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A fractional model of Harry Dym equation and its approximate solution

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Abstract The aim of present paper is to obtain the approximate analytical solution of time fractional Harry Dym equation by using homotopy perturbation method (HPM). The beauty of the paper is error analysis which shows that our approximate solution converges very rapidly to the exact solution and the numerical solution is compared with the known analytical solution which is nearly identical with the exact solution. The results show that the solution of HPM is good agreement with the exact solution. The fractional derivatives are described in the Caputo sense. The results reveal that the method is very effective and simple.
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1. Introduction

The Harry Dym equation is an important dynamical equation which is integrable and finds applications in several physical systems. The Harry Dym equation first appeared in Kruskal and Moser [1] and is attributed to an unpublished paper by Harry Dym in 1973–1974. The Dym equation represents a system in which dispersion and non-linearity are coupled together. Harry Dym is a completely integrable nonlinear evolution equation. It is interesting because it obeys an infinite number of conservation laws; it does not possess the Painleve property. The Harry Dym equation has strong links to the KdV equation and applications of this equation were found to the problems of hydrodynamics [2]. The Lax pair of the Harry Dym equation is associated with the Sturm–Liouville operator. The Liouville transformation transforms this operator spectrally into the Schrödinger operator [3]. The Harry Dym equation can be written as

$$\frac{\partial u(x,t)}{\partial t} = u^3(x,t)\frac{\partial^3 u(x,t)}{\partial x^3}. \tag{1}$$

The exact solution of the Harry Dym equation is $u(x,t) = \left(a - \frac{3\sqrt{b}}{2}(x+ct)\right)^{2/3}$ [4], where a and b are suitable constants.

Recently, many researchers studied the existence of solutions of the Harry Dym equation as exact solution of Harry

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Dym equation by Mokhtari [4]. A general formula of flow equations for Harry Dym Hierarchy by Peng et al. [5], algebraic geometric solution of the Harry Dym equation by Novikov [6], construction of coupled Harry Dym equation hierarchy by Marciniak and Blaszak [7], multi-Soliton solutions of (2 + 1)-dimensional Harry Dym equation by Dmitrieva and Khlabystova [8], explicit solution for Harry Dym equation by Fuchssteinert et al. [9], on Harry Dym equation and its solution by Ben-Yu and Roges [10], Solitons solutions of the (2 + 1) dimensional Harry Dym equation by Halim [11], scheme of constructing Solitons type solutions of the (2 + 1) dimensional Harry Dym equation by Dmitrieva and Khlabystova [12], an extended Harry Dym hierarchy by Ma [13], in this paper, an extended Harry Dym hierarchy is constructed by using eigen functions and adjoint eigen functions of the spectral problems of the Harry Dym hierarchy associated with the pseudo-differential operator. The corresponding Lax presentation possesses a self-consistent source involving squared eigen functions.

Mathematical modeling of many physical systems leads to linear and nonlinear fractional differential equations in various fields of physics and engineering. The use of fractional differentiation for the mathematical modeling of real world physical problems has been widespread in recent years, e.g. the modeling of earthquake, the fluid dynamic traffic model with fractional derivatives, measurement of viscoelastic material properties, etc. The book by Oldham and Spanier [14] has played a key role in the development of the subject. The fundamental results related with solution of fractional differential equations may be found in books [15–17].

It is significantly important in mathematical physics to search for exact solutions of nonlinear differential equations. Exact solutions play a vital role in understanding various qualitative and quantitative features of nonlinear phenomena. Exact solutions due to non-linearity present in dynamics of these physical problems are constructed by specific mathematical techniques. Among the existing theories Hirota- and bilinear technique provide a direct powerful approach [18] to nonlinear integrable equation. However, classes of Hirota-bilinear equation have also been successfully dealt by linear superposition principle [19]. A multiple exp-function method for exact multiple wave solutions of nonlinear partial differential equations is proposed by Ma et al. [20]. The main aim of this article is presents a mathematical model of nonlinear Harry Dym with fractional time derivative α $(0 < \alpha \leqslant 1)$ in the form of a rapidly convergent series with easily computable components. The homotopy perturbation method was proposed first by the Chinese researcher J.H. He in 1998 and was further developed and improved by him [21–24] and was successfully applied to solve fractional advection dispersion equation by Yildirim and Kocak [25], fractional Zakharov–Kuznetsov by Yildirim and Gulkanat [26], space and time fractional Fokker Planck equation by Yildirim [27], series solution of the Smoluchowski's coagulation equation by Yildirim and Kocak [28], generalized Berger and Bergers–Fisher equations by Rashidi et al. [29], two dimensional viscous flows in the extrusion process by Rashidi and Ganji [30], inversion of Abel integral equation by Kumar and Singh [31], analytical methods for solving the time fractional Swift–Hohenberg equation by Khan et al. [32]. The elegance of this article can be attributed to the simplistic approach in seeking the approximate analytical solution of the problem.

