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Efficient algorithms to solve singular integral equations of Abel type

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ABSTRACT

In the present paper, we obtain the approximate solution of Abel's integral equation by using the following powerful, efficient but simple methods:

(i) He's homotopy perturbation method (HPM),
(ii) Modified homotopy perturbation method (MHPM),
(iii) Adomian decomposition method (ADM) and
(iv) Modified Adomian decomposition method (MADM).

The validity and applicability of these techniques are illustrated through various particular cases which demonstrate their efficiency and simplicity in solving these types of integral equations compared with the other existing methods.

1. Introduction

The real world problems in scientific fields such as solid state physics, plasma physics, fluid mechanics, chemical kinetics and mathematical biology are nonlinear in general when formulated as partial differential equations or integral equations.

In the last two decades, many powerful and simple methods have been proposed and applied successfully to approximate various types of singular integral equations with a wide range of applications [1–19]. In this paper, we discuss the three different methods namely, He's homotopy method (HPM), Adomian decomposition method (ADM) and modified Adomian decomposition method (MADM) proposed by He [1–6], Adomian [7–12] and Wazwaz [13–16] respectively and apply these to solve singular Volterra integral equations with Abel's kernel.

Abel studied a particular integral equation of the Volterra type, in order to solve the following problem.

Let a material point of mass m move under the influence of gravity on a smooth curve lying in a vertical plane. Let the time t which is required for the point to move along the curve from the height x to the lowest point of the curve be a given function f of x. The answer to the question, "what is the equation of the curve?" leads to the integral equation

$$f(x) = \int_0^x \frac{\varphi(t)}{\sqrt{2g(x-t)}}\,\mathrm{d}t, \tag{1}$$

where g is the acceleration due to the gravity.

Abel's equation is one of the integral equations derived directly from a concrete problem of physics, without passing through a differential equation. The generalized Abel's integral equation on a finite segment was studied by Zeilon [20].

2. Homotopy perturbation method and its modification

In this method, using the homotopy technique of topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$ which is considered as a "small parameter". This method became very popular amongst the scientists and engineers,

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even though it involves continuous deformation of a simple problem into a more difficult problem under consideration. Most of the perturbation methods depend on the existence of a small perturbation parameter but many nonlinear problems have no small perturbation parameter at all. Many new methods have been proposed in the late nineties to solve such nonlinear equations devoid of such small parameters [21–24]. Late 1990s saw a surge in applications of homotopy theory in the scientific and engineering computations [1,2,25,26]. When the homotopy theory is coupled with perturbation theory it provides a powerful mathematical tool [27–29]. A review of recently developed methods of nonlinear analysis can be found in [30]. To illustrate the basic concept of HPM, consider the following nonlinear functional equation

$$A(u) = f(r), r \in \Omega, \quad \text{with the boundary conditions} : B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \partial\Omega, \tag{2}$$

where A is a general functional operator, B is a boundary operator, $f(r)$ is a known analytic function, and $\partial\Omega$ is the boundary of the domain Ω. The operator A is decomposed as $A = L + N$, where L is the linear and N is the nonlinear operator. Hence Eq. (2) can be written as

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega.$$

We construct a homotopy $v(r, p) : \Omega \times [0, 1] \to R$ satisfying

$$H(v, p) = (1 - p)\,[L(v) - L(u_0)] + p\,[A(v) - f(r)] = 0, \quad p \in [0, 1], \; r \in \Omega. \tag{3}$$

Hence,

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p\,[N(v) - f(r)] = 0, \tag{4}$$

where u_0 is an initial approximation for the solution of Eq. (2). As

$$H(v, 0) = L(v) - L(u_0) \quad \text{and} \quad H(v, 1) = A(v) - f(r),$$

it shows that $H(v, p)$ continuously traces an implicitly defined curve from a starting point $H(u_0, 0)$ to a solution $H(v, 1)$. The embedding parameter p increases monotonously from zero to one as the trivial linear part $L(u) = 0$ deforms continuously to the original problem $A(u) = f(r)$. The embedding parameter $p \in [0, 1]$ can be considered as an expanding parameter [1] to obtain

$$v = v_0 + pv_1 + p^2 v_2 + \cdots. \tag{5}$$

The solution is obtained by taking the limit as p tends to 1, in Eq. (5). Hence

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{6}$$

The series (6) converges for most cases and the rate of convergence depends on $A(u) - f(r)$ [1].

We consider the following singular Volterra integral equation of the second kind

$$y(x) = f(x) + \int_0^x \frac{y(t)}{\sqrt{x - t}}\,\mathrm{d}t, \quad 0 \le x \le 1. \tag{7}$$

To solve Eq. (7) by He's HPM, we consider the following convex homotopy:

$$(1 - p)\,[L(x) - y_0(x)] + p\left[L(x) - f(x) - \int_0^x \frac{L(t)}{\sqrt{(x - t)}}\,\mathrm{d}t\right] = 0. \tag{8}$$

We seek the solution of (8) in the following form,

$$L(x) = \sum_{i=0}^{\infty} p^i L_i(x), \tag{9}$$

where $L_i(x)$, $i = 0, 1, 2, \ldots$ are the functions to be determined. We use the following iterative scheme to evaluate $L_i(x)$.

The initial approximation to the solution $L_0(x) = y_0(x)$ is taken to be $f(x)$, therefore,

$$L_0(x) = y_0(x) = f(x).$$

Substituting (9) into (8) and equating the coefficients of p with the same power, one gets

$$p^0 \colon L_0(x) = f(x)$$
$$p^1 \colon L_1(x) - \int_0^x \frac{L_0(t)}{\sqrt{(x - t)}}\,\mathrm{d}t = 0 \Rightarrow L_1(x) = \int_0^x \frac{f(t)}{\sqrt{(x - t)}}\,dt$$
$$p^2 \colon L_2(x) - \int_0^x \frac{L_1(t)}{\sqrt{(x - t)}}\,\mathrm{d}t = 0 \Rightarrow L_2(x) = \int_0^x \frac{L_1(t)}{\sqrt{(x - t)}}\,\mathrm{d}t$$
$$p^3 \colon L_3(x) - \int_0^x \frac{L_2(t)}{\sqrt{(x - t)}}\,\mathrm{d}t = 0 \Rightarrow L_3(x) = \int_0^x \frac{L_2(t)}{\sqrt{(x - t)}}\,\mathrm{d}t, \ldots.$$

Hence, the solution of Eq. (7) is given by,

$$y(x) = \lim_{p\to 1} L(x) = \sum_{i=0}^{\infty} L_i(x). \tag{10}$$

In the modified homotopy perturbation method (MHPM), we break $f(x)$ into an infinite sum as follows

$$f(x) = \sum_{i=0}^{\infty} k_i(x), \quad \text{and define}$$

$$\psi(x; p) = \sum_{i=1}^{\infty} p^i k_i(x) \to f(x) \quad \text{as } p \to 1. \tag{11}$$

The initial approximation to the solution is taken to be $k_0(x)$. Substituting (9) and (11) into (8) and equating coefficients of p with the same power one gets the exact solution.

