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Analytical solution of a fractional diffusion equation by variational iteration method

S. Das

Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi – 221 005, India

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ABSTRACT

In the present paper the Analytical approximate solution of a fractional diffusion equation is deduced with the help of powerful Variational Iteration method. By using an initial value, the explicit solutions of the equation for different cases have been derived, which accelerate the rapid convergence of the series solution. The present method performs extremely well in terms of efficiency and simplicity. Numerical results for different particular cases of the problem are presented graphically.

1. Introduction

The fractional calculus has a tremendous use in basic sciences and engineering. Oldham and Spanier [1] have played a key role in the development of the subject. Several fundamental works solving fractional differential equations have been undertaken by Miller and Rose [2], Podlubry [3], Diethelm and Ford [4], Diethelm [5] etc. Recent applications have included solving various classes of nonlinear fractional differential equations numerically. Variational Iteration Method is one of the powerful methods by which the exact and appropriate analytical solutions for nonlinear equations can be obtained. The variational iteration method was first proposed by He [6–10] and was successfully applied to solve nonlinear systems of PDE's and nonlinear differential equations of fractional order by Shawagfeh [11], Diethelm and Ford [12], Momani and Odibat [13] etc.

The analytical fractional diffusion equation in time is governed by the equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial}{\partial x}(F(x)u(x,t)), \quad 0 < \alpha \le 1, D > 0 \tag{1}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}(\cdot)$ is the Caputo derivative of order α, $u(x,t)$ represents the probability density function of finding a particle at the x in the time t, the positive constant D depends on the temperature, the friction coefficient, the universal gas constant and finally on the Avagadro number, $F(x)$ is the external force. In the present paper, it is considered that $D = 1$, $\alpha = \frac{1}{2}$ and $F(x) = -x$. This type of problem was solved by Saha Ray and Bera [14] by using Adomian Decomposition Method. The main disadvantage of Adomian method, is that the solution procedure for calculation of Adomian polynomials is complex and difficult, as pointed out by many researchers. In this paper, the Variation iteration method is used to overcome the demerit of the Adomian method. Using the initial condition, the analytical expression of $u(x,t)$ for various values of x and t for different particular cases are derived and presented through graphs. The elegance of this method can be attributed to its simplistic approach in seeking the analytic solution of the problem.

E-mail address: subir_das08@hotmail.com.

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2. Solution of the problem

We consider the equation

$$\frac{\partial^{1/2} u(x,t)}{\partial t^{1/2}} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\left(xu(x,t)\right) \tag{2}$$

with initial condition $u(x,0) = f(x)$.

Eq. (1) can be written as

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{1/2}}{\partial t^{1/2}} \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial^{1/2}}{\partial t^{1/2}} \frac{\partial}{\partial x}\left(xu(x,t)\right). \tag{3}$$

According to the variational iteration method, we consider the correction functional in t-direction in the following form

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi) \left[\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} - \frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{\partial}{\partial x}\left(x\tilde{u}_n(x,\xi)\right) \right] \mathrm{d}\xi. \tag{4}$$

It is obvious that the successive approximation $u_j, j \geq 0$ can be established by determining Lagrange multiplier λ. The function $\tilde{u}_n$ is a restricted variation, which means $\delta\tilde{u}_n = 0$. The successive approximation $u_{n+1}(x,t), n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using Lagrange's multiplier, and by using any selective function u_0. The initial value $u(x,0)$ and $u_t(x,0)$ are usually used for selecting the zeroth approximation u_0. To find the optimal value of λ, we have

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(\xi) \frac{\partial u_n(x,\xi)}{\partial \xi} \mathrm{d}\xi = 0. \tag{5}$$

This yields the stationary condition

$$\lambda'(\xi) = 0 \tag{6}$$

$$\text{and} \quad 1 + \lambda(\xi) = 0 \tag{7}$$

$$\text{which gives} \quad \lambda = -1. \tag{8}$$

Substituting this value of Lagrangian multiplies in the Eq. (4),we get the following iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \left[\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - \frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{\partial}{\partial x}\left(xu_n(x,\xi)\right) \right] \mathrm{d}\xi. \tag{9}$$

Beginning with an initial approximation $u_0(x,t) = u(x,0) = f(x)$, we obtain the following successive approximations

$$\begin{aligned} u_1(x,t) &= u_0(x,t) - \int_0^t \left[\frac{\partial u_0(x,\xi)}{\partial \xi} - \frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{\partial^2 u_0(x,\xi)}{\partial x^2} - \frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{\partial}{\partial x}\left(xu_0(x,\xi)\right) \right] \mathrm{d}\xi \\ &= f(x) + \int_0^t \left[\frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{\partial^2 f(x)}{\partial x^2} + \frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{\partial}{\partial x}\left(xf(x)\right) \right] \mathrm{d}\xi \end{aligned}$$

$$u_2(x,t) = u_1(x,t) - \int_0^t \left[\frac{\partial u_1(x,\xi)}{\partial \xi} - \frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{\partial^2 u_1(x,\xi)}{\partial x^2} - \frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{\partial}{\partial x}\left(xu_1(x,\xi)\right) \right] \mathrm{d}\xi$$

and so on.

