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A study of generalized thermoelastic interactions in an unbounded medium with a spherical cavity

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ABSTRACT

The aim of the present paper is to study the thermoelastic interactions in an unbounded elastic medium with a spherical cavity in the context of four different theories of thermoelasticity, namely: the classical coupled dynamical thermoelasticity, the extended thermoelasticity, the temperature-rate-dependent thermoelasticity and the thermoelasticity without energy dissipation in a unified way. The cavity surface is assumed to be stress free and is subjected to a smooth and time-dependent-heating effect. The solutions for displacement, temperature and stresses are obtained with the help of the Laplace transform procedure. Firstly the short-time approximated solutions for four different theories have been obtained analytically. Then following the numerical method proposed by Bellman et al. [R. Bellmen, R.E. Kolaba, J.A. Lockette, Numerical Inversion of the Laplace Transform, American Elsevier Pub. Co., New York, 1966] for the inversion of Laplace transforms, the numerical values of the physical quantities are also computed for the copper material and results are displayed in graphical forms to compare the results obtained for the theory of thermoelasticity without energy dissipation with the results of other thermoelasticity theories.

1. Introduction

The generalized thermoelasticity theories have been developed with the aim of removing the paradox of infinite speed of heat propagation inherent in the classical coupled dynamical thermoelasticity theory, see Biot [2]. In the generalized theories, the governing equations involve thermal relaxation times and they are of hyperbolic type. The extended thermoelasticity theory by Lord and Shulman [3] which introduces one relaxation time in the thermoelastic process and the temperature-rate-dependent theory of thermoelasticity by Green and Lindsay [4] which takes into account two relaxation times are two well established generalized theories of thermoelasticity. Several experimental studies [5–7] indicate that the thermal relaxation time effects can be of relevance in the cases involving a rapidly propagating crack tip, a localized moving heat source with high intensity, shock wave propagation, laser processing technique etc. Subsequently, several investigations [8–15] based on these generalized theories were carried out.

Recently, Green and Nagdhi [16–18] have proposed three other different models of thermoelasticity in an alternative way. In one of these models [17] the significance is that the internal rate of production of entropy is taken to be identically zero, i.e., there is no dissipation of thermal energy. This theory is known as the theory of thermoelasticity without energy dissipation. In the development of this theory, the thermal displacement gradient is considered as a constitutive variable, whereas in the conventional development of thermoelasticity the temperature gradient is taken as a constitutive variable. The Uniqueness theorem in the case of a linearized version of this theory is given by Green and Nagdhi [17] and

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Nomenclature

u	displacement,
τ_{ij}	stress tensor,
Δ	dilatation,
∇^2	Laplacian operator,
λ, μ	Lame's elastic constants,
T	temperature deviation from a reference temperature T_0,
ρ	mass density,
K	thermal conductivity of the material,
c_v	specific heat at constant strain,
$\gamma = (3\lambda + 2\mu)\alpha_t, \alpha_t$	coefficient of linear thermal expansion,
δ_{ij}	Kronecker delta symbol,
α_1, α_2	thermal relaxation times.

Chandrasekharaiah [19] independently. Later on, Chandrasekharaiah [20] studied free plane harmonic waves without energy dissipation in an unbounded body. Sharma and Chauhan [21] investigated a problem concerning thermoelastic interactions without energy dissipation due to body forces and heat sources. Mukhopadhyay [22] tackled a problem concerning the thermoelastic interactions without energy dissipation in an unbounded medium with a spherical cavity subjected to harmonically varying temperature. Othman and Song [23] have investigated the effect of rotation on the reflection of magneto-thermoelastic waves under thermoelasticity without energy dissipation. In another study, Othman and Song [24] have discussed the reflection of plane waves from an elastic half-space under hydrostatic initial stress without energy dissipation.

In the present problem an attempt has been made to study the thermoelastic interactions in an isotropic elastic medium with a spherical cavity subjected to a time-dependent-heating effect in the context of four different theories of thermoelasticity in a unified way. Analytical solutions for the distributions of the field variables: displacement, temperature and stresses are found out with the help of the Laplace transform procedure. This is then followed by the numerical inversion of the transformed solution in the space–time domain. The numerical results are presented graphically to compare the nature of variations of the variables under different theories of thermoelasticity.

2. Problem formulation: Governing equations

We consider an infinitely extended homogeneous isotropic elastic medium with a spherical cavity of radius, 'a'. The center of the cavity is taken to be the origin of the spherical polar co-ordinate system (r, θ, φ). The stress strain temperature relations and the generalized heat conduction equation in the context of the theory of classical coupled thermoelasticity (CTE), extended thermoelasticity (ETE), temperature-rate-dependent thermoelasticity (TRDTE) and thermoelasticity without energy dissipation (TEWOED) can be written in a unified way as follows:

$$\tau_{ij} = \lambda \Delta \delta_{ij} + 2\mu e_{ij} - \gamma \left(T + \alpha_2 \frac{\partial T}{\partial t} \right) \delta_{ij} \tag{1}$$

$$\left(K\delta_{1k} + K^*\delta_{2k}\right) \nabla^2 T = \rho c_v \left[\delta_{1k} \left(\frac{\partial}{\partial t} + \alpha_1 \frac{\partial^2}{\partial t^2} \right) + \delta_{2k} \frac{\partial^2}{\partial t^2} \right] T + \gamma T_0 \left[\delta_{1k} \left(\frac{\partial}{\partial t} + \alpha_1 \zeta \frac{\partial^2}{\partial t^2} \right) + \delta_{2k} \frac{\partial^2}{\partial t^2} \right] \Delta. \tag{2}$$

Here K^* is an additional material constant characteristic of the theory of thermoelasticity without energy dissipation (Green and Nagdhi [17]).