It is to be noted that the rate of convergence of the series (10) depends upon the initial choice $y_0(x)$ as illustrated by the given numerical examples.

3. The Adomian decomposition method and its modification

The Adomian decomposition method has been applied to a wide class of functional equations [7–12,31] by scientists and engineers since the beginning of the 1980s. Adomian gives the solution as an infinite series usually converging to a solution. Consider the following singular Volterra integral equation of the second kind of the form

$$y(x) = f(x) + \int_0^x k(x,t)y(t)\,dt, \quad f(x) \in L^2(R). \tag{12}$$

The ADM assumes an infinite series solution for the unknown function $y(x)$, given by

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{13}$$

Substituting (13) into (12), we get

$$\sum_n y_n(x) = f(x) + \int_0^x k(x,t) \sum_n y_n(t)\,dt. \tag{14}$$

The ADM uses the following recursive relation to evaluate the various iterates $y_1, y_2, y_3, \ldots$ in (13)

$$y_0(x) = f(x), \qquad y_{n+1}(x) = \int_0^x k(x,t)y_n(t)\,dt, \quad n \geq 0. \tag{15}$$

Recently, Wazwaz [15] proposed a modification in ADM by constructing the zeroth component $y_0(x)$ of the decomposition in a slightly different way. He splitted the function $f(x)$ as the sum of two functions $f_1(x)$ and $f_2(x)$ in $L^2(R)$ and suggested the following recursive scheme:

$$y_0(x) = f_1(x), \quad y_1(x) = f_2(x) + \int_0^x k(x,t)y_0(t)\,dt, \quad \text{and}$$

$$y_{n+1}(x) = \int_0^x k(x,t)y_n(t)\,dt, \quad n \geq 1. \tag{16}$$

This type of modification provides more flexibility to the ADM in solving complicated integral equations and avoids the unnecessary complexity in calculating the Adomian polynomials. In this case, the decomposition series (14) has a rapid rate of convergence in real physical problems. The rapid convergence ensures that only a few iterations are required to get the accurate solution of the problem.

In this paper, we assume the kernel $k(x, t)$ to be Abel's kernel i.e.

$$k(x,t) = \frac{1}{\sqrt{(x-t)}}, \quad \text{and} \quad 0 \leq x \leq 1.$$

4. The application of HPM, MHPM, ADM and MADM

Example 4.1. Consider the singular Volterra integral equation [32]

$$y(x) = x^2 + \frac{16}{15}x^{5/2} - \int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt, \quad 0 \leq x \leq 1, \tag{17}$$

with x^2 as the exact solution.

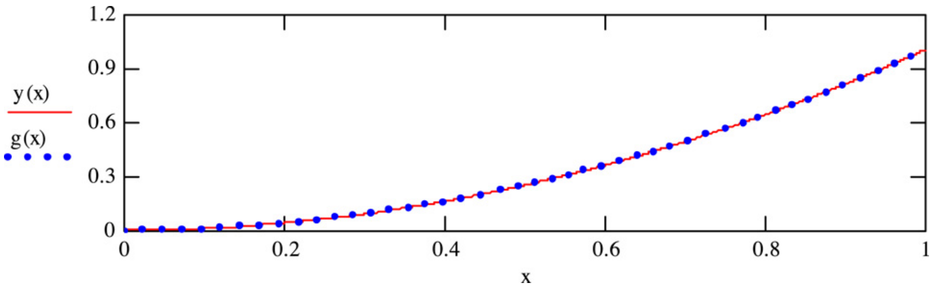

Fig. 1. The exact and the approximate solutions of the singular Volterra integral equation (17), in Example 4.1, case 1(a).

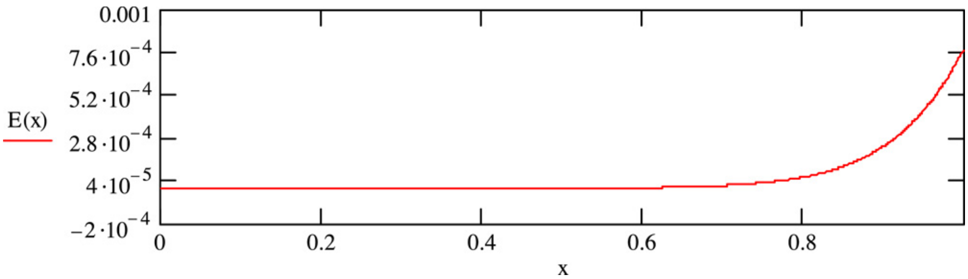

Fig. 2. The error $E(x)$ for Example 4.1, case 1(a).

Case 1(a): *Homotopy perturbation method*

A homotopy perturbation method can be constructed as follows (from Eq. (8)):

$$H(y,p)=y(x)-x^2-\frac{16}{15}x^{5/2}+p\int_0^x\frac{y(t)}{\sqrt{x-t}}\,\mathrm{d}t. \tag{18}$$