2. Basic definitions of fractional calculus

In this section, we give some basic definitions and properties of fractional calculus theory which shall be used in this paper:

Definition 2.1. A real function $f(t)$, $t > 0$ is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C(0,\infty)$ and it is said to be in the space C_n if and only if $f^{(n)} \in C_\mu$, $n \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator (J_t^α) of order α is defined as

$$\begin{aligned} &J_t^\alpha f(t) = \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau,\ (\alpha > 0,\ t > 0), \\ &J_t^\alpha f(t) = f(t). \end{aligned} \tag{2}$$

Where $f \in C_\mu$, $\mu \geqslant -1$, and $\Gamma(\cdot)$ is the Gamma function. Some of the properties of the operator (j_t^α), can be found in [13–16], we mention only the following. For $f \in C_\mu$, $\mu \geqslant -1$, $\alpha, \beta \geqslant 0$ and $\gamma \geqslant -1$:

(1) $J_t^\alpha J_t^\beta f(t) = J_t^{\alpha+\beta} f(t)$,
(2) $(J_t^\alpha J_t^\beta) f(t) = (J_t^\beta J_t^\alpha) f(t)$,
(3) $J_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}$.

It is remarkable here that Riemann–Liouville derivative has certain disadvantages when we try to model real world phenomena as fractional differential equation to overcome the problem, the fractional derivative in Caputo sense came into the picture which is defined as follows:

Definition 2.3. The fractional derivative D_t^α of $f(t)$ in Caputo sense defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \tag{3}$$

where $m-1 < \alpha \leqslant m,\ m \in N, t > 0,\ f \in C_{-1}^m$.

The following are two basic properties of the Caputo's fractional derivative:

Lemma 2.1. If $m-1 < \alpha \leqslant m,\ m \in N$ and $f \in C_\mu^n, \mu \geqslant -1$, then

$$\begin{aligned} &(D_t^\alpha J_t^\alpha) f(t) = f(t), \\ &(J_t^\alpha D_t^\alpha) f(t) = f(t) - \sum_{i=0}^{m-1} f^i(0^+)\tfrac{t^i}{i!}, \end{aligned} \tag{4}$$

3. Basic idea of homotopy perturbation method

To illustrate the basic ideas of the HPM for fractional differential equations, we consider the following problem:

$$D_{*t}^\alpha u(x,t) = v(x,t) - Lu(x,t) - Nu(x,t), \quad m-1 < \alpha \leqslant m, \quad m \in N, \quad t \geqslant 0, \quad x \in R^n \tag{5}$$

Subject to the initial and boundary conditions

$$u^{(i)}(0,0) = c_i, \quad B\left(u, \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial t}\right) = 0, \quad i = 0,1,2,\ldots,m-1, \quad j = 1,2,3,\ldots,n \tag{6}$$

where L is a linear operator, while N is a nonlinear operator, is a known analytical function and D^{α}_{*t} denotes the fractional derivative in the Caputo sense [15]. u is assumed to be a causal function of time, i.e., vanishing for $t < 0$. Also $u^{(i)}(x,t)$ is the ith derivative of u, c_i, $i = 0, 1, 2, \ldots, m-1$ are the specified initial conditions and B is a boundary operator.