Here Eqs. (1) and (2) reduce to the equations of different theories of thermoelasticity as follows:

I. CTE: $\boldsymbol{k} = \mathbf{1}, \alpha_1 = 0, \alpha_2 = 0,$

II. ETE: $\boldsymbol{k} = \mathbf{1}, \alpha_2 = 0, \alpha_1 > 0, \zeta = 1$

III. TRDTE: $\boldsymbol{k} = \mathbf{1}, \alpha_2 \geq \alpha_1 > 0, \zeta = \mathbf{0}$

IV. TEWOED: $\boldsymbol{k} = \mathbf{2}, \alpha_2 = 0.$

For our present problem (due to spherical symmetry) the displacement and temperature components are assumed to be functions of r and time t only, so that the non-zero strain components are

$$e_{rr} = \frac{\partial u}{\partial r}, \qquad e_{\theta\theta} = e_{\varphi\varphi} = \frac{u}{r}.$$

The non-zero stresses are then obtained as

$$\tau_{rr} = \lambda \left(\frac{\partial u}{\partial r} + \frac{2u}{r} \right) + 2\mu \frac{\partial u}{\partial r} - \gamma \left(1 + \alpha_2 \frac{\partial}{\partial t} \right) T \tag{3}$$

$$\tau_{\theta\theta} = \tau_{\varphi\varphi} = \lambda\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) + 2\mu\frac{u}{r} - \gamma\left(1 + \alpha_2\frac{\partial}{\partial t}\right)T. \tag{4}$$

The stress equation of motion is:

$$\frac{\partial}{\partial r}\tau_{rr} + \frac{2}{r}(\tau_{rr} - \tau_{\theta\theta}) = \rho\frac{\partial^2 u}{\partial t^2}. \tag{5}$$

Now, we introduce the following dimensionless variables and quantities

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad R = \frac{r}{a}, \quad U = \frac{u}{a}, \quad \eta = \frac{c_1}{a}t, \quad Z = \frac{T}{T_0}, \quad \alpha_1' = \frac{c_1}{a}\alpha_1,$$
$$\alpha_2' = \frac{c_1}{a}\alpha_2, \quad a_0 = \frac{aK^*}{Kc_1}, \quad a_1 = \frac{\gamma T_0}{\lambda + 2\mu}, \quad a_2 = \frac{a\rho c_v c_1}{K}, \quad a_3 = \frac{a\gamma c_1}{K},$$
$$\sigma_{RR} = \frac{\tau_{rr}}{\lambda + 2\mu}, \quad \sigma_{\varphi\varphi} = \frac{\tau_{\varphi\varphi}}{\lambda + 2\mu}, \quad \lambda_1 = \frac{\lambda}{(\lambda + 2\mu)}.$$

Eqs. (2)–(5) then reduce to the dimensionless forms as:

$$(\delta_{1k} + a_0\delta_{2k})\left(\frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R}\right)Z$$
$$= a_2\left(\delta_{1k}\left(\frac{\partial}{\partial \eta} + \alpha_1'\frac{\partial^2}{\partial \eta^2}\right) + \delta_{2k}\frac{\partial^2}{\partial \eta^2}\right)Z + a_3\left(\delta_{1k}\left(\frac{\partial}{\partial \eta} + \alpha_1'\zeta\frac{\partial^2}{\partial \eta^2}\right) + \delta_{2k}\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial U}{\partial R} + \frac{2U}{R}\right) \tag{6}$$

$$\frac{\partial}{\partial R}\left(\frac{\partial U}{\partial R} + \frac{2U}{R}\right) - a_1\left(1 + \alpha_2'\frac{\partial}{\partial \eta}\right)\frac{\partial Z}{\partial R} = \frac{\partial^2 U}{\partial \eta^2} \tag{7}$$

$$\sigma_{RR} = \frac{\partial U}{\partial R} + 2\lambda_1\frac{U}{R} - a_1\left(1 + \alpha_2'\frac{\partial}{\partial \eta}\right)Z \tag{8}$$

$$\sigma_{\varphi\varphi} = \lambda_1\frac{\partial U}{\partial R} + (\lambda_1 + 1)\frac{U}{R} - a_1\left(1 + \alpha_2'\frac{\partial}{\partial \eta}\right)Z. \tag{9}$$

3. Boundary conditions

The surface of the cavity, i.e., $R = 1$ is assumed to be stress free and is subjected to a time-dependent-heating effect so that the boundary conditions are taken as:

$$\sigma_{RR}|_{R=1} = 0 \tag{10}$$
$$Z|_{R=1} = F(\eta) \tag{11}$$

where $F(\eta)$ is a function of η.

Initially the medium is at rest and undisturbed and the initial conditions are:

$$U(R,\eta)|_{\eta=0} = \left.\frac{\partial}{\partial \eta}U(R,\eta)\right|_{\eta=0} = 0$$
$$Z(R,\eta)|_{\eta=0} = \left.\frac{\partial}{\partial \eta}Z(R,\eta)\right|_{\eta=0} = 0$$
$$\sigma_{RR}(R,\eta)|_{\eta=0} = \left.\frac{\partial}{\partial \eta}\sigma_{RR}(R,\eta)\right|_{\eta=0} = 0.$$

4. Solution of the problem

Applying the Laplace transform on time to Eqs. (6)–(9) reduces them to

$$(\delta_{1k} + a_0\delta_{2k})\left(\frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R}\right)\bar{Z} = a_2\left(\delta_{1k}(p + \alpha_1' p^2) + \delta_{2k}p^2\right)\bar{Z} + a_3\left(\delta_{1k}(p + \alpha_1'\zeta p^2) + \delta_{2k}p^2\right)\left(\frac{\partial \bar{U}}{\partial R} + \frac{2\bar{U}}{R}\right) \tag{12}$$

$$\frac{\partial \bar{e}}{\partial R} - a_1\left(1 + \alpha_2' p\right)\frac{\partial \overline{Z}}{\partial R} = p^2\overline{U} \tag{13}$$

$$\overline{\sigma}_{RR} = \bar{e} - 2(1-\lambda_1)\frac{\overline{U}}{R} - a_1(1+\alpha_2' p)\bar{Z} \tag{14}$$

$$\overline{\sigma}_{\varphi\varphi} = \lambda_1 \bar{e} + (1-\lambda_1)\frac{\overline{U}}{R} - a_1(1+\alpha_2' p)\overline{Z} \tag{15}$$

where, p is the Laplace transform parameter and $e = \frac{\partial U}{\partial R} + \frac{2U}{R}$.

