This is the smallest IFT on X containing $\mathcal{S}$ and is called the IFT generated by $\mathcal{S}$.

Definition 12 (S. J. Lee and E. P. Lee [10]). An IFS N in an IFTS (X, τ) is called an intuitionistic fuzzy neighborhood (IFN, in short) of an IFP $x_{(\alpha,\beta)}$ if $\exists U \in \tau$ such that $x_{(\alpha,\beta)} \in U \subseteq N$.

Proposition 13. *Let (X, τ) be an IFTS. Then an IFS A in X is an IFOS if and only if A is an IFN of each of IFP $x_{(\alpha,\beta)} \in A$.*

The proof is on similar lines as in ([10], Theorem 2.6) and hence is omitted.

Definition 14 (S. J. Lee and E. P. Lee [10]). A map $f : (X, \tau) \to (Y, \delta)$ between IFTSs is called intuitionistic fuzzy continuous if $f^{-1}(U) \in \tau$, for all $U \in \delta$.

Definition 15 (Abu Safia et al. [12]). Let X be a nonempty set and τ_1, τ_2 be two fuzzy topologies on X. Then (X, τ_1, τ_2) is called a bifuzzy topological space (BFTS, in short).

A map $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ between two BFTSs is said to be FP continuous if $f^{-1}(U_i) \in \tau_i$, for all $U_i \in \delta_i$, $i = 1, 2$.

Definition 16 (Bayhan and Çoker [4]). Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs in X and Y, respectively, and then $A \times B$ is the IFS in $X \times Y$ defined as follows

$$A \times B = (\mu_A \times \mu_B, \nu_A * \nu_B), \tag{1}$$

where $(\mu_A \times \mu_B)(x, y) = \min(\mu_A(x), \mu_B(y))$, for all $(x, y) \in X \times Y$ and $(\nu_A * \nu_B)(x, y) = \max(\nu_A(x), \nu_B(y))$, for all $(x, y) \in X \times Y$.

This definition can be extended to an arbitrary family of IFSs as follows.

If $\{A_i = (\mu_{A_i}, \nu_{A_i}),\ i \in J\}$ is a family of IFSs in X_i, then their product is defined as the IFS in ΠX_i given by

$$\Pi A_i = (\Pi\mu_{A_i}, \Pi^*\nu_{A_i}), \qquad (2)$$

where $\Pi\mu_{A_i}(x) = \inf \mu_{A_i}(x_i)$, for all $x = \Pi x_i \in X$ and $\Pi^*\nu_{A_i}(x) = \sup \nu_{A_i}(x_i)$,for all $x = \Pi x_i \in X$.

Definition 17 (Bayhan and Çoker [4]). Let $(x_i, \tau_i), i = 1, 2$ be two IFTSs, and then the product IFT $\tau_1 \times \tau_2$ on $X_1 \times X_2$ is defined as the IFT generated by $\{p_i^{-1}(U_i) : U_i \in \tau_i, i = 1, 2\}$ where $p_i : X_1 \times X_2 \to X_i$, $i = 1, 2$ are the projection maps, and the IFTS $(X_1 \times X_2, \tau_1 \times \tau_2)$ is called the product IFTS.

This definition can be extended to an arbitrary family of IFTSs as follows.

Let $\{(X_i, \tau_i) : i \in J\}$ be a family of IFTSs. Then the product intuitionistic fuzzy topology τ on $X = \Pi X_i$ is the one having $\{p_j^{-1}(U_j) : U_j \in \tau_j, j \in J\}$ as a subbase where $p_j : X \to X_j$ is the jth projection map. (X, τ) is called the product IFTS of the family $\{(X_i, \tau_i) : i \in J\}$.

Definition 18. A fuzzy topological space (X, τ) is called

(a) T_0 if for all $x, y \in X, x \neq y$, $\exists U \in \tau$ such that either $U(x) = 1$, $U(y) = 0$, or $U(y) = 1$, $U(x) = 0$,

(b) T_1 if for all $x, y \in X$, $x \neq y$, $\exists U, V \in \tau$ such that $U(x) = 1$, $U(y) = 0$, $V(y) = 1$, and $V(x) = 1$,

(c) T_2 (Hausdorff) if for all pair of distinct fuzzy points x_r, y_s in X, $\exists U, V \in \tau$ such that $x_r \in U$, $y_s \in V$, and $U \cap V = \underline{0}$,

(d) q-T_2 (q-Hausdorff) if for any pair of distinct fuzzy points x_r, and y_s $\exists U, V \in \tau$ such that $x_r \in U$, $y_s \in \tau$ and $U \subseteq V'$.

Here definitions (d), (c), (b), and (a) are from [5–7, 13], respectively.

Definition 19. Let (X, τ_1, τ_2) be a BFTS. Then it is called

(a) T_0 if for all $x, y \in X, x \neq y$, $\exists U \in \tau_1 \cup \tau_2$ such that $U(x) = 1$, $U(y) = 0$ or $U(x) = 0$, $U(y) = 1$,

(b) T_1 if for all $x, y \in X, x \neq y$, $\exists U \in \tau_1$ and $V \in \tau_2$ such that $U(x) = 1$, $U(y) = 0$ and $V(x) = 0$, $V(y) = 1$,

(c) T_2 if for all pair of distinct fuzzy points x_r, y_s in X, $\exists U \in \tau_1$, $V \in \tau_2$ such that $x_r \in U$, $y_s \in V$ and $U \cap V = \underline{0}$.

Here definitions (a) and (b) are from [14], and (c) is from [15].

For the categorical concepts used here, we refer the reader to [16].