We construct the following homotopy

$$(1-p)D^{\alpha}_{*t}u(x,t) + p\big(D^{\alpha}_{*t}u(x,t) + Lu(x,t) + Nu(x,t) - v(x,t)\big) = 0, \quad p \in [0,1] \tag{7}$$

which is equivalent to

$$D^{\alpha}_{*t}u(x,t) + p(Lu(x,t) + Nu(x,t) - v(x,t)) = 0, \quad p \in [0,1] \tag{8}$$

The homotopy parameter p always changes from zero to unity. In case $p = 0$, Eq. (8) becomes

$$D^{\alpha}_{*t}u(x,t) = 0, \tag{9}$$

when $p = 1$, Eq. (8) turns out to be the original fractional differential equation. The homotopy parameter p is used to expand the solution in the following form:

$$u(x,t) = u_0(x,t) + pu_1(x,t) + p^2u_2(x,t) + pu_3(x,t) + \cdots. \tag{10}$$

For nonlinear problems, we set $Nu(x,t) = S(x,t)$. Substituting Eq. (10) into Eq. (8) and equating the terms with identical power of p, we obtain a sequence of equations of the form

$$\begin{aligned} p^0 &: D^{\alpha}_{*t}u_0(x,t) = 0,\\ p^1 &: D^{\alpha}_{*t}u_1(x,t) = -Lu_0(x,t) - S_0(u_0(x,t)) + v(x,t),\\ p^2 &: D^{\alpha}_{*t}u_2(x,t) = -Lu_1(x,t) - S_1(u_0(x,t),\ u_1(x,t)),\\ p^j &: D^{\alpha}_{*t}u_j(x,t) = -Lu_{j-1}(x,t) - S_{j-1}(u_0(x,t),\\ &u_1(x,t),\ u_2(x,t), \ldots, u_{j-1}(x,t)), \quad j = 2,3,4,\ldots \end{aligned} \tag{11}$$

The functions $S_0, S_1, S_2, \ldots$ satisfy the following equation:

$$S\big(u_0(x,t) + pu_1(x,t) + p^2u_2(x,t) + p^3u_3(x,t) + \cdots\big) = S_0(u_0(x,t)) + pS_1(u_0(x,t),\ u_1(x,t)) + p^2S_2(u_0(x,t),\ u_1(x,t),\ u_2(x,t)) + \cdots. \tag{12}$$

Applying the inverse operator J^{α}_t on both sides of the Eq. (9) and considering the initial and boundary conditions, the various components of the series solution are given by

$$\begin{aligned} u_0(x,t) &= \sum_{i=0}^{n-1} c_i \frac{t^i}{i!},\\ u_1(x,t) &= -J^{\alpha}_t(Lu_0(x,t)) - J^{\alpha}_t S_0(u_0(x,t)) + J^{\alpha}_t(x,t),\\ u_j(x,t) &= -J^{\alpha}_t(Lu_{j-1}(x,t)) - J^{\alpha}_t S_{j-1}(u_0(x,t),\ u_1(x,t),\\ &u_2(x,t), \ldots, u_{j-1}(x,t)), \quad j = 2,3,4,\ldots. \end{aligned} \tag{13}$$

Hence, the HPM solution $u(x,t)$ is given by

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t). \tag{14}$$

4. Solution of the problem by HPM

We first consider the following time fractional Harry Dym equation

$$D^{\alpha}_t u(x,t) = u^3(x,t)D^3_x u(x,t), \tag{15}$$

with initial conditions $u(x,0) = \left(a - \frac{3\sqrt{b}}{2}x\right)^{2/3}$.

According to the HPM [17–20], we construct the following homotopy

$$(1-p)D^{\alpha}_t u + p(D^{\alpha}_t u - u^3D^3_x u) = 0, \quad 0 < \alpha \leqslant 1, \tag{16}$$

Substituting (10) into (16) and equating the coefficients of like powers of p, we get the following sets of differential equations:

$$p^0 : D^{\alpha}_t u_0(x,t) = 0,$$

$$p^1 : D^{\alpha}_t u_1(x,t) = u_0^3 D^3_x u_0,$$

$$p^2 : D^{\alpha}_t u_2(x,t) = u_0^3 D^3_x u_1 + 3u_0^2u_1D^3_x u_0,$$

$$p^3 : D^{\alpha}_t u_3(x,t) = u_0^3 D^3_x u_2 + 3u_0^2u_1D^3_x u_1 + (3u_0u_1^2 + 3u_0^2u_2)D^3_x u_0,$$