One can now try to obtain a solution of Eq. (16) in the form of

$$L(x)=L_0(x)+pL_1(x)+p^2L_2(x)+\cdots \tag{19}$$

where $L_i(x), i=0,1,2,\ldots$ are functions yet to be determined. From Eqs. (18) and (19) the approximations are:

$$p^0: L_0(x)=x^2+\frac{16}{15}x^{5/2}$$

$$p^1: L_1(x)+\int_0^x\frac{L_0(t)}{\sqrt{(x-t)}}\,\mathrm{d}t=0\Rightarrow L_1(x)=-\frac{16}{15}x^{5/2}-\frac{\pi x^3}{3}$$

$$p^2: L_2(x)+\int_0^x\frac{L_1(t)}{\sqrt{(x-t)}}\,\mathrm{d}t=0\Rightarrow L_2(x)=\frac{\pi x^3}{3}+\frac{32}{105}\pi x^{7/2}$$

$$p^3: L_3(x)+\int_0^x\frac{L_2(t)}{\sqrt{(x-t)}}\,\mathrm{d}t=0\Rightarrow L_3(x)=-\frac{32}{105}\pi x^{7/2}-\frac{\pi^2x^4}{12},\ldots,$$

$$p^{18}: L_{18}(x)+\int_0^x\frac{L_{18}(t)}{\sqrt{(x-t)}}\,\mathrm{d}t=0\Rightarrow L_{18}(x)=\frac{\pi^9x^{11}}{19958400}+\frac{8192}{316234143225}\pi^9x^{23/2}.$$

Hence, from Eq. (10), the solution is

$$y(x)=\sum_{i=0}^{\infty}L_i(x)=\sum_{i=0}^{n}L_i(x)+O(x^{3+\frac{n}{2}})=x^2+O(x^{3+n/2})=x^2\quad\text{as } n\to\infty. \tag{20}$$

Figs. 1–3 show the comparison between the exact solution $y(x)$ (solid line) and the approximate solution $g(x)$ (dotted line) obtained by truncating (20) at level $n=18$, the error $E(x)=g(x)-y(x)$ and the relative error $R(x)=\frac{g(x)-y(x)}{y(x)}\cdot 100$ respectively.

Case 1(b) Choosing the initial guess $L_0(x)=x$, the following iterates of the solution are obtained

$$L_0(x)=x,\qquad L_1(x)=-x+x^2-\frac{4}{3}x^{3/2}+\frac{16}{15}x^{5/2},\qquad L_2(x)=\frac{4}{3}x^{3/2}-\frac{16}{15}x^{5/2}+\frac{1}{2}\pi x^2-\frac{1}{3}\pi x^3$$

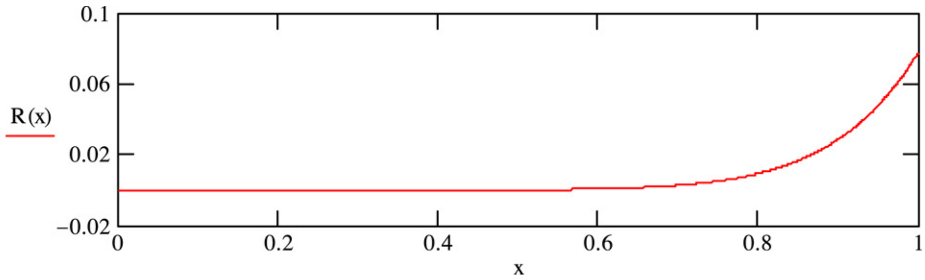

Fig. 3. The relative error $R(x)$ for Example 4.1, case 1(a).

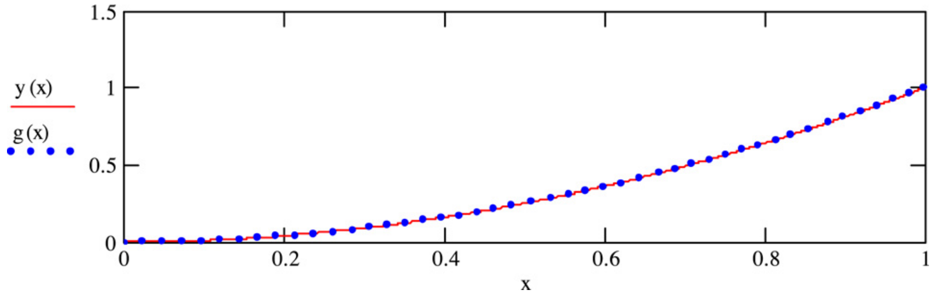

Fig. 4. The exact and the approximate solutions of the singular Volterra integral equation (17), in Example 4.1, case 1(b).

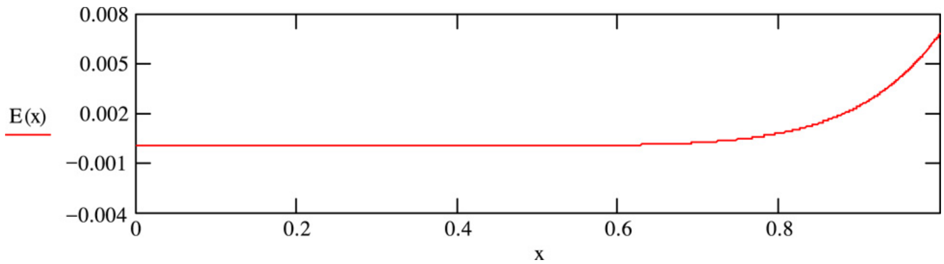

Fig. 5. The error $E(x)$ for Example 4.1, case 1(b).

$$L_3(x) = -\frac{1}{2}\pi x^2 + \frac{1}{3}\pi x^3 - \frac{8}{15}\pi x^{5/2} + \frac{32}{105}\pi x^{7/2}, \ldots$$

$$L_{18}(x) = \frac{1}{3628800}\pi^9 x^{10} - \frac{1}{19958400}\pi^9 x^{11} + \frac{81024}{654729075}\pi^8 x^{19/2} - \frac{4096}{13749310575}\pi^8 x^{21/2}.$$

Therefore, the solution is given by,

$$y(x) = \sum_{i=0}^{\infty} L_i(x) = \begin{cases} \sum_{i=0}^{n} L_i(x) + O(x^{m+2}) = x^2 + O(x^{m+2}), & n = 2m \\ \sum_{i=0}^{n} L_i(x) + O(x^{m+5/2}) = x^2 + O(x^{m+5/2}), & n = 2m+1 \end{cases} \to x^2 \quad \text{as } n \to \infty.$$

The above series is truncated again at level $n = 18$ to obtain Figs. 4–6 conveying the same information for case 1(b) as Figs. 1–3 did for the case 1(a).