3. Separation Axioms in Intuitionistic Fuzzy Topological Spaces

Definition 20. An IFTS (X, τ) is called

(a) T_0 if for all $x, y \in X$, $x \neq y$, $\exists U = (\mu_U, \nu_U)$, $V = (\mu_V, \nu_V) \in \tau$ such that $(\mu_U, \nu_U)(x) = (1, 0)$, $(\mu_U, \nu_U)(y) = (0, 1)$ or $(\mu_V, \nu_V)(x) = (0, 1)$, $(\mu_V, \nu_V)(y) = (1, 0)$,

(b) (Bayhan and Çoker [4]). T_1 if for all $x, y \in X$, $x \neq y$, $\exists U = (\mu_U, \nu_U)$, $V = (\mu_V, \nu_V) \in \tau$ such that $(\mu_U, \nu_U)(x) = (1, 0)$, $(\mu_U, \nu_U)(y) = (0, 1)$, $(\mu_V, \nu_V)(x) = (0, 1)$ and $(\mu_V, \nu_V)(y) = (1, 0)$,

(c) T_2 (Hausdorff) if for all pair of distinct intuitionistic fuzzy points $x_{(\alpha,\beta)}, y_{(\gamma,\delta)}$ in X, $\exists U, V \in \tau$ such that $x_{(\alpha,\beta)} \in U$, $y_{(\gamma,\delta)} \in V$ and $U \cap V = 0_\sim$,

(d) q-T_2 if for every pair of distinct intuitionistic fuzzy points $x_{(\alpha,\beta)}, y_{(\gamma,\delta)}$ in X, $\exists U$ and $V \in \tau$ such that $x_{(\alpha,\beta)} \in U$, $y_{(\gamma,\delta)} \in V$ and $U \subseteq V'$.

Example 21. Let $X = \{a, b\}$ and let $\tau = \{0_\sim, A, B, 1_\sim\}$, where $A = \langle x, (a/1, b/0), (a/0, b/1)\rangle$ and $B = \langle x, (a/1, b/0), (a/0, b/1)\rangle$, then (X, τ) is an IFTS, and it is T_0, T_1, T_2 (Hausdorff) and q-T_2.

We have $T_2 \Rightarrow T_1 \Rightarrow T_0$ and $T_2 \Rightarrow q$-T_2, but none of the implication are reversible.

Now we associate a BFTS with an IFTS and vice versa on parallel lines as in Bayhan and Çoker [11].

Let (X, τ) be an IFTS and $\tau_1 = \{\mu_A \mid \exists \nu_A \in I^X$ such that $(\mu_A, \nu_A) \in \tau\}$, $\tau_2 = \{(\underline{1} - \nu_A) \mid \exists \mu_A \in I^X$ such that $(\mu_A, \nu_A) \in \tau\}$. It is easy to see that (X, τ_1) and (X, τ_2) are fuzzy topological spaces in Chang's sense.

(X, τ_1, τ_2) is called the bifuzzy topological space associated with the IFTS (X, τ).

Proposition 22. *Let (X, τ_1, τ_2) be a BFTS and $\tau_{\tau_1,\tau_2} = \{(U, V') \mid U \in \tau_1, V \in \tau_2$ and $U \subseteq V\}$. Then $(X, \tau_{\tau_1,\tau_2})$ is an IFTS and $(\tau_{\tau_1,\tau_2})_1 = \tau_1$, $(\tau_{\tau_1,\tau_2})_2 = \tau_2$.*

Proof. Clearly members of τ_{τ_1,τ_2} are intuitionistic fuzzy sets, and $0_\sim$ and $1_\sim$ belong to it. Now let $(U_i, V_i') \in \tau_{\tau_1,\tau_2}$, $i = 1, 2$ then $(U_1, V_1') \cap (U_2, V_2') = (U_1 \cap U_2, V_1' \cup V_2') = (U_1 \cap U_2, (V_1 \cap V_2)') \in \tau_{\tau_1,\tau_2}$. Further let $\{(U_i, V_i') : i \in J$, where J is arbitrary$\} \subseteq \tau_{\tau_1,\tau_2}$. Then $\bigcup\{(U_i, V_i')\} = (\bigcup U_i, \bigcap V_i') = (\bigcup U_i, (\bigcup V_i)') \in \tau_{\tau_1,\tau_2}$. Thus $(X, \tau_{\tau_1,\tau_2})$ is an IFTS.

Now let $U \in \tau_1$ then $(U, \phi) \in \tau_{\tau_1,\tau_2}$. Therefore $\tau_1 \subseteq (\tau_{\tau_1,\tau_2})_1$. Conversely let $U \in (\tau_{\tau_1,\tau_2})_1$ then $\exists V \in I^X$ such that $(U, V) \in \tau_{\tau_1,\tau_2} \Rightarrow U \in \tau_1$, so $(\tau_{\tau_1,\tau_2})_1 \subseteq \tau_1$. Thus $(\tau_{\tau_1,\tau_2})_1 = \tau_1$. Similarly we can show that $(\tau_{\tau_1,\tau_2})_2 = \tau_2$. □

The IFTS $(X, \tau_{\tau_1,\tau_2})$ is called the IFTS associated with the BFTS (X, τ_1, τ_2).

Proposition 23. *Let (X, τ) and (Y, δ) be two IFTSs and $f : (X, \tau) \to (Y, \delta)$ be IF-continuous. Then $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ is FP-continuous (Here (X, τ_1, τ_2) and (Y, δ_1, δ_2) are BFTSs associated with (X, τ) and (Y, δ), resp.).*

Proof. Let $f : (X, \tau) \to (Y, \delta)$ be IF-continuous. To show that $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ is FP-continuous, take $U \in \delta_1$ then $\exists V \in I^Y$ such that $(U, V) \in \delta$. Hence since f is IF-continuous, $f^{-1}(U, V) \in \tau$, that is, $(f^{-1}(U), f^{-1}(V)) \in \tau \Rightarrow f^{-1}(U) \in \tau_1$. Further take $V_1 \in \delta_2$ then $\exists U_1 \in I^Y$ such that $(U_1, V_1') \in \delta \Rightarrow f^{-1}(U_1, V_1') \in \tau \Rightarrow (f^{-1}(U_1), f^{-1}(V_1')) \in \tau \Rightarrow (f^{-1}(U_1), (f^{-1}(V_1))') \in \tau \Rightarrow f^{-1}(V_1) \in \tau_2$. Thus f is FP-continuous. □

