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# Generalized Continuous Nondifferentiable Fractional Programming Problems with Invexity

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The concept of invexity has allowed the convexity requirements in a variety of mathematical programming problems to be weakened. We extend a number of Kuhn–Tucker type sufficient optimality criteria for a class of continuous nondifferentiable minmax fractional programming problems that involves several ratios in the objective with a nondifferentiable term in the numerators. As an application of these optimality results, various Mond–Weir type duality results are proved under a variety of generalized invexity assumptions. These results extend many well-known duality results and also give a dynamic generalization of those of finite dimensional nonlinear programming problems recently explored. © 1995 Academic Press, Inc.

## 1. INTRODUCTION

Duality for a class of nondifferentiable mathematical programming was studied first by Mond [8]; subsequently Chandra *et al.* [3] weakened the convexity requirements for duality by giving a Mond–Weir type dual and assuming that the objective function is pseudo-convex. Further, Mond and Smart [11] established duality results for a class of nondifferentiable programming problems with invexity assumptions in the single objective case, which extends an earlier work of Chandra *et al.* [2].

Recently, Mond *et al.* [10] established duality results for nondifferentiable multiobjective programs with convexity assumptions. Mukherjee and Mishra [12] weakened the convexity requirements and extended the work of [10] for the case of multiobjective variational problems.

Crouzeix *et al.* [7] obtained duality results for generalized minmax fractional programming involving several ratios in the objective. Bector *et al.*



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[1] used a parametric approach to establish duality theorems for minmax fractional programming problems under convexity assumptions, which extends some part of an earlier work of Crouzeix *et al.* [6].

The purpose of this paper is to establish sufficient optimality criteria and duality theorems for nondifferentiable minmax fractional programming problems with a variety of invexity assumptions. This work extends the work of Bector *et al.* [1] to the nondifferentiable case along with relaxation of the convexity requirements also.

## 2. PREREQUISITES AND MAIN PROBLEM

Let $I = [a, b]$ be a real interval, $\phi: I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, and $g: I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function. In order to consider $\phi(t, x, \dot{x})$, where $x: I \to \mathbb{R}^n$ is differentiable with derivative $\dot{x}$, denote the partial derivatives of $\phi$ by $\phi_t$,

$$\phi_x = [\partial\phi/\partial x^1, \ldots, \partial\phi/\partial x^p], \qquad \phi_{\dot{x}} = [\partial\phi/\partial\dot{x}^1, \ldots, \partial\phi/\partial\dot{x}^p].$$

The partial derivatives of other functions used will be written similarly. Denote by $X$ the space of piecewise smooth functions $x: I \to \mathbb{R}^n$ with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator is given by $u = Dx$, $x = \alpha + \int_a^t u(s)\, ds$, where $a$ is a value given at the boundary, thus giving $D = d/dt$ except at discontinuities. Let

$$J[x] = \int_a^b \phi(t, x(t), \dot{x}(t))\, dt$$

be Fréchet differentiable. For notational simplicity we shall write, as and when necessary, $x(t)$ and $\dot{x}(t)$ as $x$ and $\dot{x}$, respectively, and so on.

We now give some definitions from [9] that we shall use in the sequel. At a point $u \in X$ we define a functional $J$ to be:

(i) *Invex* with respect to $\eta$ if there exists a differentiable vector function $\eta(t, x, u)$ with $\eta(t, x, x) = 0$ such that for all $x \in X$

$$J[x] - J[u] \geq \int_a^b \left\{ \eta(t, x, u) f_x(t, u, \dot{u}) + \left(\frac{d}{dt}\eta(t, x, u)\right) f_{\dot{x}}(t, u, u) \right\} dt,$$

or strict invex if strict inequality holds.