Case 1(c): *Modified Homotopy perturbation method*

Writing $f(x) = \sum_{i=0}^{\infty} k_i(x)$, where $k_0(x) = x^2$, $k_1(x) = \frac{16}{15}x^{5/2}$ and $k_i(x) = 0$ for $i \geq 2$, we get $L_0(x) = x^2$. Hence, the various iterates are as follows:

$$p^0: L_0(x) = x^2, \qquad p^1: L_1(x) = \frac{16}{15}x^{5/2} - \int_0^x \frac{L_0(t)}{\sqrt{(x-t)}}\,dt \Rightarrow L_1(x) = 0$$

$$p^2: L_2(x) = \int_0^x \frac{L_1(t)}{\sqrt{(x-t)}}\,dt \Rightarrow L_2(x) = 0.$$

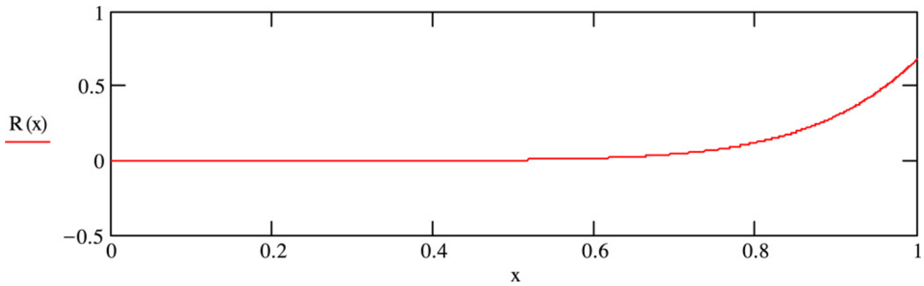

Fig. 6. The relative error $R(x)$ for Example 4.1, case 1(b).

Therefore, one can see that $L_n(x) = 0$, for all $n \geq 1$ and hence,

$$y(x) = L_0(x) + L_1(x) + L_2(x) + \cdots = x^2, \quad \text{is the exact solution.} \tag{21}$$

Case 1(d): *Adomian decomposition method*
From the recursive scheme (15), we get

$$y_0(x) = x^2 + \frac{16}{15}x^{5/2}, \qquad y_1(x) = -\int_0^x \frac{y_0(t)}{\sqrt{x-t}}\,dt = -\frac{16}{15}x^{5/2} - \frac{\pi x^3}{3},$$
$$y_2(x) = -\int_0^x \frac{y_1(t)}{\sqrt{x-t}}\,dt = \frac{\pi x^3}{3} + \frac{32}{105}\pi x^{7/2}, \qquad y_3(x) = -\int_0^x \frac{y_2(t)}{\sqrt{x-t}}\,dt = -\frac{32}{105}\pi x^{7/2} - \frac{\pi^2 x^4}{12},$$
$$y_4(x) = \frac{\pi^2 x^4}{12} + \frac{64}{945}\pi^2 x^{9/2}, \ldots, \qquad y_{18}(x) = \frac{\pi^9 x^{11}}{19958400} + \frac{8192}{316234143225}\pi^9 x^{23/2}.$$

These iterates are the same as obtained from HPM. Hence, the solution is given by

$$y(x) = \sum_{i=0}^{n} y_i(x) + O(x^{3+n/2}) = x^2 \quad \text{as } n \to \infty. \tag{22}$$

Case 1(e): *Modified Adomian decomposition method*
As suggested before, we split $f(x)$ into two parts $f_1(x) = x^2$ and $f_2(x) = \frac{16}{15}x^{5/2}$ and obtain

$$y_0(x) = x^2, \qquad y_1(x) = \frac{16}{15}x^{5/2} - \int_0^x \frac{y_0(t)}{\sqrt{x-t}}\,dt = 0, \quad \text{therefore,}$$
$$y_{n+1}(x) = 0, \quad n \geq 0. \tag{23}$$

Hence,

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = x^2, \quad \text{which is the exact solution, as well.} \tag{24}$$

Example 4.2. Consider the singular Volterra integral equation [33]

$$y(x) = x + \frac{4}{3}x^{3/2} - \int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt, \quad 0 \leq x \leq 1, \tag{25}$$

which has x as the exact solution.
Case 1(a): *Homotopy perturbation method*
A homotopy perturbation method can be constructed as:

$$H(y,p) = y(x) - x - \frac{4}{3}x^{3/2} + p\int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt, \tag{26}$$

giving various $L_i(x)$ as follows:

$$L_0(x) = x + \frac{4}{3}x^{3/2}, \qquad L_1(x) = -\frac{4}{3}x^{3/2} - \frac{\pi x^2}{2}, \qquad L_2(x) = \frac{\pi x^2}{2} + \frac{8}{15}\pi x^{5/2}, \ldots$$
$$L_{18}(x) = \frac{1}{3628800}\pi^9 x^{10} + \frac{2048}{13749310575}\pi^9 x^{21/2}.$$

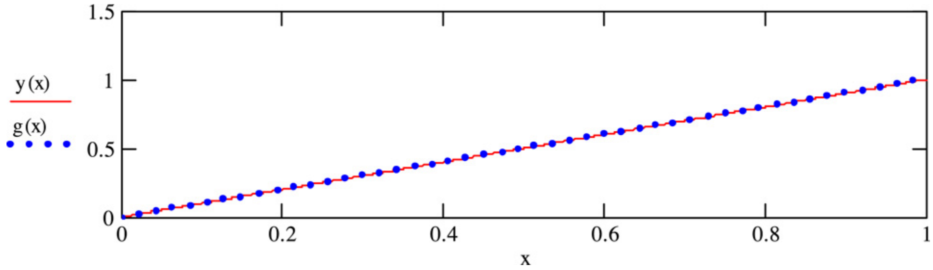

Fig. 7. The exact and the approximate solutions of the singular Volterra integral equation (25), in Example 4.2, case 1(a).

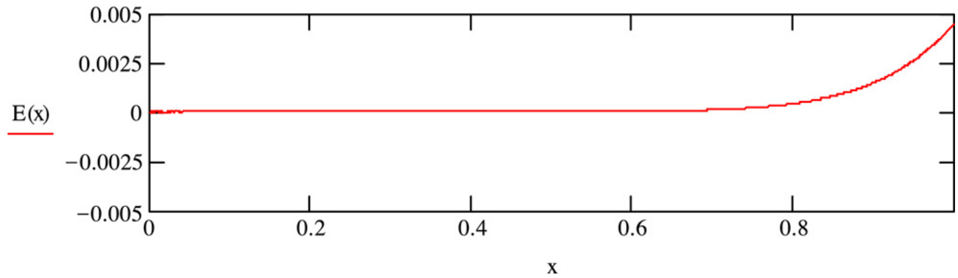

Fig. 8. The error $E(x)$ for Example 4.2, case 1(a).