Proposition 24. *Let (X, τ_1, τ_2) and (Y, δ_1, δ_2) be two BFTSs and $(X, \tau_{\tau_1,\tau_2})$ and $(Y, \delta_{\delta_1,\delta_2})$ be the associated IFTSs respectively. Then $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ is FP-continuous if and only if $f : (X, \tau_{\tau_1,\tau_2}) \to (Y, \delta_{\delta_1,\delta_2})$ is IF-continuous.*

Proof. Let $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ be FP-continuous. To show that $f : (X, \tau_{\tau_1,\tau_2}) \to (Y, \delta_{\delta_1,\delta_2})$ is IF continuous, take any $U = (\mu_U, \nu_U') \in \delta_{\delta_1,\delta_2}$, where $\mu_U \in \delta_1$, $\nu_U \in \delta_2$ and $\mu_U \subseteq \nu_U$. Now using FP-continuity of f, $f^{-1}(\mu_U) \in \tau_1$ and $f^{-1}(\nu_U) \in \tau_2$. So $(f^{-1}(\mu_U), (f^{-1}(\nu_U))') \in \tau_{\tau_1,\tau_2} \Rightarrow (f^{-1}(\mu_U), f^{-1}(\nu_U')) \in \tau_{\tau_1,\tau_2} \Rightarrow f^{-1}(\mu_U, \nu_U') \in \tau_{\tau_1,\tau_2}$, that is, $f^{-1}(U) \in \tau_{\tau_1,\tau_2}$, showing that f is IF-continuous.

The converse follows from the previous Proposition 23 in view of the fact that $(\tau_{\tau_1,\tau_2})_1 = \tau_1$, $(\tau_{\tau_1,\tau_2})_2 = \tau_2$ and $(\delta_{\delta_1,\delta_2})_1 = \delta_1$ and $(\delta_{\delta_1,\delta_2})_2 = \delta_2$. □

The category of all BFTS together with FP-continuous functions will be denoted by BF-Top and the category of all IFTS together with IF-continuous function will be denoted by IF-Top.

Now we define $\mathcal{B}$: IF-Top $\to$ BF-Top as follows:

$\mathcal{B}(X, \tau) = (X, \tau_1, \tau_2)$, $\mathcal{B}(f) = f$, for all morphism f and $\mathcal{D}$: BF-Top $\to$ IF-Top as follows:

$\mathcal{D}(X, \tau_1, \tau_2) = (X, \tau_{\tau_1,\tau_2})$, $\mathcal{D}(f) = f$, for all morphism f.

It can be checked easily that $\mathcal{B}$ and $\mathcal{D}$ are covariant functors, and in view of Proposition 22 we have the following remark.

Remark 25. $\mathcal{B} \circ \mathcal{D} = Id_{\text{BF-Top}}$, the identity functor.

Theorem 26. *The functor $\mathcal{D}$: BF-Top $\to$ IF-Top is left adjoint to the functor $\mathcal{B}$: IF-Top $\to$ BF-Top.*

The proof is on parallel lines as in ([11], Theorem 3.10) and hence is omitted.

Proposition 27. *The following statements are equivalent in an IFTS (X, τ):*

(1) *(X, τ) is T_1,*

(2) *$(\{x\}, \{x\}')$ is intuitionistic fuzzy closed in (X, τ), for all $x \in X$.*

Proof. (1) $\Rightarrow$ (2) We show that $(\{x\}', \{x\})$ is intuitionistic fuzzy open in X. Choose any IFP $y_{(\alpha,\beta)}$ in $(\{x\}', \{x\})$ then, $y \neq x$. Hence $\exists$ IFOSs $U, V \in \tau$ such that $U(x) = (\mu_U(x), \nu_U(x)) = (1, 0)$, $U(y) = (\mu_U(y), \nu_U(x)) = (0, 1)$ and $V(x) = (\mu_V(x), \nu_V(x)) = (0, 1)$, $V(y) = (\mu_V(y), \nu_V(y)) = (1, 0)$. Now $y_{(\alpha,\beta)} \in V \subseteq (\{x\}', \{x\})$. Therefore in view of Proposition 13, $(\{x\}', \{x\})$ is an IFOS.

(2) $\Rightarrow$ (1) Choose $x, y \in X$ such that $x \neq y$. Then $(\{x\}, \{x\}')$ and $(\{y\}, \{y\}')$ are IFCSs and hence $U = (\{x\}', \{x\})$, $V = (\{y\}', \{y\})$ are IFOSs such that $U(x) = (0, 1)$, $U(y) = (1, 0)$, $V(y) = (0, 1)$ and $V(x) = (1, 0)$ showing that (X, τ) is T_1. □

Proposition 28. *If an IFTS (X, τ) is T_1, then its associated BFTS is T_1.*

Proof. Let (X, τ) be T_1. Then for all $x \neq y$, $\exists U = (\mu_U, \nu_U)$ and $V = (\mu_V, \nu_V) \in \tau$ such that $(\mu_U, \nu_U)(x) = (1, 0)$, $(\mu_U, \nu_U)(y) = (0, 1)$, $(\mu_V, \nu_V)(x) = (0, 1)$, and $(\mu_V, \nu_V)(y) = (1, 0)$. Now $(\mu_U, \nu_U)(x) = (1, 0) \Rightarrow \mu_U(x) = 1$, $\nu_U(x) = 0$.

Similarly, $\mu_U(y) = 0$, $\nu_U(y) = 1$, $\mu_V(x) = 0$, $\nu_V(x) = 1$, and $\mu_V(y) = 1$, $\nu_V(y) = 0$. Now $\mu_U \in \tau_1$, $(\underline{1} - \nu_V) \in \tau_2$ and further $\mu_U(x) = 1$, $\mu_U(y) = 0$, $(\underline{1} - \nu_V)(x) = 0$, and $(\underline{1} - \nu_V)(y) = 1$ showing that (X, τ_1, τ_2) is a T_1 space. □

Similarly it can be shown that if an IFTS (X, τ) is T_0, then its associated BFTS is T_0.