$$\begin{aligned} p^4 : D^{\alpha}_t u_4(x,t) &= u_0^3D^3_xu_3 + 3u_0^2u_1D^3_xu_2 + (3u_0u_1^2 + 3u_0^2u_2)D^3_xu_1 + (u_1^3 + 3u_0^2u_3\\ &+ 6u_0u_1u_2)D^3_xu_0, \end{aligned}$$

$$\begin{aligned} p^5 : D^{\alpha}_t u_5(x,t) &= u_0^3D^3_xu_4 + 3u_0^2u_1D^3_xu_3 + (3u_0u_1^2 + 3u_0^2u_2)D^3_xu_2 + (u_1^3 + 3u_0^2u_3\\ &+ 6u_0u_1u_2)D^3_xu_1 + (3u_0u_2^2 + 3u_0^2u_4 + 3u_1^2u_2\\ &+ 6u_0u_1u_3)D^3_xu_0, \end{aligned}$$

The above system of nonlinear equations can be easily solved by applying the operator J^{α}_t to obtain the various components $u_n(x,t)$, thus enabling the series solution to be entirely determined. The first few components of the homotopy perturbation solutions for the Eq. (10) are given as follows:

$$u_0(x,t) = u_0(x,0) = \left(a - \frac{3\sqrt{b}}{2}x\right)^{2/3},$$

$$u_1(x,t) = -b^{3/2}\left((a - \frac{3\sqrt{b}}{2}x\right)^{-1/3}\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = -\frac{b^3}{2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-4/3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,t) = b^{9/2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-7/3}\left(\frac{15\Gamma(2\alpha+1)}{2(\Gamma(\alpha+1))^2} - 16\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

In this manner the rest of components of the homotopy perturbation solution can be obtained. Thus the solution $u(x,t)$ of the Eq. (15) is given as

$$u(x,t) = \lim_{N\to\infty}\sum_{n=0}^{N} u_n(x,t). \tag{17}$$

The series solution converges very rapidly. The rapid convergence means only few terms are required to get analytic function.

Fig. 1 shows the exact solution $u(x,t)$ of the Harry Dym equation given by Mokhtari [4] for constants value of $a = 4$ and $b = 1$. Fig. 2 shows the approximate solution for the standard Harry Dym equation i.e. for $\alpha = 1$ by HPM. It can be seen from Fig. 2 that the solution obtained by the present method is nearly identical to the exact solution with high accuracy.

5. Numerical result and discussion

In this section, we have discussed the error analysis between the exact solution and approximate solution which is depicted through Fig. 3 with high accuracy.

The simplicity and accuracy of the proposed method is illustrated by computing the absolute error $E(u_6) = |u(x,t) - \tilde{u}(x,t)|$ at the constants value of $a = 4$ and $b = 1$, where $u(x, t)$ and $\tilde{u}(x,t)$ are exact and approximate solutions of (1) respectively. Fig. 3 shows the absolute error between the exact and approximate solution at level $N = 6$, which is significantly small thus indicates the convergence of series solution very rapidly. Here, during the all numerical computation only six order term of the series solution is considered. The accuracy of the result can be improved by introducing more terms of the approximate solutions. It achieves a high level of accuracy in only six order term of approximations. The behavior of the approximate solutions depicted through graphically.

Fig. 4 shows the behavior of the approximate solution $\tilde{u}_6(x,t)$ for different value of $\alpha = 0.97, 0.98, 0.99$ and for standard Harry Dym equation i.e. $\alpha = 1$ at the value of $t = 1$. It is seen from Fig. 4 that the $u(x, t)$ decreases very rapidly with the increases in t.

6. Conclusion

In this paper, the homotopy perturbation method is applied to obtain approximate solution of the time fractional Harry Dym equation. In HPM, a homotopy with an embedding parameter $p \in [0, 1]$ is constructed, and the embedding parameter is considered as a "small parameter", which can take full advantages of the traditional perturbation methods and homotopy techniques. This method contains the homotopy parameter p, which provides us with a simple way to control the convergence region of solution series for large values of t. The obtained results demonstrate the reliability of the algorithm and its wider applicability to nonlinear fractional partial differential equations.