Finally the exact solution is obtained by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{10}$$

In other words, the correction functional (4) will give several approximations and, therefore, the exact solution is obtained at the limit of the resulting successive approximations.

3. Illustrative examples

Example 1. Let us consider $f(x) = 1$, then

$$u_0(x,t) = 1$$

$$u_1(x,t) = 1 + \frac{2\sqrt{t}}{\sqrt{\pi}}$$

$$u_2(x,t) = 1 + \frac{\sqrt{t}}{\sqrt{\pi}} + t$$

$$u_3(x,t) = 1 + \frac{2\sqrt{t}}{\sqrt{\pi}} + t + \frac{4t^{3/2}}{3\sqrt{\pi}}.$$

Thus,

$$u_n(x,t) = 1 + \frac{2\sqrt{t}}{\sqrt{\pi}} + t + \frac{4t^{3/2}}{3\sqrt{\pi}} + \cdots.$$

The exact solution is

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\ &= 1 + \frac{t^{1/2}}{\Gamma(3/2)} + \frac{t}{\Gamma(2)} + \frac{t^{3/2}}{\Gamma(5/2)} + \cdots \\ &= \sum_{r=0}^{\infty} \frac{t^{r/2}}{\Gamma(r/2+1)} \\ &= E_{1/2}(\sqrt{t}) \end{aligned} \tag{11}$$

where $E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}$ $(\alpha > 0)$ is the Mittag–Leffler function in one parameter.

Example 2. Let us consider $f(x) = x$, we then obtain

$$\begin{aligned} u_0(x,t) &= x \\ u_1(x,t) &= x + \frac{4x\sqrt{t}}{\sqrt{\pi}} \\ u_2(x,t) &= x + \frac{4x\sqrt{t}}{\sqrt{\pi}} + 4xt \\ u_3(x,t) &= x + \frac{4x\sqrt{t}}{\sqrt{\pi}} + 4xt + \frac{32xt^{3/2}}{3\sqrt{\pi}} \\ u_4(x,t) &= x + \frac{4x\sqrt{t}}{\sqrt{\pi}} + 4xt + \frac{32xt^{3/2}}{3\sqrt{\pi}} + 8xt^2. \end{aligned}$$

Therefore,

$$u_n(x,t) = x + \frac{4x\sqrt{t}}{\sqrt{\pi}} + 4xt + \frac{32xt^{3/2}}{3\sqrt{\pi}} + 8xt^2 + \cdots.$$

The exact solution is

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\ &= x + \frac{2xt^{1/2}}{\Gamma(3/2)} + \frac{2^2xt}{\Gamma(2)} + \frac{2^3xt^{3/2}}{\Gamma(5/2)} + \frac{2^4xt^2}{\Gamma(3)} + \cdots \\ &= \sum_{r=0}^{\infty} \frac{2^r xt^{r/2}}{\Gamma(r/2+1)} \\ &= xE_{1/2}(2\sqrt{t}). \end{aligned} \tag{12}$$

The above result is in complete agreement with Saha Ray and Bera [14].

Example 3. Now consider $f(x) = x^2$, then

$$\begin{aligned} u_0(x,t) &= x^2 \\ u_1(x,t) &= x^2 + \frac{2(2+3x^2)\sqrt{t}}{\sqrt{\pi}} \\ u_2(x,t) &= x^2 + \frac{2(2+3x^2)\sqrt{t}}{\sqrt{\pi}} + (8+9x^2)t \\ u_3(x,t) &= x^2 + \frac{2(2+3x^2)\sqrt{t}}{\sqrt{\pi}} + (8+9x^2)t + \frac{4(26+27x^2)t^{3/2}}{3\sqrt{\pi}}. \end{aligned}$$

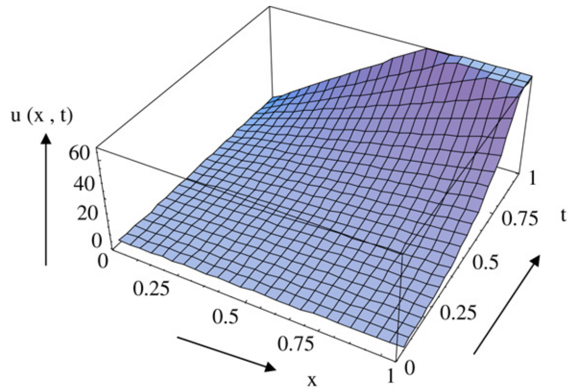

Fig. 1. Three dimensional figure for $u(x, t)$ with respect to x and time t.