Theorem 29. *Let $\{(X_i, \tau_i) : i \in J\}$ be a family of T_1 IFTSs and (X, τ) be their product IFTS. Then (X, τ) is T_1 if and only if (X_i, τ_i) is T_1, for all $i \in J$.*

Proof. Let (X_i, τ_i) be T_1 for all $i \in J$. To show that (X, τ) is T_1, choose $x, y \in X$, $x \neq y$. Let $x = \Pi x_i$, $y = \Pi y_i$ then $\exists j \in J$ such that $x_j \neq y_j$. Now since (X_j, τ_j) is T_1, $\exists U_j = (\mu_{U_j}, \nu_{U_j})$ and $V_j = (\mu_{V_j}, \nu_{V_j}) \in \tau_j$ such that $\mu_{U_j}(x_j) = 1$, $\nu_{U_j}(x_j) = 0$, $\mu_{U_j}(y_j) = 0$, $\nu_{U_j}(y_j) = 1$, and $\mu_{V_j}(y_j) = 1$, $\nu_{V_j}(y_j) = 0$, $\mu_{V_j}(x_j) = 0$, $\nu_{V_j}(x_j) = 1$. Now consider the basic IFOSs ΠU_k and ΠV_k where $U_k = V_k = (\underline{1}, \underline{0})$ for $k \in J, k \neq j$ and $U_k = U_J$ and $V_k = V_J$ when $k = j$. Then $\Pi U_k(x) = (\inf \mu_{U_k}(x_k), \sup \nu_{U_k}(x_k)) = (1, 0)$, $\Pi U_k(y) = (\inf \mu_{U_k}(y_k), \sup \nu_{U_k}(y_k)) = (0, 1)$. Similarly we can show that $\Pi V_k(y) = (1, 0)$ and $\Pi V_k(x) = (0, 1)$. Therefore (X, τ) is T_1.

Conversly let (X, τ) be T_1. To show that (X_j, τ_j) is T_1, choose $x_j, y_j \in X_j$ such that $x_j \neq y_j$. Now consider $x = \Pi x_i$, $y = \Pi y_i$ where $x_i = y_i$, $i \neq j$ and the jth coordinate of x, y are x_j and y_j, respectively. Then $x \neq y$, therefore since (X, τ) is T_1, $\exists U = (\mu_U, \nu_U)$ and $V = (\mu_V, \nu_V) \in \tau$ such that $U(x) = (1, 0)$, $U(y) = (0, 1)$, $V(x) = (0, 1)$, and $V(y) = (1, 0)$. Now consider the intuitionistic fuzzy points $x_{(\alpha,1-\alpha)} \in U$ and $y_{(\gamma,1-\gamma)} \in V$. Then $\exists$ basic IFOSs ΠU_i^{α} and ΠV_i^{γ} in X such that $x_{(\alpha,1-\alpha)} \in \Pi U_i^{\alpha} \subseteq U$ and $y_{(\gamma,1-\gamma)} \in \Pi V_i^{\gamma} \subseteq V$:

$$\begin{gathered} x_{(\alpha,1-\alpha)} \in \Pi U_i^{\alpha} \subseteq U \Longrightarrow \alpha < \inf_i \mu_{U_i^{\alpha}}(x_i), \\ 1 - \alpha > \sup_i \nu_{U_i^{\alpha}}(x_i) \Longrightarrow \alpha < \mu_{U_i^{\alpha}}(x_i), \\ (1 - \alpha) > \nu_{U_i^{\alpha}}(x_i), \quad \forall i \in J. \end{gathered} \tag{3}$$

Similarly we can show that

$$y_{(\gamma,1-\gamma)} \in \Pi V_i^{\gamma} \subseteq V \Longrightarrow \gamma < \mu_{V_i^{\gamma}}(y_i), (1 - \gamma) > \nu_{V_i^{\gamma}}(y_i), \quad \forall i \in J. \tag{4}$$

Now $V_j = \cup\{U_J^{\gamma} : \gamma \in (0,1)\}$ is such that $\mu_{V_j}(y_j) = 1$, $\nu_{V_j}(y_j) = 0$. Further, since $x_i = y_i$ for $i \neq j$, from (3) and (4), it follows that

$$\alpha < \mu_{U_i^\alpha}(y_i), \qquad (1-\alpha) > \nu_{U_i^\alpha}(y_i), \quad \forall i \in J, \quad i \neq j,\ \alpha \in (0,1), \tag{5}$$

$$\gamma < \mu_{V_i^\gamma}(x_i), \qquad (1-\gamma) > \nu_{V_i^\gamma}(x_i), \quad \forall i \in J, \quad i \neq j,\ \gamma \in (0,1). \tag{6}$$

Therefore

$$\begin{aligned} U(y) = (0,1) &\Longrightarrow \Pi U_i^\alpha(y) = (0,1) \\ &\Longrightarrow \inf_i \mu_{U_i^\alpha}(y_i) = 0, \\ \sup_i \nu_{U_i^\alpha}(y_i) = 1, &\quad \forall \alpha \in (0,1) \\ &\Longrightarrow \mu_{U_j^\alpha}(y_j) = 0, \quad \nu_{U_j^\alpha}(y_j) = 1, \quad \forall \alpha \in (0,1) \\ &\qquad \text{(in view of (5))} \\ &\Longrightarrow \sup_\alpha \mu_{U_j^\alpha}(y_j) = 0, \qquad \inf_\alpha \nu_{U_j^\alpha}(y_j) = 1. \end{aligned} \tag{7}$$

Thus $\mu_{U_j}(y_j) = 0$ and $\nu_{U_j}(y_j) = 1$. That is, $U_j(y_j) = (0,1)$. Similarly it can be shown that $V_J(x_J) = (0,1)$. Hence (X_j, τ_j) is T_1. □

The following theorem can be proved in a similar way.