(ii) *Pseudoinvex* $(PIX)$ with respect to $\eta$ if there exists a differentiable vector function $\eta(t, x, u)$ with $\eta(t, x, x) = 0$ such that for $x \in X$

$$\int_a^b \left\{ \eta(t,x,u) f_x(t,u,\dot{u}) + \left(\frac{d}{dt}\eta(t,x,u)\right) f_{\dot{x}}(t,u,\dot{u}) \right\} dt \geq 0 \Rightarrow J[x] \geq J[u]$$

or equivalently,

$$J[x] < J[u] \Rightarrow \int_a^b \left\{ \eta(t,x,u) f_x(t,u,\dot{u}) + \left(\frac{d}{dt}\eta(t,x,u)\right) f_{\dot{x}}(t,u,\dot{u}) \right\} dt < 0.$$

(iii) *Strictly Pseudoinvex* (*SPIX*) with respect to $\eta$ if there exists a differentiable vector function $\eta(t, x, u)$ with $\eta(t, x, x) = 0$ such that for all $x \in X$

$$\int_a^b \left\{ \eta(t,x,u) f_x(t,u,\dot{u}) + \left(\frac{d}{dt}\eta(t,x,u)\right) f_{\dot{x}}(t,u,\dot{u}) \right\} dt \geq 0 \Rightarrow J[x] > J[u]$$

or equivalently,

$$J[x] \leq J[u] \Rightarrow \int_a^b \left\{ \eta(t,x,u) f_x(t,u,\dot{u}) + \left(\frac{d}{dt}\eta(t,x,u)\right) f_{\dot{x}}(t,u,\dot{u}) \right\} dt < 0.$$

(iv) *Quasi-invex* (*QIX*) with respect to $\eta$ if there exists a differentiable vector function $\eta(t, x, u)$ with $\eta(t, x, x) = 0$ such that

$$\int_a^b \left\{ \eta(t,x,u) f_x(t,u,\dot{u}) + \left(\frac{d}{dt}\eta(t,x,u)\right) f_{\dot{x}}(t,u,\dot{u}) \right\} dt > 0 \Rightarrow J[x] \geq J[u]$$

or equivalently,

$$J[x] < J[u] \Rightarrow \int_a^b \left\{ \eta(t,x,u) f_x(t,u,\dot{u}) + \left(\frac{d}{dt}\eta(t,x,u)\right) f_{\dot{x}}(t,u,\dot{u}) \right\} dt \leq 0.$$

This QIX is equivalent to QIX of Mond and Husain [9], which can be seen from [1]. In the above definitions, $d\eta/dt$ is the vector whose $i$th component is $(d/dt)\eta^i(t, x, u)$. Here if $f$ is independent of $t$ and $\eta(t, x, u) = (x - u)$, definitions (i)–(iv) reduce to convexity, pseudoconvexity, strict pseudoconvexity, and quasiconvexity.

We now consider the following generalized continuous nondifferentiable minmax fractional programming problem:

*Primal Problem*
(P)

$$v^* = \min_{x} \max_{1\le i\le p} \left\{ \frac{\int_a^b [f^i(t,x,\dot{x}) + (x^T B_i(t) x)^{1/2}]\, dt}{\int_a^b h^i(t,x,\dot{x})\, dt} \right\}$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta \tag{1}$$

$$g^j(t,x,\dot{x}) \le 0, \qquad t \in I, j = 1,2,\ldots,m, \tag{2}$$

where $\int_a^b h^i(t, x, \dot{x})\, dt > 0$, $i = 1, 2, \ldots, p$, and $x \in C_p$ is the set of feasible solutions of (P). The notation $C_D$ will have a similar meaning for the problem (D). Each $B_i$, $i = 1, \ldots, p$, is an $n \times n$ positive semi-definite (symmetric) matrix.

In view of [1] we consider the following continuous nondifferentiable minmax parametric programming problem in $v$:
($P_v$)

$$\min_{x} \max_{1\le i\le p} \int_a^b [f^i(t,x,\dot{x}) + (x^T B_i(t) x)^{1/2} - v h^i(t,x,\dot{x})]\, dt \tag{3}$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta \tag{4}$$

$$g^j(t,x,\dot{x}) \le 0, \qquad t \in I, j = 1,2,\ldots,m \tag{5}$$

and in the spirit of [1] state the following lemma:

LEMMA 1. *If* (P) *has an optimal solution* $x^*$ *with optimal value of the* (P)*-objective equal to* $v^*$, *then* $F(v^*) = 0$ *and conversely, if* $F(v^*) = 0$, *then* (P) *and* ($P_{v^*}$) *have the same optimal solution set.*