The Laplace transform of (10) and (11) yields:

$$\overline{\sigma}_{RR}|_{R=1} = 0 \tag{16}$$

$$\bar{Z}\big|_{R=1} = \bar{F}(p), \tag{17}$$

$\overline{F}(p)$ being the Laplace transform of the function $F(\eta)$.

Decoupling Eqs. (12) and (13) and solving we get the general solutions for $\bar{e}$ and $\bar{Z}$ bounded at infinity as

$$\bar{e} = \frac{1}{\sqrt{R}}\left[A_1 K_{1/2}(m_1 R) + A_2 K_{1/2}(m_2 R)\right] \tag{18}$$

$$\overline{Z} = \frac{1}{\sqrt{R}}\left[B_1 K_{1/2}(m_1 R) + B_2 K_{1/2}(m_2 R)\right]. \tag{19}$$

Here A_i, B_i are arbitrary constants, independent of R, $K_{1/2}(m_i R)$ are modified Bessel functions and m_1, m_2 satisfy the following equation:

$$\begin{aligned}&(\delta_{1k} + a_0\delta_{2k})\, m^4 - \left[\delta_{1k}\left\{a_2(1+\varepsilon)p + (1 + a_2\alpha_1' + a_2\varepsilon\alpha_1'\zeta + a_2\varepsilon\alpha_2')p^2\right\} + \delta_{2k}\left\{a_0 + (1+\varepsilon)a_2\right\}p^2\right] m^2 \\ &\quad + a_2 p^3\left\{\delta_{1k} + \left(\delta_{1k}\alpha_1' + \delta_{2k}\right)p\right\} = 0\end{aligned} \tag{20}$$

where, $\varepsilon = \frac{\gamma^2 T_0}{\rho^2 c_v c_1^2}$ is the thermoelastic coupling constant.

Now we use the relation

$$\frac{\mathrm{d}}{\mathrm{d}R}K_{1/2}(mR) = \frac{1}{2R}K_{1/2}(mR) - mK_{3/2}(mR)$$

and Eqs. (18), (19) and (13) to obtain the relations between the constants A_i and B_i as

$$B_i = F_i A_i, \quad i = 1, 2$$

$$\text{where } F_i = \frac{m_i^2 - p^2}{a_1(1+\alpha_2' p)m_i^2}, \quad i = 1, 2. \tag{21}$$

The solutions for displacement and stresses are obtained from Eqs. (12)–(15), (18) and (19) as follows:

$$\overline{U} = -\frac{1}{\sqrt{R}}\left[\frac{1}{m_1}A_1 K_{3/2}(m_1 R) + \frac{1}{m_2}A_2 K_{3/2}(m_2 R)\right] \tag{22}$$

$$\overline{\sigma}_{RR} = \frac{1}{\sqrt{R}}\left[A_1\sigma_{R1} + A_2\sigma_{R2}\right] \tag{23}$$

$$\overline{\sigma}_{\varphi\varphi} = \frac{1}{\sqrt{R}}\left[A_1\sigma_{\varphi 1} + A_2\sigma_{\varphi 2}\right] \tag{24}$$

where,

$$\sigma_{Ri} = \left[1 - a_1(1+\alpha_2' p)F_i\right]K_{1/2}(m_i R) + \frac{2(1-\lambda_1)}{m_i R}K_{3/2}(m_i R), \quad i = 1, 2$$

$$\sigma_{\varphi i} = \left[\lambda_1 - a_1(1+\alpha_2' p)F_i\right]K_{1/2}(m_i R) - \frac{(1-\lambda_1)}{m_i R}K_{3/2}(m_i R), \quad i = 1, 2.$$

The constants A_1 and A_2 are obtained with the help of the boundary conditions (16) and (17) as follows:

$$A_1 = -\frac{\sigma_{R2}^0 \overline{F}(p)}{\sigma_{R1}^0 F_2 K_{1/2}(m_2) - \sigma_{R2}^0 F_1 K_{1/2}(m_1)},$$

$$A_2 = \frac{\sigma_{R1}^0 \overline{F}(p)}{\sigma_{R1}^0 F_2 K_{1/2}(m_2) - \sigma_{R2}^0 F_1 K_{1/2}(m_1)}$$

where,

$$\sigma_{Ri}^0 = \left[1 - a_1(1+\alpha_2' p)F_i\right]K_{1/2}(m_i) + \frac{2(1-\lambda_1)}{m_i}K_{3/2}(m_i), \quad i = 1, 2.$$

5. Short-time approximated solution

The distributions of displacement, temperature and stresses in the physical domain (R, η) are determined by inverting the expressions for $\bar{U}$, $\bar{Z}$, $\bar{\sigma}_{RR}$, $\bar{\sigma}_{\varphi\varphi}$ and using the inverse Laplace transforms. But because of the dependency of m_1, and m_2 on p, it is extremely difficult to carry out this operation exactly for all values of p. Therefore we will now confine our attention to obtain the short-time approximated solutions of the field variables for which we assume that p is large.