Theorem 30. *Let $\{(X_i, \tau_i) : i \in J\}$ be a family of T_0 IFTSs and (X, τ) be their product IFTS. Then (X, τ) is T_0 if and only if (X_i, τ_i) is T_0, for all $i \in J$.*

Proposition 31. *An IFTS (X, τ) is Hausdorff, and then its associated BFTS (X, τ_1, τ_2) is Hausdorff.*

Proof. Let (X, τ) be Hausdorff. To show that (X, τ_1, τ_2) is Hausdorff, choose any two distinct fuzzy points x_α, $y_{1-\delta}$ in X. Now choose $\beta, \gamma \in (0,1)$, and then $x_{(\alpha,\beta)}$ and $y_{(\gamma,\delta)}$ are distinct intuitionistic fuzzy points in X. Since (X, τ) is Hausdorff, $\exists U, V \in \tau$ such that $x_{(\alpha,\beta)} \in U$, $y_{(\gamma,\delta)} \in V$ and $U \cap V = 0_\sim$. Let $U = (\mu_U, \nu_U)$, $V = (\mu_V, \nu_V)$, then

$$\begin{aligned} x_{(\alpha,\beta)} \in U &\Longrightarrow \alpha < \mu_U(x), \quad \beta > \nu_U(x) \\ y_{(\gamma,\delta)} \in V &\Longrightarrow \gamma < \mu_V(y), \\ \delta > \nu_V(y) &\Longleftrightarrow 1-\delta < 1-\nu_V(y), \end{aligned} \tag{8}$$

and also we have $\mu_U + \nu_U \subseteq \underline{1}$ and $\mu_V + \nu_V \subseteq \underline{1}$.

Now

$$U \cap V = 0_\sim \Longleftrightarrow \mu_U \cap \mu_V = \underline{0}, \qquad \nu_U \cup \nu_V = \underline{1}. \tag{9}$$

From (8) we have $x_\alpha \in \mu_U$ and $y_{1-\delta} \in \underline{1} - \nu_V$. Now we show that $\mu_U \cap (\underline{1} - \nu_V) = \underline{0}$ as follows.

We have, $\nu_U(x) < 1 \Rightarrow \nu\_V(x) = 1$ (in view of (9)), $\Rightarrow 1 - \nu_V(x) = 0 \Rightarrow (\mu_U \cap (\underline{1} - \nu_V))(x) = 0$.

Further, $\nu_V(y) < 1 \Rightarrow \nu_U(y) = 1$ (in view of (9)), $\Rightarrow \mu_U(y) = 0$ since $\mu_U(y) + \nu_U(y) \leq 1 \Rightarrow (\mu_U \cap (\underline{1} - \nu_V))(y) = 0$.

Now take $z \in X$ such that $z \neq x, y$. If $\mu_U(z) = 0$, then obviously $(\mu_U \cap (\underline{1} - \nu_V))(z) = 0$, and if $\mu_U(z) \neq 0$, then $\nu_U(z) < 1$ (since $\mu_U + \nu_U \subseteq \underline{1}$) $\Rightarrow \nu_V(z) = 1 \Rightarrow 1 - \nu_V(z) = 0 \Rightarrow (\mu_U \cap (\underline{1} - \nu_V))(z) = 0$.

Thus we have $\mu_U \in \tau_1$, $(\underline{1} - \nu_V) \in \tau_2$ such that $x_\alpha \in \mu_U$, $y_{1-\delta} \in (\underline{1} - \nu_V)$ and $\mu_U \cap (\underline{1} - \nu_V) = \underline{0}$, and hence (X, τ_1, τ_2) is Hausdorff. □

Definition 32. Let (X, τ) be an IFTS and $Y \subseteq X$. Then $(Y, \tau \mid Y)$ is called a subspace of (X, τ) where $\tau \mid Y = \{U \mid Y = (\mu_U \mid Y, \nu_U \mid Y) : U \in \tau\}$.

Proposition 33. *If an IFTS (X, τ) is T_i, $i = 0,1,2$, then its subspace, $(Y, \tau \mid Y)$ is also T_i, $i = 0,1,2$.*

The proofs are easy and hence are omitted.

Proposition 34. *The product IFTS (X, τ) of $\{(X_j, \tau_j) : j \in J\}$ is initial with respect to the family of projections $\{p_j : X \to X_j, j \in J\}$, that is, for any IFTS (Y, η), a map $g : (Y, \eta) \to (X, \tau)$ is IF continuous if and only if the map $p_j \circ g : (Y, \eta) \to (X_j, \tau_j)$ is IF continuous for all $j \in J$.*

Proof. Since projection maps are IF continuous and composition of IF-continuous maps are IF-continuous, so $p_j \circ g$ is IF continuous for all $j \in J$.

Conversely, if $p_j \circ g$ is IF-continuous for all $j \in J$, then $(p_j \circ g)^{-1}(U_j) = g^{-1}(p_j^{-1}(U_j))$ is IFO in (Y, η) for all $U_j \in \tau_j, j \in J$ showing that inverse image of every subbasic IFOS in X is IFO in Y which implies that g is IF continuous. □

Proposition 35. *Let $\{(X_j, \tau_j) : j \in J\}$ be a family of IFTSs, $(X_j, (\tau_j)_1, (\tau_j)_2)$ be the BFTS associated with (X_j, τ_j), and (X, τ_1, τ_2) be the BFTS associated with the product IFTS (X, τ). Then $\tau_1 = \Pi(\tau_j)_1$ and $\tau_2 = \Pi(\tau_j)_2$.*

Proof. The product space (X, τ) is generated by $\{p_j^{-1}(U_j) : U_j \in \tau_j, j \in J\}$ where p_j's are projection maps. Let $U_j = (\mu_{U_j}, \nu_{U_j})$, and then $p_j^{-1}(U_j) = (p_j^{-1}(\mu_{U_j}), p_j^{-1}(\nu_{U_j}))$.