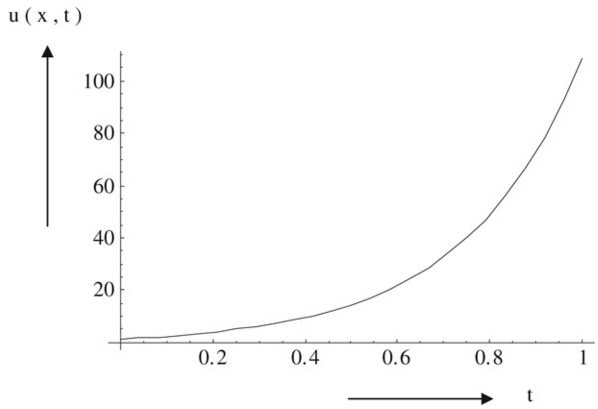

Fig. 2. Plot of $u(x, t)$ vs. time t when $x = 1$.

And finally the exact solution is

$$\begin{aligned}
u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\
&= x^2 + \frac{2(2+3x^2)\sqrt{t}}{\sqrt{\pi}} + (8+9x^2)t + \frac{4(26+27x^2)t^{3/2}}{3\sqrt{\pi}} + \cdots \\
&= x^2 + \frac{(2+3x^2)t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + \frac{(8+9x^2)t}{\Gamma(2)} + \frac{(26+27x^2)t^{3/2}}{\Gamma(\frac{5}{2})} + \cdots \\
&= \sum_{r=0}^{\infty} \frac{k^r t^{r/2}}{\Gamma(r/2+1)}, \quad \text{where } k^r = x^2 + (1+x^2)(3^r - 1) \\
&= E_{1/2}(k\sqrt{t}).
\end{aligned} \tag{13}$$

4. Numerical results and discussion

In this section, numerical results of the displacement $u(x, t)$ for various values of x and t for Examples 2 and 3 are made, which are presented through Figs. 1–4.

Figs. 1 and 2 respectively represent three a dimensional figure for $u(x, t)$ w.r.t x & t and a two dimensional figure for $u(x, t)$ for different values of t at $x = 1$ for Example 2. Figs. 3 and 4 are those for Example 3. It is seen from the figures that, in both the cases, $u(x, t)$ increases with the increase of x and t. But the increase of $u(x, t)$ is much higher for Example 3 in comparison with that of Example 2. This implies that if the degree of the polynomial, which mathematically expresses the initial condition, increases, then the increase of the values of the $u(x, t)$ w.r.t both x & t becomes higher. Here, all the computations and Figures are made using the Mathematica (Version 5.2) Software.

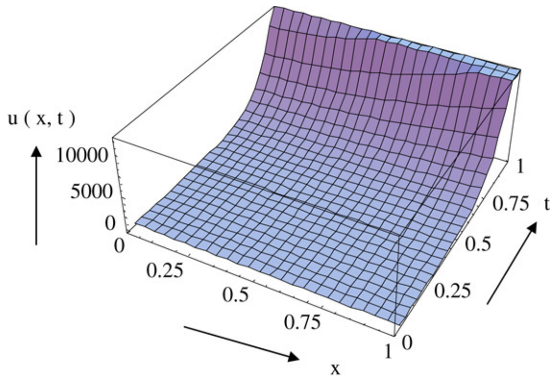

Fig. 3. Three dimensional figure for $u(x, t)$ with respect to x and time t.

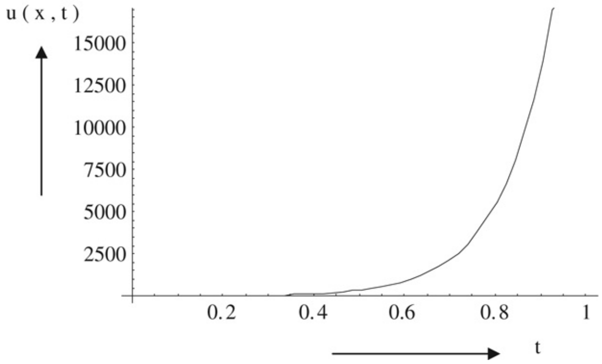

Fig. 4. Plot of $u(x, t)$ vs. time t when $x = 1$.

5. Conclusion

By using variational iteration method we obtain the analytical solution of the fractional diffusion equation. This technique is very powerful in finding solutions for various physical problems. Showing its application for finding solutions in fractional diffusion equation, we may conclude that this method will be very useful for solving many Engineering problems, both analytically and numerically.

The main advantage of the method is its fast convergence to the solution. The numerical results obtained here, conform to its high degree of accuracy. Moreover, it avoids the volume of calculations required by the Adomian polynomials, for finding the solution by Adomian decomposition method (Saha Ray and Bera [14]).

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