With the help of Maclaurin's series expansions and neglecting the higher powers of small terms we get the roots m_1, m_2 of Eq. (20) as follows:

(I) For the case of ETE, TRDTE and TEWOED:

$$m_i = b_i^0 p + b_i^1 + b_i^2 \frac{1}{p}, \quad i = 1, 2. \tag{25}$$

(II) For the case of CTE:

$$m_1 = b_1^0 p + b_1^1 + b_1^2 \frac{1}{p} \tag{26}$$

$$m_2 = \left[v_1 p^{\frac{1}{2}} + \frac{v_2}{\sqrt{p}} + \frac{v_3}{\sqrt{p^3}} \right] \tag{27}$$

where,

$$b_i^0 = \sqrt{\left(a_i^0\right)}, \quad i = 1, 2$$

$$b_i^1 = \frac{a_i^1}{2\left(a_i^0\right)^{\frac{1}{2}}}, \quad i = 1, 2$$

$$b_i^2 = \frac{4a_i^2 a_i^0 - \left(a_i^1\right)^2}{8\left(a_i^0\right)^{\frac{3}{2}}}, \quad i = 1, 2$$

$$a_i^0 = \frac{A_2 + (-1)^{i+1}\sqrt{C}}{2A_5}, \quad i = 1, 2$$

$$a_i^1 = \frac{A_1\sqrt{C} + (-1)^{i+1}\left(A_1A_2 - 2A_3A_5a_2\right)}{2A_5\sqrt{C}}, \quad i = 1, 2$$

$$a_i^2 = (-1)^{i+1}\frac{1}{4A_5}\left[\frac{(A_1)^2}{\sqrt{C}} - \frac{\left(A_1A_2 - 2A_3A_5a_2\right)^2}{\sqrt{C^3}}\right], \quad i = 1, 2$$

$$v_1 = \sqrt{\left(a_2^1\right)}, \qquad v_2 = \frac{1}{2}\frac{a_2^2}{\sqrt{a_2^1}}, \qquad v_3 = -\frac{1}{8}\frac{\left(a_2^2\right)^2}{\left(a_2^1\right)^{\frac{3}{2}}}$$

$$A_1 = \delta_{1k}a_2(1+\varepsilon), \qquad A_2 = \delta_{1k}\left(1 + a_2\alpha_1' + a_2\varepsilon\alpha_1'\zeta + a_2\varepsilon\alpha_2'\right) + \delta_{2k}\left(a_0 + (1+\varepsilon)a_2\right), \qquad A_3 = \delta_{1k}$$

$$A_4 = \left(\delta_{1k}\alpha_1' + \delta_{2k}\right), \qquad A_5 = \delta_{1k} + a_0\delta_{2k}, \qquad C = (A_2)^2 - 4a_2A_4A_5.$$

Now substituting the value of m_1 and m_2 from Eqs. (25)–(27) in the expressions for displacement, temperature and stresses and using expression

$$K_v(z) \approx \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}\left[1 + \frac{4v^2 - 1}{8z} + \frac{\left(4v^2-1\right)\left(4v^2-9\right)}{2!\,(8z)^2} + \cdots\right]$$

we obtain, after a long calculation, the short-time approximated solutions for the distributions of displacement, temperature and stresses in the Laplace transform domain (R, p) as follows:

(I) For the case of ETE, TRDTE & TEWOED:

$$\bar{U} = \frac{2a_1}{RD_1}\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+1}\,\mathrm{e}^{-m_i(R-1)}\left[\frac{U_{ij}}{p^{j+2}}\right]$$

$$\bar{Z} = \frac{2}{RD_1}\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i}\,\mathrm{e}^{-m_i(R-1)}\left[\frac{Z_{ij}}{p^{j+2}}\right]$$

$$\bar{\sigma}_{RR} = \frac{2a_1}{RD_1} \sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^i \, e^{-m_i(R-1)} \left[\frac{S_{ij}^R}{p^{j+1}} \right]$$

$$\bar{\sigma}_{\varphi\varphi} = \frac{2a_1}{RD_1} \sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^i \, e^{-m_i(R-1)} \left[\frac{S_{ij}^\varphi}{p^{j+1}} \right].$$

(II) For the case of CTE:

$$\bar{U} = \frac{2a_1}{D_{C1}R} \left\{ e^{-m_1(R-1)} \left[\frac{U_{C11}}{p^4} + \frac{U_{C12}}{p^5} \right] - e^{-m_2(R-1)} \left[\frac{U_{C21}}{p^{\frac{9}{2}}} + \frac{U_{C22}}{p^5} \right] \right\}$$

$$\bar{Z} = \frac{2}{D_{C1}R} \left\{ -e^{-m_1(R-1)} \left[\frac{Z_{C11}}{p^3} + \frac{Z_{C12}}{p^4} \right] + e^{-m_2(R-1)} \left[\frac{Z_{C21}}{p^3} + \frac{Z_{C22}}{p^4} \right] \right\}$$

$$\bar{\sigma}_{RR} = \frac{2a_1}{D_{C1}R} \left\{ -e^{-m_1(R-1)} \left[\frac{X_{11}}{p^3} + \frac{X_{12}}{p^4} \right] + e^{-m_2(R-1)} \left[\frac{X_{21}}{p^3} + \frac{X_{22}}{p^4} \right] \right\}$$

$$\bar{\sigma}_{\varphi\varphi} = \frac{2a_1}{D_{C1}R} \left\{ -e^{-m_1(R-1)} \left[\frac{Y_{11}}{p^3} + \frac{Y_{12}}{p^4} \right] + e^{-m_2(R-1)} \left[\frac{Y_{21}}{p^3} + \frac{Y_{22}}{p^4} \right] \right\}$$

where, we take the expression for the heating effect $F(\eta)$ of the boundary condition (11) as:

$$F(\eta) = \eta^2 e^{-\alpha\eta} \quad \text{for } \eta \geq 0, \alpha > 0$$

(the positive constant α being responsible for the velocity of changes in temperature on the boundary).