Now members of τ are of the form:

$$\begin{aligned} &\bigcup_{l \in L(\text{arbitrary})} \ \bigcap_{k \in K\,(\text{finite})} p_{l_k}^{-1}(U_{l_k}) \\ &= \bigcup_l \bigcap_k p_{l_k}^{-1}\left(\mu_{U_{l_k}}, \nu_{U_{l_k}}\right) \\ &= \bigcup_l \bigcap_k \left(p_{l_k}^{-1}\left(\mu_{U_{l_k}}\right), p_{l_k}^{-1}\left(\nu_{U_{l_k}}\right)\right) \\ &= \left(\bigcup_l \bigcap_k p_{l_k}^{-1}\left(\mu_{U_{l_k}}\right), \bigcap_l \bigcup_k p_{l_k}^{-1}\left(\nu_{U_{l_k}}\right)\right). \end{aligned} \tag{10}$$

So, members of τ_1 are of the form $\bigcup_l \bigcap_k p_{l_k}^{-1}(\mu_{U_{l_k}})$, hence $\tau_1 = \Pi(\tau_j)_1$, and members of τ_2 are of the form $\underline{1} - \bigcap_l \bigcup_k p_{l_k}^{-1}(\nu_{U_{l_k}}) = \bigcup_l \bigcap_k p_{l_k}^{-1}(\underline{1} - \nu_{U_{l_k}})$, hence $\tau_2 = \Pi(\tau_j)_2$. □

Theorem 36. *An IFTS (X,τ) is Hausdorff if and only if (Δ_X,Δ'_X) is IFCS in $(X\times X,\tau\times\tau)$.*

Proof. We show that (Δ'_X,Δ_X) is IFOS in X. Choose any $(x,y)_{(\alpha,\beta)} \in (\Delta'_X,\Delta_X)$ then $x\neq y$ and $\alpha < \Delta'_X(x,y)$, $\beta > \Delta_X(x,y)$. Now $x_{(\alpha,\beta)}$ and $y_{(\alpha,\beta)}$ are distinct intuitionistic fuzzy points in X. Since (X,τ) is Hausdorff, $\exists$ IFOSs $U=(\mu_U,\nu_U)$ and $V=(\mu_V,\nu_V)$ in τ such that $x_{(\alpha,\beta)}\in U$, $y_{(\alpha,\beta)}\in V$ and $U\cap V=0_\sim$, that is, $(\mu_U\cap\mu_V,\nu_U\cup\nu_V)=(\underline{0},\underline{1})$.

Now consider $U\times V=(\mu_U\times\mu_V,\nu_U*\nu_V)$, and then $(x,y)_{(\alpha,\beta)}\in U\times V\subseteq(\Delta'_X,\Delta_X)$ as shown below: $(x,y)_{(\alpha,\beta)}\in(\mu_U\times\mu_V,\nu_U*\nu_V)$ as $\alpha<(\mu_U\times\mu_V)(x,y)=\inf(\mu_U(x),\mu_V(y))$ (since $\alpha<\mu_U(x),\alpha<\mu_V(y)$ both) and $\beta>(\nu_U*\nu_V)(x,y)=\sup(\nu_U(x),\nu_V(y))$ (since $\beta>\nu_U(x),\beta>\nu_V(y)$ both).

Further, $U\times V\subseteq(\Delta'_X,\Delta_X)$ since $U\times V=(\mu_U\times\mu_V,\nu_U*\nu_V)$ and $\mu_U\times\mu_V\subseteq\Delta'_X$ as $(\mu_U\times\mu_V)(x,x)=\inf(\mu_U(x),\mu_V(x))=0$, for all $x\in X$ and $\nu_U*\nu_V\supseteq\Delta_X$ as $(\nu_U\times\nu_V)(x,x)=\sup(\nu_U(x),\nu_V(x))=1$, for all $x\in X$.

Conversely, let (Δ_X,Δ'_X) be IFCS in (X,τ). Then (Δ'_X,Δ_X) is IFOS in X. To show that (X,τ) is Hausdorff choose any two distinct intuitionistic fuzzy points $x_{(\alpha,\beta)}$, $y_{(\gamma,\delta)}$ in X then $x\neq y$. Let max $(\alpha,\gamma)=\alpha$ and min $(\beta,\delta)=\beta$. Consider the intuitionistic fuzzy point $(x,y)_{(\alpha,\beta)}$ in $X\times X$. Then $(x,y)_{(\alpha,\beta)}\in(\Delta'_X,\Delta_X)$ and hence $\exists$ a basic IFOS $U\times V$ in $(X\times X,\tau\times\tau)$ such that $(x,y)_{(\alpha,\beta)}\in U\times V\subseteq(\Delta'_X,\Delta_X)$.

Now, $(x,y)_{(\alpha,\beta)}\in U\times V\Rightarrow\alpha<(\mu_U\times\mu_V)(x,y)=\inf(\mu_U(x),\mu_V(y))\leq\mu_U(x)$, and $\beta>(\nu_U*\nu_V)(x,y)=\sup(\nu_U(x),\nu_V(y))\geq\nu_U(x)$ implies that $x_{(\alpha,\beta)}\in U$, similarly $y_{(\alpha,\beta)}\in V$ which implies that $y_{(\gamma,\delta)}\in V$. Now we show that $U\cap V=0_\sim$. Since $U\times V\subseteq(\Delta'_X,\Delta_X)$, $(\mu_U\times\mu_V)(x,x)\leq\Delta'_X(x,x)=0\Rightarrow\inf(\mu_U(x),\mu_V(x))=0$, for all $x\in X$ and $(\nu_U\times\nu_V)(x,x)\geq\Delta_X(x,x)=1\Rightarrow\sup(\nu_U(x),\nu_V(x))=1$, for all $x\in X$. Thus $U\cap V=(\mu_U\cap\mu_V,\nu_U\cup\nu_V)=(\underline{0},\underline{1})=0_\sim$. □

Definition 37. A BFTS (X,τ_1,τ_2) is called q-T_2 if for any two distinct fuzzy points x_r and y_s there exists $U\in\tau_1$, $V\in\tau_2$ such that $x_r\in U$, $y_s\in V$ and $U\subseteq V'$.