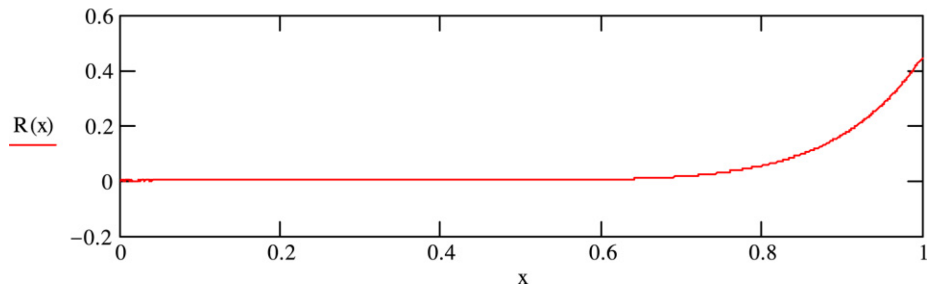

Fig. 9. The relative error $R(x)$ for Example 4.2, case 1(a).

Hence, from Eq. (10), the solution is

$$y(x) = \sum_{i=0}^{n} L_i(x) + O(x^{2+n/2}) \to x \quad \text{as } n \to \infty. \tag{27}$$

Figs. 7–9 show the comparison between the exact and approximate solutions, the error between them and the relative error respectively when the series (27) is truncated at level $n = 18$.

Case 1(b): *Modified Homotopy perturbation method*

Arguments similar to Example 4.1 case 1(c) suggest to choose $L_0(x) = x$, thus obtaining the various components as,

$$L_0(x) = x, \qquad L_n(x) = 0 \quad \text{for all } n \geq 1.$$

Hence, the solution is given as

$$y(x) = L_0(x) + L_1(x) + L_2(x) + \cdots = x. \tag{28}$$

Case 1(c): *Adomian decomposition method*

As explained in Eq. (15), we have

$$y_0(x) = x + \frac{4}{3}x^{3/2}, \qquad y_1(x) = -\frac{4}{3}x^{3/2} - \frac{\pi}{2}x^2, \qquad y_2(x) = \frac{\pi}{2}x^2 + \frac{8\pi}{15}x^{5/2},$$
$$y_3(x) = -\frac{8\pi}{15}x^{5/2} - \frac{\pi^2}{6}x^3, \qquad y_4(x) = \frac{\pi^2}{6}x^3 + \frac{16\pi^2}{105}x^{7/2}, \ldots$$
$$y_{18}(x) = \frac{1}{3628800}\pi^9 x^{10} + \frac{2048}{13749310575}\pi^9 x^{21/2}.$$

Hence, the approximate solution is given by

$$y(x)=\sum_{i=0}^{\infty}y_i(x)=\sum_{i=0}^{n}y_i(x)+O(x^{2+n/2})\approx x. \tag{29}$$

Case 1(d): *Modified Adomian decomposition method*
By splitting $f(x)$ into two parts $f_1(x)=x$ and $f_2(x)=\frac{4}{3}x^{3/2}$, we get

$$y_0(x)=x, \quad \text{therefore, } y_{n+1}(x)=0, \quad n\geq 0. \tag{30}$$

Hence,

$$y(x)=\sum_{n=0}^{\infty}y_n(x)=x, \quad \text{which is the exact solution, as well.} \tag{31}$$

Example 4.3. Consider the singular Volterra integral equation [32]

$$y(x)=2\sqrt{x}-\int_0^x\frac{y(t)}{\sqrt{x-t}}\,\mathrm{d}t, \quad 0\leq x\leq 1, \tag{32}$$

which has $y(x)=1-\mathrm{e}^{\pi x}\mathit{erfc}(\sqrt{\pi x})$ as the exact solution, where the complementary error function *erfc* is defined as, $\mathit{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^{\infty}\mathrm{e}^{-u^2}\,\mathrm{d}u$.
Case 1(a): *Homotopy perturbation method*
A homotopy perturbation method is constructed as follows:

$$H(y,p)=y(x)-2\sqrt{x}+p\int_0^x\frac{y(t)}{\sqrt{x-t}}\,\mathrm{d}t. \tag{33}$$

The various $L_i(x)$, $=0,1,2,\ldots$, are:

$$L_0(x)=2\sqrt{x}, \qquad L_1(x)=-\pi x, \qquad L_2(x)=\frac{4}{3}\pi x^{3/2}, \qquad L_3(x)=-\frac{\pi^2x^2}{2}, \qquad L_4(x)=\frac{8}{15}\pi^2x^{5/2},\ldots.$$

Hence, the solution is given as

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty}L_n(x)\\ &= 2\sqrt{x}-\pi x+\frac{4}{3}\pi x^{3/2}-\frac{1}{2}\pi^2x^2+\frac{8}{15}\pi^2x^{5/2}-\frac{1}{6}\pi^3x^3+\frac{16}{105}\pi^3x^{7/2}-\cdots\\ &= \sum_{r=1}^{\infty}\frac{(-1)^{r-1}(\pi x)^{r/2}}{\Gamma\left(\frac{r}{2}+1\right)}=1-E_{\frac{1}{2}}\left(-\sqrt{\pi x}\right),\\ &= 1-\mathrm{e}^{\pi x}\mathit{erfc}(\sqrt{\pi x}) \quad \text{(the exact solution)}, \end{aligned} \tag{34}$$

where $E_\alpha(z)=\sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+1)}$, $(\alpha>0)$ is the Mittag–Leffler function in one parameter.
Case 1(b): *Adomian decomposition method*
For this problem, the various components $y_i(x)$ are given as:

$$y_0(x)=2\sqrt{x}, \qquad y_1(x)=-\pi x, \qquad y_2(x)=\frac{4}{3}\pi x^{3/2}, \qquad y_3(x)=-\frac{\pi^2x^2}{2},$$
$$y_4(x)=\frac{8}{15}\pi^2x^{5/2}, \qquad y_5(x)=-\frac{\pi^3x^3}{6}, \qquad y_6(x)=\frac{16}{105}\pi^3x^{7/2},\ldots \quad \text{and so on.}$$

Thus the solution $y(x)$ is obtained as,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty}y_n(x)=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}(\pi x)^{r/2}}{\Gamma\left(\frac{r}{2}+1\right)},\\ &= 1-\mathrm{e}^{\pi x}\mathit{erfc}(\sqrt{\pi x}) \quad \text{(the exact solution)}. \end{aligned} \tag{35}$$

From the Figs. 10 and 11 (plotted by taking 24 iterates and $x=0,0.01\ldots,1$), we conclude that the approximate solution of problem (32), given by ((34)/ (35)), is quite accurate.