In the above solutions the following notations for $i = 1, 2$ have been used:

$$F_{i1} = \frac{(a_i^0 - 1)}{a_i^0}, \qquad F_{i2} = \frac{a_i^0}{\left(a_i^1\right)^2}, \quad i = 1, 2$$

$$S_{i1} = 1 - F_{i1}, \qquad S_{i2} = \frac{2(1-\lambda_1)}{b_i^0 R} - \left(F_{i2} + \frac{1}{2} \frac{b_i^1}{b_i^0} S_{i1} \right), \quad i = 1, 2$$

$$S_{ij}^0 = S_{ij}\big|_{R=1}, \quad i, j = 1, 2$$

$$D_1 = S_{11}^0 F_{21} - S_{21}^0 F_{11}$$

$$D_2 = S_{11}^0 F_{22} + S_{12}^0 F_{21} - S_{22}^0 F_{11} - S_{21}^0 F_{12} - \frac{1}{2} \left[S_{11}^0 F_{21} \frac{b_2^1}{b_2^0} - S_{21}^0 F_{11} \frac{b_1^1}{b_1^0} \right]$$

$$B_{11} = S_{21}^0, \qquad B_{21} = S_{11}^0, \qquad B_{12} = S_{22}^0 - 3\alpha S_{21}^0 - \frac{D_2}{D_1} S_{21}^0, \qquad B_{22} = S_{12}^0 - 3\alpha S_{11}^0 - \frac{D_2}{D_1} S_{11}^0$$

$$U_{i1} = \frac{\alpha_2' B_{i1}}{b_i^0}, \qquad U_{i2} = \frac{B_{i1}}{\left(b_i^0\right)^2} \left[\alpha_2' \left(\frac{1}{R} - \frac{b_i^1}{2} - b_i^1 \right) + b_i^0 \right] + B_{i2} \frac{\alpha_2'}{b_i^0}, \quad i = 1, 2$$

$$Z_{i1} = F_{i1} B_{i1}, \qquad Z_{i2} = \left(F_{i2} - \frac{b_i^1}{2b_i^0} \right) B_{i1} + F_{i1} B_{i2}, \quad i = 1, 2$$

$$S_{i1}^R = S_{i1} \alpha_2' B_{i1}, \qquad S_{i2}^R = B_{i1}(S_{i1} + \alpha_2' S_{i2}) + B_{i2} \alpha_2' S_{i1}, \quad i = 1, 2$$

$$S_{i1}^\varphi = \varphi_{i1} \alpha_2' B_{i1}, \qquad S_{i2}^\varphi = B_{i1}(\varphi_{i1} - \alpha_2' \varphi_{i2}) + B_{i2} \alpha_2' \varphi_{i1}, \quad i = 1, 2$$

$$D_{C1} = \frac{S_{11}^0 - F_{11}}{\left(a_2^1\right)^2}$$

$$D_{C2} = F_{21} S_{12}^0 + \frac{1}{2\left(a_2^1\right)^2} \left[\frac{\left(v_2 b_1^0 - v_1 b_1^0 \left(a_2^1\right)^2 - a_2^2 v_1 b_1^0 \right) S_{11}^0 - b_1^0 F_{11}(2a_2^2 - v_2) - 2v_1 b_1^0 F_{12} + v_1 b_1^1 F_{11}}{v_1 b_1^0} \right]$$

$$U_{C11} = \left(a_2^1\right)^{\frac{-5}{2}}, \qquad U_{C12} = \left(a_2^1\right)^{\frac{-5}{2}} \left[\left(\frac{2 - b_1^1 R}{2b_1^0 R \left(a_2^1\right)^{\frac{1}{2}}} \right) + \frac{(2a_2^2 - v_2) D_{C1} - 6\alpha D_{C1} v_1 - 2v_1 D_{C2}}{2b_1^0 v_1 D_{C1}} - \frac{b_1^1}{\left(b_1^0\right)^2} \right]$$

$$U_{C21} = \left(a_2^1\right)^{-\frac{1}{2}} S_{11}^0, \qquad U_{C22} = \left(a_2^1\right)^{-\frac{1}{2}} S_{11}^0 \frac{1}{Rv_1}$$

$$Z_{C11} = \frac{F_{11}}{\left(a_2^1\right)^2}, \qquad Z_{C12} = \frac{F_{11}}{2(a_2^1)^2} \left[\frac{(2a_2^2 - v_1) D_{C1} - 6\alpha D_{C1} v_1 - 2v_1 D_{C2}}{v_1 D_{C1}} - \frac{b_1^1}{b_1^0} \right] + \frac{F_{12}}{\left(a_2^1\right)^2}$$

$$Z_{C21} = -\frac{S_{11}^0}{\left(a_2^1\right)^2}, \qquad Z_{C22} = \frac{1}{2\left(a_2^1\right)^2}\left[\left[\frac{v_2}{v_1} + 6\alpha + \frac{2D_{C2}}{D_{C1}} + 2\left[\left(a_2^1\right)^2 + a_2^2\right]\right] S_{11}^0 - 2S_{12}^0\right]$$

$$X_{11} = \frac{S_{11}}{\left(a_2^1\right)^2}, \qquad X_{12} = \frac{S_{12}}{\left(a_2^1\right)^2} + \frac{(2a_2^2 - v_2)D_{C1} - 6\alpha D_{C1} v_1 - 2v_1 D_{C2}}{2(a_2^1)^2 v_1 D_{C1}} S_{11}$$

$$X_{21} = \frac{S_{11}^0}{\left(a_2^1\right)^2}, \qquad X_{22} = \frac{1}{\left(a_2^1\right)^2} S_{11}^0 \left[\frac{2a_2^2 - v_2}{2v_1} - 3\alpha - \frac{D_{C2}}{D_{C1}}\right] + \frac{1}{\left(a_2^1\right)^2} S_{12}^0$$

$$Y_{11} = \frac{\varphi_{11}}{\left(a_2^1\right)^2}, \qquad Y_{12} = \frac{(2a_2^2 - v_2)D_{C1} - 6\alpha D_{C1} v_1 - 2v_1 D_{C2}}{2(a_2^1)^2 v_1 D_{C1}} \varphi_{11} - \frac{\varphi_{12}}{\left(a_2^1\right)^2}$$

$$Y_{21} = \frac{S_{11}^0}{\left(a_2^1\right)^2}, \qquad Y_{22} = \frac{1}{2\left(a_2^1\right)^2 v_1}\left[2(\lambda_1 - 1) v_1 \left(a_2^1\right)^2 + 2a_2^2 v_1 + v_2 \left(a_2^1\right)^2 + 2v_1 S_{11}^0\right].$$