Proposition 38. *An IFTS (X,τ) is q-T_2, and then its associated BFTS (X,τ_1,τ_2) is q-T_2.*

Proof. Let x_α and $y_{1-\delta}$ be any two distinct fuzzy points in X. Since (X,τ) is q-T_2 and $x_{(\alpha,\beta)}$, $y_{(\gamma,\delta)}$ are distinct intuitionistic fuzzy points in X, there exist $G_1=(\mu_{G_1},\nu_{G_1})$ and $G_2=(\mu_{G_2},\nu_{G_2})$ such that $x_{(\alpha,\beta)}\in G_1$, $y_{(\gamma,\delta)}\in G_2$ and $\mu_{G_1}\subseteq\mu'_{G_2}$, $\nu_{G_1}\supseteq\nu'_{G_2}$. $\Rightarrow\alpha<\mu_{G_1}(x)$ and $\delta>\nu_{G_2}(y)\Rightarrow x_\alpha\in\mu_{G_1}$ and $y_{1-\delta}\in(\underline{1}-\nu_{G_2})$. Thus for fuzzy points x_α and $y_{1-\delta}$ in X, $\exists\mu_{G_1}\in\tau_1$ and $(\underline{1}-\nu_{G_2})\in\tau_2$. Further, since $\mu_{G_1}+\nu_{G_1}\subseteq\underline{1}$, we have $\mu_{G_1}\subseteq(\underline{1}-\nu_{G_1})\subseteq\nu_{G_2}$ (since $\nu_{G_1}\supseteq\nu'_{G_2}$). Hence (X,τ_1,τ_2) is q-T_2. □

Theorem 39. *Let $\{(X_i,\tau_i):i\in J\}$ be a family of IFTs and (X,τ) be their product IFTS. Then (X,τ) is Hausdorff if and only if each coordinate space (X_i,τ_i) is Hausdorff.*

Proof. Let $x_{(\alpha,\beta)}$ and $y_{(\gamma,\delta)}$ be two distinct intuitionistic fuzzy points in X. Let $x=\Pi x_i$, $y=\Pi y_i$ then $x\neq y$ and hence $\exists k\in J$ such that $x_k\neq y_k$. Consider the distinct intuitionistic fuzzy points $(x_k)_{(\alpha,\beta)}$ and $(y_k)_{(\gamma,\delta)}$ in (X_k,τ_k). Since (X_k,τ_k) is Hausdorff, $\exists$ disjoint U_k,V_k in τ_k such that $(x_k)_{(\alpha,\beta)}\in U_k$, $(y_k)_{(\gamma,\delta)}\in V_k$ and $U_k\cap V_k=0_\sim$. Let $U_k=(\mu_{U_k},\nu_{U_k})$ and $V_k=(\mu_{V_k},\nu_{V_k})$, and then $\alpha<\mu_{U_k}(x_k)$, $\beta>\nu_{U_k}(x_k)$ and $\gamma<\mu_{V_k}(y_k)$, $\delta>\nu_{V_k}(y_k)$. Equivalently, we have, $\alpha<p_k^{-1}(\mu_{U_k})(x)$, $\beta>p_k^{-1}(\nu_{U_k})(x)$, that is, $x_{(\alpha,\beta)}\in p_k^{-1}(U_k)$ and $\gamma<p_k^{-1}(\mu_{V_k})(y)$, $\delta>p_k^{-1}(\nu_{V_k})(y)$, that is, $y_{(\gamma,\delta)}\in p_k^{-1}(V_k)$ and $p_k^{-1}(U_k)\bigcap p_k^{-1}(V_k)=0_\sim$, since $U_k\bigcap V_k=0_\sim$. □

Conversely, let (X,τ) be Hausdorff. To show that (X_j,τ_j) is Hausdorff, and consider two distinct intuitionistic fuzzy points $(x_j)_{(\alpha,\beta)}$ and $(y_j)_{(\gamma,\delta)}$ in X_j, then $x_j\neq y_j$. Now consider $x=\Pi x_i$ and $y=\Pi y_i$ in X where $x_i=y_i$ for $i\neq j$ and the jth coordinate of x,y are x_j and y_j, respectively. Since (X,τ) is Hausdorff, $\exists U,V\in\tau$ such that $x_{(\alpha,\beta)}\in U$, $y_{(\gamma,\delta)}\in V$ and $U\cap V=0_\sim$. Now consider the basic IFOSs ΠU_i and ΠV_i in X such that $x_{(\alpha,\beta)}\in\Pi U_i\subseteq U$ and $y_{(\gamma,\delta)}\in\Pi V_i\subseteq V$. Here the IFS $U_i=(\underline{1},\underline{0})$ except for finitely many i's say $i_1,i_2,\ldots,i_k$ and $V_i=(\underline{1},\underline{0})$ except for finitely many i's say $l_1,l_2,\ldots,l_{k_1}$.

Now, $x_{(\alpha,\beta)}\in\Pi U_i\subseteq U\Rightarrow\alpha<\inf\{\mu_{U_{i_1}}(x_{i_1}),\mu_{U_{i_2}}(x_{i_2}),\ldots,\mu_{U_{i_k}}(x_{i_k})\}$,

$$\beta>\sup\left\{\nu_{U_{i_1}}(x_{i_1}),\nu_{U_{i_2}}(x_{i_2}),\ldots,\nu_{U_{i_k}}(x_{i_k})\right\}$$
$$\Longrightarrow\alpha<\mu_{U_{i_m}}(x_{i_m}),\qquad\beta>\nu_{U_{i_m}}(x_{i_m}),\quad\forall m=1,2,\ldots,k. \tag{11}$$

Similarly,

$$y_{(\gamma,\delta)}\in\Pi V_i\subseteq V\Longrightarrow\gamma<\mu_{V_{l_n}}(y_{l_n}),\delta>\nu_{V_{l_n}}(y_{l_n}),\quad\forall n=1,2,\ldots,k_1. \tag{12}$$

We claim that $j\in\{i_1,i_2,\ldots,i_k\}$ and also $j\in\{l_1,l_2,\ldots,l_{k_1}\}$. If it is not so, then $x_{i_m}=y_{i_m}$ and $x_{l_n}=y_{l_n}$, for all $m=1,2,\ldots,k$ and for all $n=1,2,\ldots,k_1$ and hence from (11) and (12) we have

$$\begin{aligned}\alpha<\mu_{U_{i_m}}(x_{i_m}),\qquad&\beta>\nu_{U_{i_m}}(x_{i_m}),\quad\forall m=1,2,\ldots,k,\\ \gamma<\mu_{V_{l_n}}(x_{l_n}),\qquad&\delta>\nu_{V_{l_n}}(x_{l_n}),\quad\forall n=1,2,\ldots,k_1\end{aligned} \tag{13}$$

implying that $\Pi\mu_{U_i}(x)>0$, $\Pi\mu_{V_i}(x)>0$, therefore $(\Pi\mu_{U_i}\cap\Pi\mu_{V_i})\neq\underline{0}\Rightarrow\mu_U\cap\mu_V\neq\underline{0}$ which is a contradiction to the fact that $U\cap V=0_\sim$.