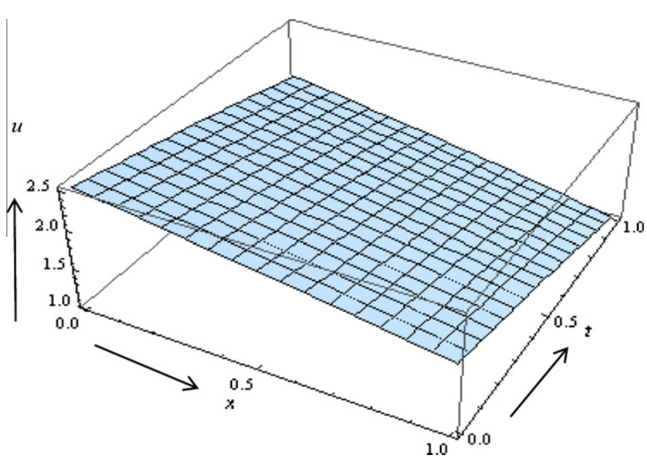

Figure 1 The exact solution $u(x, t)$ for $\alpha = 1$.

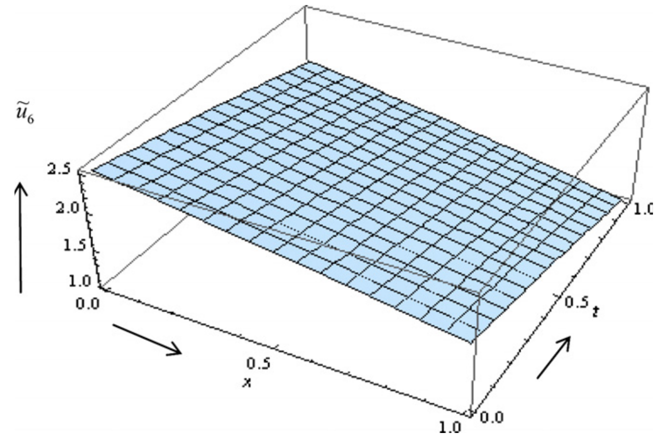

Figure 2 The approximate solution $\tilde{u}_6(x,t)$ for $\alpha = 1$.

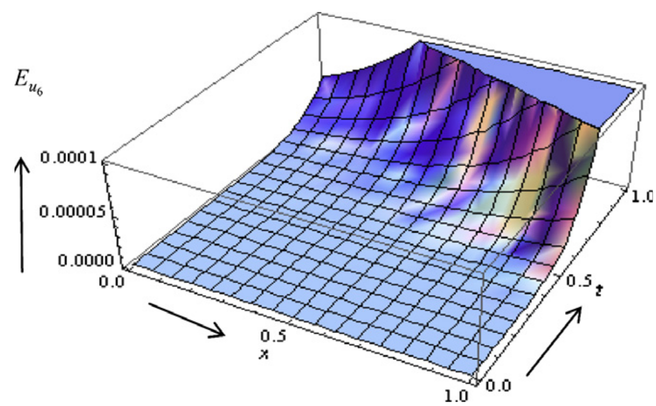

Figure 3 The absolute error $E_{u_6}(x,t)$.

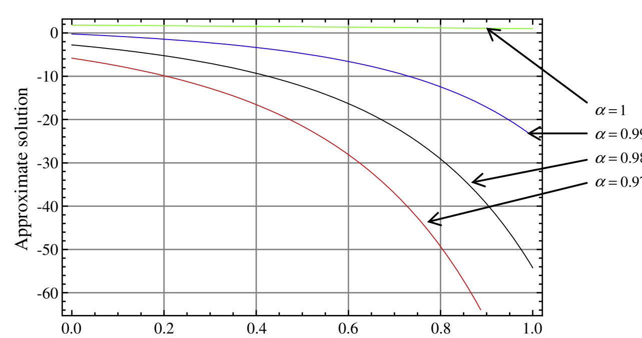

Figure 4 The $\tilde{u}_6(x,t)$ for different value of α at $\beta = 1$.

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References

[1] Kruskal MD, Moser J. Dynamical systems, theory and applications, lecturer notes physics. Berlin: Springer; 1975. p. 310.

[2] Vosconcelos GL, Kaclanoff LP. Stationary solution for the Saffman Taylor problem with surface tension. Phys Rev A 1991;44:6490–5.

[3] Gesztesy F, Unterkofler K. Isospectral deformations for Strumi Liuville and Dirac-type operators and associated nonlinear evolution equations. Rep Math Phys 1992;31:13–137.