In subsequent analysis we will require the generalized Schwarz inequality [8]

$$x^T B_i w \le (x^T B_i x)^{1/2} (w^T B_i w)^{1/2}, \qquad x, x \in R^n, i = 1, 2, \ldots, p.$$

The following proposition is the analogue of Proposition 3 of [8] in our setting:

PROPOSITION 1. *If $x^* \in X$ is an optimal solution of the primal problem* (P), *then there exist multiples $y^{i*} \in \mathbb{R}_+$, $i = 1, 2, ..., p$, and piecewise smooth $z^{j*}: I \to \mathbb{R}^m_+$, $j = 1, 2, ..., m$, $w^*: I \to \mathbb{R}^n_+$, with ( , ) not all zero, such that*

$$\sum_{i=1}^{p} y^{i*}[f^i_x(t, x^*, \dot{x}^*) + B_i(t)w^* - v^*h^i_x(t, x^*, \dot{x}^*)] + \sum_{j=1}^{m} z^{j*}g^j(t, x^*, \dot{x}^*)$$

$$= D\left\{\sum_{i=1}^{p} y^{i*}[f^i_{\dot{x}}(t, x^*, \dot{x}^*) - v^*h^i_{\dot{x}}(t, x^*, \dot{x}^*)] + \sum_{j=1}^{m} z^{j*}g^{j*}_{\dot{x}}(t, x^*, \dot{x}^*)\right\},$$

*where*

$$z^{j*}g^j(t, x^*, \dot{x}^*) = 0, \qquad t \in I, j = 1, 2, ..., m$$
$$w^{*T}B_i(t)w^* \leq 1, \qquad i = 1, 2, ..., p$$
$$x^{*T}B_i(t)w^* = (x^{*T}B_i(t)x^*(t))^{1/2}, \qquad i = 1, 2, ..., p.$$

## 3. OPTIMALITY CONDITIONS

To derive the optimality conditions and duality we shall make use of ($P_v$). Using (3)–(5) we get the following continuous programming problem, which is equivalent to ($P_v$) for a given $v \in \mathbb{R}$.

$$(\text{EP}_v) \text{ Minimize } q \tag{6}$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta \tag{7}$$

$$\int_a^b \{f^i(t, x, \dot{x}) + (x^T B_i(t)x)^{1/2} - vh^i(t, x, \dot{x})\}\, dt \leq q, \qquad i = 1, ..., p \tag{8}$$

$$g^j(t, x, \dot{x}) \leq 0, \qquad t \in I, j = 1, ..., m. \tag{9}$$

We now state Lemma 2 of [1]:

LEMMA 2 [1]. *$x^*$ is (P)-optimal with the corresponding optimal value of the (P)-objective equal to $v^*$ if and only if $(x^*, v^*, q^*)$ is ($EP_v$)-optimal with the corresponding value of the ($EP_v$)-objective equal to zero, that is, $q = 0$.*

Clarke [4, 5] has given necessary conditions for a simple problem subject to a differential inequality for the form $q(t, \zeta(t), \dot{\zeta}(t)) \leq 0$ in terms of generalized subdifferential $\partial q(t, \zeta(t), \dot{\zeta}(t))$. In fact, the results of (10)–(16) of the following theorem are obtained by putting the problem ($EP_v$) into

Clarke's form. Also, in such a process one would need a representation for the subdifferential. The proof of the necessary part as depicted in Theorem 1, below, applies a known Fritz John theorem for constrained minimization in abstract spaces. In that approach also questions of representation of subdifferential arise.