5.1. Laplace inversion

For the inversion of the Laplace transforms obtained above we use the convolution theorem of the Laplace transform and the following formulae [25].

$$L^{-1}\left[\frac{e^{-\frac{a}{p}}}{p^{v+1}}\right] = \left(\frac{t}{a}\right)^{v/2} J_v\left(2\sqrt{at}\right), \quad \mathrm{Re}(v) > -1, a > 0$$

$$L^{-1}\left[\frac{e^{\frac{a}{p}}}{p^{v+1}}\right] = \left(\frac{t}{a}\right)^{\frac{v}{2}} I_v\left(2\sqrt{at}\right), \quad \mathrm{Re}(v) > -1, a > 0$$

$$L^{-1}\left[\frac{e^{-a\sqrt{p}}}{p^{\frac{v}{2}+1}}\right] = (4t)^{\frac{v}{2}}\, i^v \mathrm{erfc}\left(\frac{a}{2\sqrt{t}}\right), \quad v = 0, 1, 2, \ldots$$

where, J_v and I_v are the Bessel function and modified Bessel functions of order v and of the first kind respectively.

It can be proved that $b_1^2 > 0$ and $b_2^2 < 0$ for the cases of ETE & TRDTE, $b_1^1 = b_2^1 = b_1^2 = b_2^2 = 0$ for the case of TEWOED and $b_2^0 = 0, b_2^1 \to \infty, b_2^2 \to \infty$ for the case of CTE. Therefore the final solutions for the field variables in the physical domain (R, η) are obtained as follows:

(I) For the case of ETE & TRDTE:

$$U = \frac{2a_1}{RD_1}\sum_{j=1}^{2}\left\{e^{-b_1^1(R-1)}\left[\frac{\eta_1}{b_1^2(R-1)}\right]^{\frac{j+1}{2}} J_{j+1}(z_1) H(\eta_1) U_{1j} - e^{-b_2^1(R-1)}\left[\frac{\eta_2}{-b_2^2(R-1)}\right]^{\frac{j+1}{2}} I_{j+1}(z_2) H(\eta_2) U_{2j}\right\}$$

$$Z = \frac{2}{RD_1}\sum_{j=1}^{2}\left\{-e^{-b_1^1(R-1)}\left[\frac{\eta_1}{b_1^2(R-1)}\right]^{\frac{j+1}{2}} J_{j+1}(z_1) H(\eta_1) Z_{1j} + e^{-b_2^1(R-1)}\left[\frac{\eta_2}{-b_2^2(R-1)}\right]^{\frac{j+1}{2}} I_{j+1}(z_2) H(\eta_2) Z_{2j}\right\}$$

$$\sigma_{RR} = \frac{2a_1}{RD_1}\sum_{j=1}^{2}\left\{-e^{-b_1^1(R-1)}\left[\frac{\eta_1}{b_1^2(R-1)}\right]^{\frac{j}{2}} J_j(z_1) H(\eta_1) S_{1j}^R + e^{-b_2^1(R-1)}\left[\frac{\eta_2}{-b_2^2(R-1)}\right]^{\frac{j}{2}} I_j(z_2) H(\eta_2) S_{2j}^R\right\}$$

$$\sigma_{\varphi\varphi} = \frac{2a_1}{RD_1}\sum_{j=1}^{2}\left\{-e^{-b_1^1(R-1)}\left[\frac{\eta_1}{b_1^2(R-1)}\right]^{\frac{j}{2}} J_j(z_1) H(\eta_1) S_{1j}^\varphi + e^{-b_2^1(R-1)}\left[\frac{\eta_2}{-b_2^2(R-1)}\right]^{\frac{j}{2}} I_j(z_2) H(\eta_2) S_{2j}^\varphi\right\}.$$

(II) For the case of TEWOED:

$$U = \frac{2a_1}{RD_1}\sum_{j=1}^{2}\left\{\frac{\eta_1^{j+1}}{(j+1)!} H(\eta_1) U_{1j} - \frac{\eta_2^{j+1}}{(j+1)!} H(\eta_2) U_{2j}\right\}$$

$$Z = \frac{2}{RD_1}\sum_{j=1}^{2}\left\{-\frac{\eta_1^{j+1}}{(j+1)!} H(\eta_1) Z_{1j} + \frac{\eta_2^{j+1}}{(j+1)!} H(\eta_2) Z_{2j}\right\}$$

$$\sigma_{RR} = \frac{2a_1}{RD_1}\sum_{j=1}^{2}\left\{-\frac{\eta_1^{j}}{(j)!} H(\eta_1) S_{1j}^R + \frac{\eta_2^{j}}{(j)!} H(\eta_2) S_{2j}^R\right\}$$

$$\sigma_{\varphi\varphi} = \frac{2a_1}{RD_1}\sum_{j=1}^{2}\left\{-\frac{\eta_1^{j}}{(j)!} H(\eta_1) S_{1j}^\varphi + \frac{\eta_2^{j}}{(j)!} H(\eta_2) S_{2j}^\varphi\right\}.$$