[4] Mokhtari R. Exact solutions of the Harry-Dym equation. Commun Theor Phys 2011;55:204–8.

[5] Peng CJ, Jing-Song HE, Hong WL. A general formula of flow equations for Harry-Dym hierarchy. Commun Theor Phys 2011;55:193–8.

[6] Novikov DP. Algebraic geometric solution of the Harry Dym equation. Siberian Math J 1999;40(1):136–40.

[7] Marciniak K, Blaszak M. Construction of coupled Harry Dym equation hierarchy and its solutions from Stackel system. Nonlinear Anal: Theory Method Appl 2010;73:3004–17.

[8] Dmitrieva L, Khlabystova M. Multi-Solitons solutions of (2 + 1) – dimensional Harry Dym equation. Phys Lett A 1998;237:369–80.

[9] Fuchssteinert B, Schulzet T, Carllot S. Explicit solutions for Harry Dym equation. J Phys A: Math Gen 1992;25:223–30.

[10] Ben-Yu G, Roges C. On Harry Dym equation and its solution. Science 1989;32(3):283–95 (China).

[11] Halim AA. Soliton solutions of the (2 + 1) dimensional Harry Dym equation via Darboux transformation. Chaos Solitons Fract 2008;36:646–53.

[12] Dmitrieva L, Khlabystova M. Scheme of constructing Solitons type solutions of the (2 + 1) dimensional Harry Dym equation. Comput Math Appl 1991;34:75–93.

[13] Ma WX. An extended Harry Dym hierarchy. J Phys A: Math Theor 2010;43:165202.

[14] Oldham KB, Spanier J. The fractional calculus. New York: Academic Press; 1974.

[15] Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. New York: John Willey and Sons, Inc.; 2003.

[16] Podlubny I. Fractional differential equations. New York: Academic Press; 1999.

[17] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and application of fractional differential equations. Amsterdam: Elsevier; 2006.

[18] Hirota R. The direct method in Soliton theory. Cambridge University Press; 2004.

[19] Ma WX, Fan E. Linear superposition principle applying to Hirota bilinear equations. Comput Math Appl 2011;61:950–9.

[20] Ma WX, Huang T, Zhang Y. A multiple exp-function method for nonlinear differential equations and its application. Phys Scr 2010;82:065003.

[21] He JH. Homotopy perturbation technique. Comput Method Appl Mech Eng 1999;178:257–62.

[22] He JH. A coupling of method of homotopy technique and perturbation technique for nonlinear problems. Int J Nonlinear Mech 2000;35(1):37–43.

[23] He JH. Comparison of homotopy perturbation method and homotopy analysis method. Appl Math Comput 2004;156:527–39.

[24] He JH. Homotopy perturbation method for solving boundary value problems. Phys Lett A 2006;350:87–8.

[25] Yildirim A, Kocak H. Homotopy perturbation method for solving the space–time fractional advection–dispersion equation. Adv Water Resour 2009;32:1711–6.

[26] Yildirim A, Gulkanat Y. Analytical approach to fractional Zakharov–Kuznetsov equations by He's homotopy perturbation method. Commun Theor Phys 2010;53:1005.

[27] Yildirim A. Analytical approach to Fokker–Planck equation with space and time fractional derivatives by means of the homotopy perturbation method. J King Saud Univ 2010;22:257–64.

[28] Yildirim A, Kocak H. Series solution of the Smoluchowski's coagulation equation. J King Saud Univ 2011;23:183–9.

[29] Rashidi MM, Ganji DD, Dinarvand S. Explicit analytical solutions of the generalized burger and Burger–Fisher equations by homotopy perturbation method. Numer Method Part Diff Equat 2009;25(2):409–17.

[30] Rashidi MM, Ganji DD. Homotopy perturbation combined with Padé approximation for solving two dimensional viscous flows in the extrusion process. Int J Nonlinear Sci 2009;7(4):387–94.

[31] Kumar S, Singh Om P. Numerical inversion of Abel integral equation by homotopy perturbation method. Z Naturforsch 2010;65a:677–82.

[32] Khan NA, Khan NU, Ayaz M, Mahmood A. Analytical methods for solving the time-fractional Swift–Hohenberg (S–H) equation. Comput Math Appl 2011;61:2182–5.

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