So, $\alpha<\mu_{U_j}(x_j)$, $\beta>\nu_{U_j}(x_j)$, $\gamma<\mu_{V_j}(y_j)$ and $\delta>\nu_{V_j}(y_j)$ in view of (11) and (12) showing that $(x_j)_{(\alpha,\beta)}\in U_j$, $(y_j)_{(\gamma,\delta)}\in V_j$.

Now it remains to show that $U_j\cap V_j=0_\sim$, that is, $\mu_{U_j}\cap\mu_{V_j}=\underline{0}$ and $\nu_{U_j}\cup\nu_{V_j}=\underline{1}$. Let $\exists z_{j_1}$ and z_{j_2} in X_j such that $(\mu_{U_j}\cap\mu_{V_j})(z_{j_1})>0$ and $(\nu_{U_j}\cup\nu_{V_j})(z_{j_2})<1$. Consider $z_1=\Pi z_i$ in X where $z_i=x_i(=y_i)$ for $i\neq j$ and $z_j=z_{j_1}$.

Now

$$\left(\mu_{U_j}\cap\mu_{V_j}\right)\left(z_{j_1}\right)>0\Longrightarrow\mu_{U_j}\left(z_{j_1}\right)>0,\quad\mu_{V_j}\left(z_{j_1}\right)>0. \tag{14}$$

In view of (11), (12), and (14) we have $(\Pi\mu_{U_i} \cap \Pi\mu_{V_i})(z_1) > 0$ implying that $U \cap V \neq 0_\sim$ which is a contradiction. Hence $\mu_{U_j} \cap \mu_{V_j} = \underline{0}$. Now let $z_2 = \Pi z_i$ where $z_i = x_i(= y_i)$ for $i \neq j$ and $z_j = z_{j_2}$. Since $(\nu_{U_j} \cup \nu_{V_j})(z_{j_2}) < 1$,

We have

$$\nu_{U_j}\left(z_{j_2}\right) < 1, \quad \nu_{V_j}\left(z_{j_2}\right) < 1. \tag{15}$$

Therefore from (11), (12), and (15) we have $(\Pi\nu_{U_j} \cup \Pi\nu_{V_j})(z_2) < 1$ implying that $U \cap V \neq 0_\sim$, again a contradiction. Hence (X_j, τ_j) is Hausdorff.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[3] D. Çoker, "An introduction to intuitionistic fuzzy topological spaces," *Fuzzy Sets and Systems*, vol. 88, no. 1, pp. 81–89, 1997.

[4] S. Bayhan and D. Coker, "On fuzzy separation axioms in intuitionistic fuzzy topological spaces," *Busefal*, vol. 67, pp. 77–87, 1996.

[5] F. G. Lupianez, "Hausdorffness in intuitionistic fuzzy topological spaces," *Mathware and Soft Computing*, vol. 10, pp. 17–22, 2003.

[6] R. Srivastava, S. N. Lal, and A. K. Srivastava, "Fuzzy Hausdorff topological spaces," *Journal of Mathematical Analysis and Applications*, vol. 81, no. 2, pp. 497–506, 1981.

[7] R. Srivastava, S. N. Lal, and A. K. Srivastava, "On fuzzy T_1—topological spaces," *Journal of Mathematical Analysis and Applications*, vol. 136, no. 1, pp. 124–130, 1988.

[8] C. K. Wong, "Fuzzy points and local properties of fuzzy topology," *Journal of Mathematical Analysis and Applications*, vol. 46, no. 2, pp. 316–328, 1974.

[9] D. Coker and M. Demirci, "On intuitionistic fuzzy points," *Notes on Intuitionistic Fuzzy Sets*, vol. 1, no. 2, pp. 79–84, 1995.

[10] S. J. Lee and E. P. Lee, "The category of intuitionistic fuzzy topological spaces," *Bulletin of the Korean Mathematical Society*, vol. 37, pp. 63–76, 2000.

[11] S. Bayhan and D. Coker, "Pairwise separation axioms in intuitionistic topological spaces," *Hacettepe Journal of Mathematics and Statistics*, vol. 34S, pp. 101–114, 2005.

[12] A. S. Abu Safiya, A. A. Fora, and M. W. Warner, "Fuzzy separation axioms and fuzzy continuity in fuzzy bitopological spaces," *Fuzzy Sets and Systems*, vol. 62, no. 3, pp. 367–373, 1994.

[13] R. Srivastava, S. N. Lal, and A. K. Srivastava, "On fuzzy T_0 and R_0 topological spaces," *Journal of Mathematical Analysis and Applications*, vol. 136, pp. 66–73, 1988.

[14] M. Srivastava and R. Srivastava, "On fuzzy pairwise—T_0 and fuzzy pairwise—T_1 bitopological spaces," *Indian Journal of Pure and Applied Mathematics*, vol. 32, no. 3, pp. 387–396, 2001.

[15] R. Srivastava and M. Srivastava, "On pairwise Hausdorff fuzzy bitopological spaces," *Journal of Fuzzy Mathematics*, vol. 5, pp. 553–564, 1997.

[16] M. A. Arbib and E. G. Manes, *Arrows, Structures and Functors*, Academic Press, New York, NY, USA, 1975.