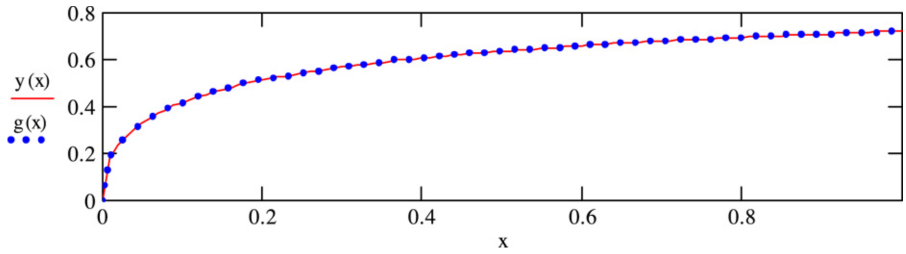

Fig. 10. The exact and the approximate solutions of the singular Volterra integral equation (32) in Example 4.3 are represented by $y(x)$ (solid line) and $g(x)$ (dotted line) respectively.

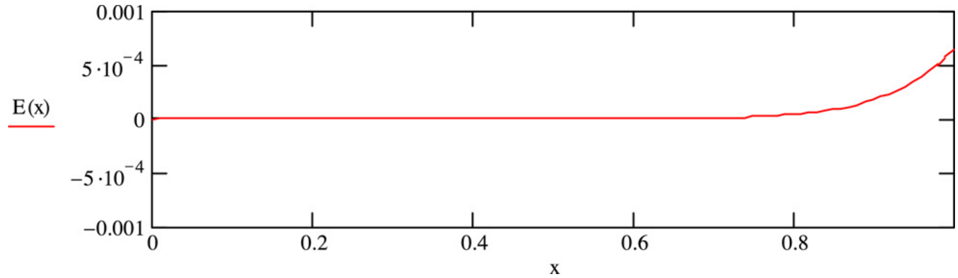

Fig. 11. The error $E(x) = y(x) - g(x)$ for the singular Volterra integral equation (32) in Example 4.3.

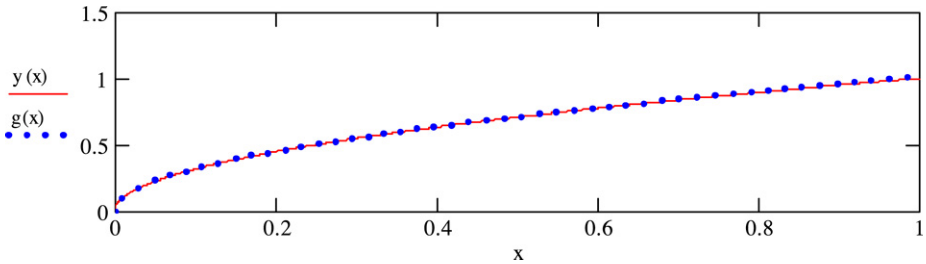

Fig. 12. The exact and the approximate solutions of the singular Volterra integral equation (36), in Example 4.4, case 1(a).

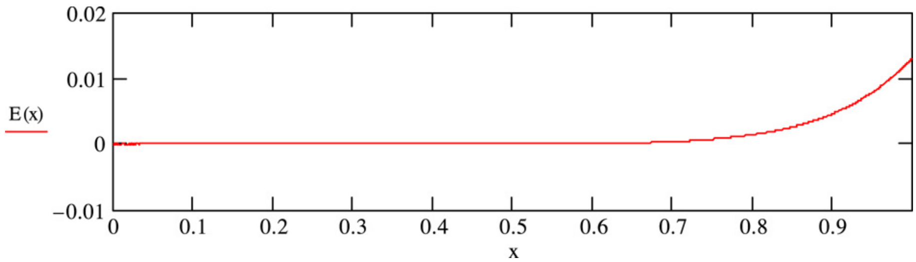

Fig. 13. The error $E(x)$ for Example 4.4, case 1(a).

Example 4.4. Consider the singular Volterra integral equation

$$y(x) = \sqrt{x} + \frac{\pi x}{2} - \int_0^x \frac{y(t)}{\sqrt{x-t}}\,\mathrm{d}t, \quad 0 \le x \le 1, \tag{36}$$

which has $y(x) = \sqrt{x}$ as the exact solution.

Case 1(a): *Homotopy perturbation method* (Figs. 12–14).

A homotopy perturbation method can be constructed as follows (from Eq. (8)):

$$H(y, p) = y(x) - \sqrt{x} - \frac{\pi x}{2} + p\int_0^x \frac{y(t)}{\sqrt{x-t}}\,\mathrm{d}t. \tag{37}$$

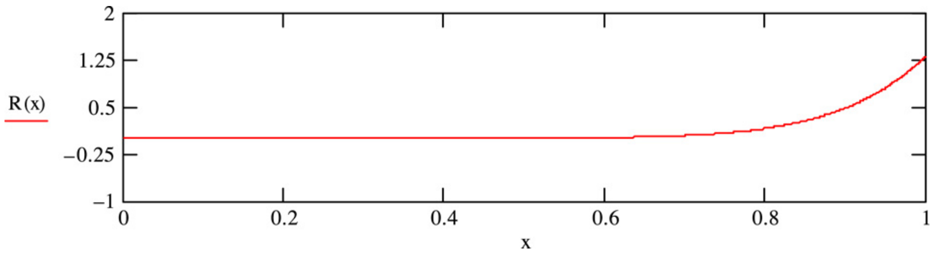

Fig. 14. The relative error $R(x)$ for Example 4.4, case 1(a).