(III) For the case CTE:

$$U = \frac{2a_1}{D_{C1}R}\left\{e^{-b_1^1(R-1)}H(\eta_1)\left[\frac{(\eta_1)^3}{6}U_{C11} + \frac{(\eta_1)^4}{24}U_{C12}\right] - \left[(4\eta)^{\frac{7}{2}}i^7\mathrm{erfc}(\eta_3)U_{C21} + (4\eta)^4 i^8\mathrm{erfc}(\eta_3)U_{C22}\right]\right\}$$

$$Z = \frac{2}{D_{C1}R}\left\{e^{-b_1^1(R-1)}H(\eta_1)\left[\frac{(\eta_1)^2}{2}Z_{C11} + \frac{(\eta_1)^3}{6}Z_{C12}\right] + \left[(4\eta)^2 i^4\mathrm{erfc}(\eta_3)Z_{C21} + (4\eta)^3 i^6\mathrm{erfc}(\eta_3)Z_{C22}\right]\right\}$$

$$\sigma_{RR} = \frac{2a_1}{D_{C1}R}\left\{e^{-b_1^1(R-1)}H(\eta_1)\left[\frac{(\eta_1)^2}{2}X_{11} + \frac{(\eta_1)^3}{6}X_{12}\right] + \left[(4\eta)^2 i^4\mathrm{erfc}(\eta_3)X_{21} + (4\eta)^3 i^6\mathrm{erfc}(\eta_3)X_{22}\right]\right\}$$

$$\sigma_{\varphi\varphi} = \frac{2a_1}{D_{C1}R}\left\{e^{-b_1^1(R-1)}H(\eta_1)\left[\frac{(\eta_1)^2}{2}Y_{11} + \frac{(\eta_1)^3}{6}Y_{12}\right] + \left[(4\eta)^2 i^4\mathrm{erfc}(\eta_3)Y_{21} + (4\eta)^3 i^6\mathrm{erfc}(\eta_3)Y_{22}\right]\right\}$$

where,

$$\eta_i = \eta - b_i^0(R-1), \qquad z_i = 2\sqrt{(-1)^{i+1} b_i^2 (R-1)\left[\eta - b_i^0(R-1)\right]}, \quad \text{for } i = 1, 2$$

$$\eta_3 = \frac{v_1(R-1)}{2\sqrt{\eta}}$$

and erfc(x) is the associated complementary error function of nth degree defined by:

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2}\mathrm{d}u,$$

$$i^n\mathrm{erfc}(x) = \int_x^{\infty} i^{n-1}\mathrm{erfc}(\xi)\mathrm{d}\xi, \quad n = 1, 2, \ldots$$

$$i^0\mathrm{erfc}(x) = \mathrm{erfc}(x).$$

From the solutions obtained as above we observe that solutions for the distributions of displacement, temperature and stresses in the cases of ETE, TRDTE and TEWOED consist of two coupled waves propagating with the finite speeds $\frac{1}{b_i^0}$, for $i = 1, 2$. The wave propagating with speed $\frac{1}{b_1^0}$ is predominantly an elastic wave and the wave propagating with the speed $\frac{1}{b_2^0}$ is predominantly a thermal wave. The solutions of all the variables are continuous in nature. However, the analytical solutions of the field variables indicate that as compared to the cases of ETE and TRDTE, in the case of TEWOED the waves propagate without attenuation, which is obviously a characteristic feature of this theory.

For the case of CTE, it is noticed from the expressions of b_i^0 that $b_2^0 = 0$, which implies that the thermal wave propagates with infinite speed in the case of CTE. For this case the solutions of all the variables consist of two parts. The first parts involving the term $H(\eta)$ represent the contribution due to the elastic wave front traveling with finite speed $\frac{1}{b_1^0}$ with exponential attenuation, whereas the second parts of the solutions are of diffusive nature which is because of the parabolic nature of heat transport equation for this case.

6. Numerical results and conclusions

Now for the illustration of the problem we follow the numerical method proposed by Bellman et al. [1] (see the Appendix) for the inversion of Laplace transforms and compute the numerical values of physical quantities like displacement, temperature and stresses from Eqs. (19) and (22)–(24) for different values of R ($R > 1$) (programming language C++ has been used here). The results are plotted in Figs. 1–4. The material chosen is the copper material, the physical data for which is given as:

$$\varepsilon = 0.0168, \qquad \lambda = 1.387 \times 10^{12}\ \text{dyne/cm}^2, \qquad \mu = 0.448 \times 10^{12}\ \text{dyne/cm}^2.$$

We take $\alpha_1' = 0.01$, $\alpha_2' = 0.02$.

In case of TEWOED theory, K^* is a material constant, characteristic of the theory. We have chosen $K^* = \frac{c_v(\lambda+2\mu)}{4}$ (Chandrasekharaiah and Srinath [26]).

Figs. 1–4 represent the radial distributions of displacement, temperature, radial stress and circumferential stress for four theories of thermoelasticity (i.e., (TEWOED), (ETE), (TRDTE) and (CTE)) at the non-dimensional time $\eta = 0.5$ and $\alpha = 2$. It is observed from these figures that the distinction of (TEWOED) from other three theories is very prominent for all the

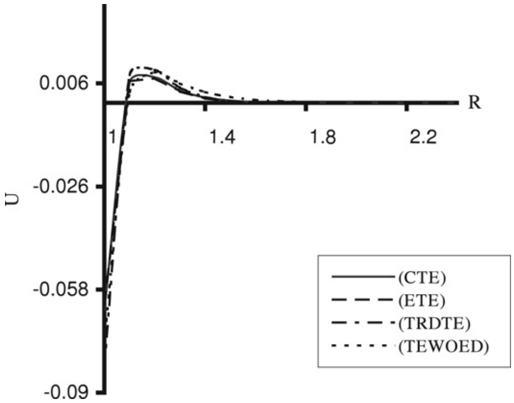

Fig. 1. Radial distribution of displacement at $\eta = 0.5$.