One can now try to obtain a solution of Eq. (37) in the form of

$$L(x) = L_0(x) + pL_1(x) + p^2L_2(x) + \cdots \tag{38}$$

where $L_i(x)$, $i = 0, 1, 2, \ldots$ are functions yet to be determined. From Eqs. (37) and (38) the approximations are:

$$p^0: L_0(x) = \sqrt{x} + \frac{\pi}{2}x$$

$$p^1: L_1(x) + \int_0^x \frac{L_0(t)}{\sqrt{(x-t)}}\,\mathrm{d}t = 0 \Rightarrow L_1(x) = -\frac{1}{2}\pi x - \frac{2}{3}\pi x^{3/2}.$$

Similarly,

$$L_2(x) = \frac{1}{4}\pi^2 x^2 + \frac{2}{3}\pi x^{3/2}, \qquad L_3(x) = -\frac{1}{4}\pi^2 x^2 - \frac{4}{15}\pi^2 x^{5/2}, \ldots$$

$$L_{18}(x) = \frac{1}{7257600}\pi^{10}x^{10} + \frac{512}{654729075}\pi^9 x^{19/2}.$$

Hence, from Eq. (10), the solution is given by

$$y(x) = \sum_{i=0}^{\infty} L_i(x) = \sqrt{x}. \tag{39}$$

Case 1(b): *Modified Homotopy perturbation method*
A modified homotopy perturbation method is constructed as follows:

$$p^0: L_0(x) = \sqrt{x}, \qquad p^1: L_1(x) = \frac{\pi}{2}x - \int_0^x \frac{L_0(t)}{\sqrt{(x-t)}}\,\mathrm{d}t \Rightarrow L_1(x) = 0,$$

$$p^2: L_2(x) = \int_0^x \frac{L_1(t)}{\sqrt{(x-t)}}\,\mathrm{d}t \Rightarrow L_2(x) = 0.$$

Therefore, one can see that $L_n(x) = 0$, for all $n \geq 1$ and hence,

$$y(x) = L_0(x) + L_1(x) + L_2(x) + \cdots = \sqrt{x}, \quad \text{is the exact solution.}$$

As Adomian decomposition method gives the same series solution as given by HPM, we will skip this method from now onwards.

Case 1(c): *Modified Adomian decomposition method*
By splitting $f(x)$ into two parts $f_1(x) = \sqrt{x}$ and $f_2(x) = \frac{\pi}{2}x$, we get

$$y_0(x) = \sqrt{x}, \quad \text{therefore, } y_{n+1}(x) = 0, \quad n \geq 0. \tag{40}$$

Hence,

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = \sqrt{x}, \quad \text{which is the exact solution, as well.} \tag{41}$$

Example 4.5. Consider the singular Volterra integral equation

$$y(x) = \frac{1}{\sqrt{x}} + \pi - \int_0^x \frac{y(t)}{\sqrt{x-t}}\,\mathrm{d}t, \quad 0 \leq x \leq 1, \tag{42}$$

which has $y(x) = \frac{1}{\sqrt{x}}$ as the exact solution.

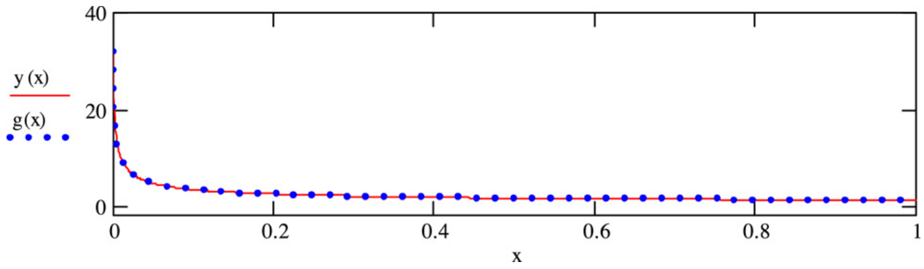

Fig. 15. The exact and the approximate solutions of the singular Volterra integral equation (42), in Example 4.5, case 1(a).

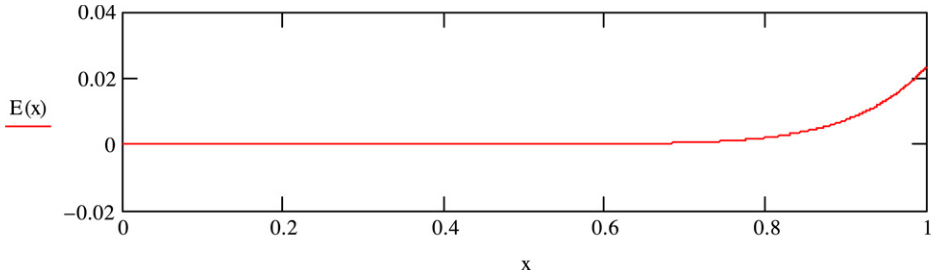

Fig. 16. The error $E(x)$ for Example 4.5, case 1(a).

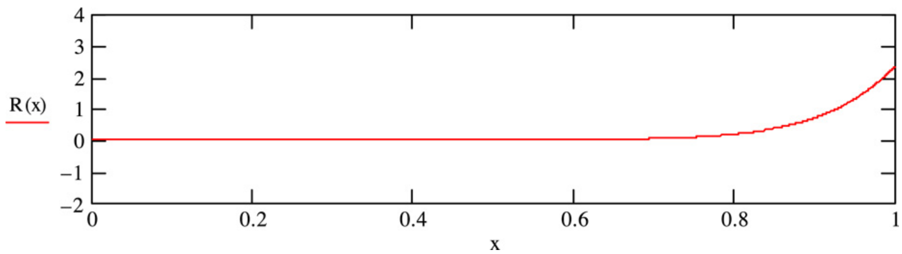

Fig. 17. The relative error $R(x)$ for Example 4.5, case 1(a).

Case 1(a): *Homotopy perturbation method*

A homotopy perturbation method can be constructed as follows (from Eq. (8)):

$$H(y,p) = y(x) - \frac{1}{\sqrt{x}} - \pi + p\int_0^x \frac{y(t)}{\sqrt{x-t}}\,\mathrm{d}t. \tag{43}$$

The various iterates are given as:

$$L_0(x) = \frac{1}{\sqrt{x}} + \pi, \qquad L_1(x) = -\pi - 2\pi x^{1/2}, \qquad L_2(x) = \pi^2 x + 2\pi x^{1/2},$$

$$L_3(x) = -\pi^2 x - \frac{4}{3}\pi^2 x^{3/2}, \ldots$$

$$L_{22}(x) = \frac{1}{39916800}\pi^{12}x^{11} + \frac{2048}{13749310575}\pi^{11}x^{21/2}.$$

Hence,

$$y(x) = \sum_{i=0}^{\infty} L_i(x) = \frac{1}{\sqrt{x}}. \tag{44}$$

The Figs. 15–17 are self explanatory.