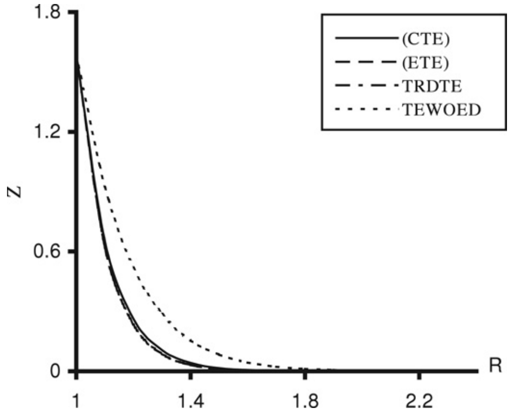

Fig. 2. Radial distribution of temperature at $\eta = 0.5$.

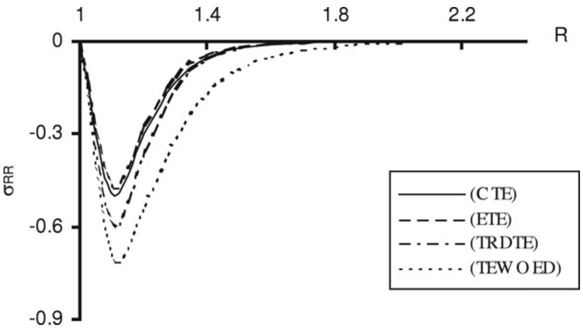

Fig. 3. Radial distribution of radial stress at $\eta = 0.5$.

field variables. The temperature field shows no significant distinction for cases of (ETE) and (TRDTE), but the disagreement of these two theories with (CTE) and (TEWOED) is significant. The values of all the variables in the case of (TEWOED) are very high as compared to the values in the cases of other theories.

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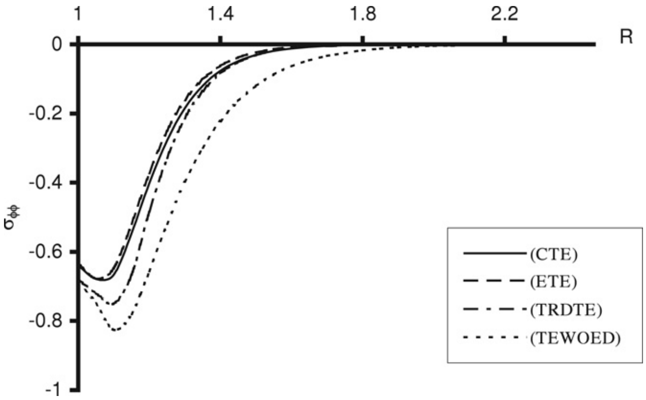

Fig. 4. Radial distribution of circumferential stress at $\eta = 0.5$.

Appendix

Let the Laplace transform of the function $u(t)$ be given by

$$\bar{F}(p) = \int_0^\infty u(t)\mathrm{e}^{-pt}\mathrm{d}t, \quad p > 0. \tag{A.1}$$

We assume that $u(t)$ is sufficiently smooth to permit the approximate method we apply.

Putting $x = \mathrm{e}^{-t}$ in Eq. (A.1) we get

$$\bar{F}(p) = \int_0^1 x^{p-1} g(x)\mathrm{d}x \tag{A.2}$$

where $u(-\log x) = g(x)$.

Applying the Gaussian quadrature formula to (A.2) yields

$$\sum_{i=1}^{N} W_i x_i^{p-1} g(x_i) = \bar{F}(p) \tag{A.3}$$

where x_i 's $(i = 1, 2, 3 \dots N)$ are the roots of the shifted Legendre polynomial $P_N(x) = 0$ and W_i's $(I = 1, 2, 3 \dots N)$ are the corresponding weights.

Eq. (A.3) can be written as :

$$W_1 x_1^{p-1} g(x_1) + W_2 x_2^{p-1} g(x_2) + W_3 x_3^{p-1} g(x_3) + \cdots W_N x_N^{p-1} g(x_N) = \bar{F}(p). \tag{A.4}$$

We now put $p = 1, 2, \dots, N$ in Eq. (A.4), then the resulting equations become

$$\begin{aligned} &W_1 g(x_1) + W_2 g(x_2) + W_3 g(x_3) + \cdots W_N g(x_N) = \bar{F}(1) \\ &W_1 x_1 g(x_1) + W_2 x_2 g(x_2) + W_3 x_3 g(x_3) + \cdots W_N x_N g(x_N) = \bar{F}(2) \\ &\dots \\ &W_1 x_1^{N-1} g(x_1) + W_2 x_2^{N-1} g(x_2) + W_3 x_3^{N-1} g(x_3) + \cdots W_N x_N^{N-1} g(x_N) = \bar{F}(N). \end{aligned} \tag{A.5}$$

Therefore from (A.5) we get $g(x_i)$ as:

$$\begin{bmatrix} g(x_1) \\ g(x_2) \\ \dots \\ g(x_N) \end{bmatrix} = \begin{bmatrix} W_1 & W_2 & \dots & W_N \\ W_1 x_1 & W_2 x_2 & \dots & W_N x_N \\ \dots & \dots & \dots & \dots \\ W_1 x_1^{N-1} & W_2 x_2^{N-1} & \dots & W_N x_N^{N-1} \end{bmatrix}^{-1} \begin{bmatrix} \bar{F}(1) \\ \bar{F}(2) \\ \dots \\ \bar{F}(N) \end{bmatrix}. \tag{A.6}$$

Therefore $g(x_1), g(x_2), \dots, g(x_N)$ are known.

For $N = 7$ we have:

The roots of the shifted Legendre polynomial	Corresponding weights
$x_1 = 0.02555604382862$	0.06474248308443
$x_2 = 0.12923440720030$	0.13985269574463
$x_3 = 0.29707742431130$	0.19091502525255
$x_4 = 0.500000000000000$	0.20897959183673
$x_5 = 0.702922575688698$	0.190915025252559
$x_6 = 0.870765592799697$	0.139852695744638
$x_7 = 0.974553956171379$	0.064742483084435.

From equations in (A.6) we can calculate the discrete values of $g(x_i)$ i.e. $u(t_i)$ and finally using interpolation we obtain the displacement component $u(t)$.

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