Case 1(b): *Modified Homotopy perturbation method*

A modified homotopy perturbation method is constructed as follows:

$$H(y,p) = y(x) - \frac{1}{\sqrt{x}} - \pi + p\int_0^x \frac{y(t)}{\sqrt{x-t}}\,\mathrm{d}t. \tag{45}$$

$$p^0: L_0(x) = \frac{1}{\sqrt{x}}, \qquad p^1: L_1(x) = \pi - \int_0^x \frac{L_0(t)}{\sqrt{(x-t)}}\,dt \Rightarrow L_1(x) = 0.$$

Therefore, one can see that $L_n(x) = 0$, for all $n \geq 1$ and hence,

$$y(x) = L_0(x) + L_1(x) + L_2(x) + \cdots = \frac{1}{\sqrt{x}}, \quad \text{is the exact solution.}$$

Case 1(c): *Modified Adomian decomposition method*
By splitting $f(x)$ into two parts $f_1(x) = \frac{1}{\sqrt{x}}$ and $f_2(x) = \pi$, we get

$$y_0(x) = \frac{1}{\sqrt{x}}, \quad \text{therefore, } y_{n+1}(x) = 0 \quad n \geq 0. \tag{46}$$

Hence, $y(x) = \sum_{n=0}^{\infty} y_n(x) = \frac{1}{\sqrt{x}}$, which is the exact solution, as well.

As MHPM is more convenient and simple to use, we solve the following examples by using MHPM and MADM.

Example 4.6. Consider the singular Volterra integral equation

$$y(x) = \frac{1}{1+x} + \frac{2\,arc\sinh\sqrt{x}}{\sqrt{1+x}} - \int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt, \quad 0 \leq x \leq 1, \tag{47}$$

which has $y(x) = \frac{1}{1+x}$ as the exact solution.
Case 1(a): *Modified Homotopy perturbation method*
A modified homotopy perturbation method is constructed as follows:

$$H(y,p) = y(x) - \frac{1}{1+x} - \frac{2\,arc\sinh\sqrt{x}}{\sqrt{1+x}} + p\int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt. \tag{48}$$

$$p^0: L_0(x) = \frac{1}{1+x} \qquad , p^1: L_1(x) = \frac{2\,arc\sinh\sqrt{x}}{\sqrt{1+x}} - \int_0^x \frac{L_0(t)}{\sqrt{(x-t)}}\,dt \Rightarrow L_1(x) = 0$$

$$p^2: L_2(x) = \int_0^x \frac{L_1(t)}{\sqrt{(x-t)}}\,dt \Rightarrow L_2(x) = 0 \Rightarrow L_i(x) = 0 \quad \text{for } i > 2.$$

Hence,

$$y(x) = L_0(x) + L_1(x) + L_2(x) + \cdots = \frac{1}{1+x}, \quad \text{is the exact solution.}$$

Case 1(b): *Modified Adomian decomposition method*
By splitting $f(x)$ into two parts $f_1(x) = \frac{1}{1+x}$ and $f_2(x) = \frac{2\,arc\sinh\sqrt{x}}{\sqrt{1+x}}$, we get

$$y_0(x) = \frac{1}{1+x}, \quad \text{therefore, } y_{n+1}(x) = 0, \quad n \geq 0. \tag{49}$$

$$\text{Therefore, } y(x) = \sum_{n=0}^{\infty} y_n(x) = \frac{1}{1+x}, \quad \text{which is the exact solution, as well.} \tag{50}$$

Example 4.7. Consider the singular Volterra integral equation

$$y(x) = |x-c| + \frac{2}{3}\left|\sqrt{x}\,(2x-3c)\right| - \int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt, \quad 0 \leq x \leq 1, \tag{51}$$

which has $y(x) = |x-c|$ as the exact solution.
Case 1(a): *Modified Homotopy perturbation method*
A modified homotopy perturbation method is constructed as follows:

$$H(y,p) = y(x) - |x-c| - \frac{2}{3}\left|\sqrt{x}\,(2x-3c)\right| + p\int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt. \tag{52}$$

Writing $f(x) = \sum_{i=0}^{\infty} k_i(x)$, where $k_0(x) = |x-c|$, $k_1(x) = \frac{2}{3}\left|\sqrt{x}\,(2x-3c)\right|$ and $k_i(x) = 0$ for $i \geq 2$, we get $L_0(x) = |x-c|$.

Hence, the various iterates are as follows:

$$p^0: L_0(x) = |x-c|, \qquad p^1: L_1(x) = \frac{2}{3}\left|\sqrt{x}\,(2x-3c)\right| - \int_0^x \frac{L_0(t)}{\sqrt{(x-t)}}\,\mathrm{d}t \Rightarrow L_1(x) = 0.$$

Therefore, one can see that $L_n(x) = 0$, for all $n \geq 1$ and hence,

$$y(x) = L_0(x) + L_1(x) + L_2(x) + \cdots = |x-c|, \quad \text{is the exact solution.}$$

Case 1(b): *Modified Adomian decomposition method*
By splitting $f(x)$ into two parts $f_1(x) = |x-c|$ and $f_2(x) = \frac{2}{3}\left|\sqrt{x}\,(2x-3c)\right|$, we get

$$y_0(x) = |x-c|, \quad \text{it is easy to see that, } y_{n+1}(x) = 0, \quad \geq 0. \tag{53}$$

$$\text{Hence, } y(x) = \sum_{n=0}^{\infty} y_n(x) = |x-c|, \quad \text{which is the exact solution, as well.} \tag{54}$$

5. Conclusion

From the above examples, it is obvious that HPM and ADM give the same approximate solutions, where as when we apply MHPM, the iterates become zero (as seen from the examples) from second iterates itself as is the case with MADM. So MADM and HPM (with different choice for $L_0(x)$) are much more efficient and simpler than HPM and ADM. The rate of convergence for the series representing the solution obtained by HPM depends upon the initial choice $L_0(x)$.

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