THEOREM 1 (Necessary optimality condition). *Let $x^*$ be an optimal solution of* (P) *with the optimal value of* (P)*-objective equal to $v^*$. Let the normality condition* [11] *hold. Then there exist $q^* \in \mathbb{R}$, $y^* \in \mathbb{R}^p$, and $z^*$: $I \to \mathbb{R}^m$ piecewise smooth such that $(x^*, y^*, v^*, w^*, z^*)$ satisfies*

$$\sum_{i=1}^{p} y^{i*}(x)[f_x^i(t, x^*, \dot{x}^*) + B_i(t)w^*(t) - v^*h_x^i(t, x, \dot{x}^*)] + \sum_{j=1}^{m} z^{j*}g_x^j(t, x^*, \dot{x}^*)$$
$$= D\left\{\sum_{i=1}^{p} y^{i*}[f_{\dot{x}}^i(t, x^*, \dot{x}^*) - v^*h_{\dot{x}}^i(t, x^*, \dot{x}^*)] + \sum_{j=1}^{m} z^{j*}g_{\dot{x}}^j(t, x^*, \dot{x}^*)\right\} \tag{10}$$

$$\int_a^b y^{i*}[f^i(t, x^*, \dot{x}^*) + B_i(t)w^*(t) - v^*h^i(t, x^*, \dot{x}^*)]\, dt = 0 \qquad \forall i = 1, 2, \ldots, p \tag{11}$$

$$z^{j*}g^j(t, x^*, \dot{x}^*) = 0, \qquad t \in I, j = 1, 2, \ldots, m \tag{12}$$

$$\int_a^b \{f^i(t, x^*, \dot{x}^*) + B_i(t)w^*(t) - v^*h^i(t, x^*, \dot{x}^*)\}\, dt \leq 0 \qquad \forall i = 1, 2, \ldots, p \tag{13}$$

$$g^j(t, x^*, \dot{x}^*) \leq 0, \qquad t \in I, \forall j = 1, 2, \ldots, m \tag{14}$$

$$w^{*T}B_i(t)w^* \leq 1, \qquad t \in I, \forall i = 1, \ldots, p \tag{15}$$

$$\sum_{i=1}^{p} y^{i*} = 1 \tag{16}$$

$$q^* = 0 \tag{17}$$

$$v^* \in \mathbb{R}, \qquad y^* \in \mathbb{R}^p, \qquad z^* \in \mathbb{R}^m, \qquad y^*, z^* \geq 0, \qquad t \in I. \tag{18}$$

*Proof.* Since $x^*$ is (P)-optimal with the corresponding optimal value of the (P)-objective equal to $v^*$, therefore, by Lemma 2, $(x^*, v^*, q^*)$ is $(\mathrm{EP}_v)$-optimal with the corresponding optimal value of the $(\mathrm{EP}_v)$-objective equal to zero. The theorem now follows by applying the Kuhn–Tueker condition and Proposition 1 at $(x^*, v^*, q^*)$ to $(\mathrm{EP}_v)$. ∎

THEOREM 2 (Sufficient optimality conditions). *Let $(x^*, v^*, y^*, w^*, z^*)$ satisfy* (10)–(18) *and at $x^*$ let*

$$\theta(x) = \sum_{i=1}^{p} y^{i*} \int_a^b [f^i(t,x,\dot{x}) + \cdot^T B_1(t)w - v^* h^i(t,x,\dot{x})]\, dt \text{ be } PIX$$

*and*

$$\sum_{j=1}^{m} \int_a^b z^{j*} g^j(t,x,\dot{x})\, dt \text{ be } QIX \qquad \text{for all } x \in {}^C P_{v^*}.$$

*Then $x^*$ is* (P)*-optimal with the corresponding optimal value $v^*$.*

*Proof.* From (14), $x^*$ is (P)-feasible and by (13) and (14), $x^* \in {}^C P_{v^*}$. From (9), $x^* \in {}^C EP_{v^*}$ (i.e., $x^* \in {}^C P$) we have

$$\int_a^b \sum_{j=1}^{m} z^{j*} g^j(t,x,\dot{x})\, dt \leq \int_a^b \sum_{j=1}^{m} z^{j*} g^j(t,x^*,\dot{x}^*)\, dt.$$

Then quasi-invexity of $\sum_{j=1}^{m} \int_a^b z^{j*} g^j(t,x,\dot{x})\, dt$ gives

$$\int_a^b \left\{ \eta(t,x,x^*)^T \sum_{j=1}^{m} z^{j*} g_x^j(t,x^*,\dot{x}^*) + (D\eta(t,x,x^*))^T \times \left[ \sum_{j=1}^{m} z^{j*} g_{\dot{x}}^j(t,x^*,\dot{x}^*) \right] \right\} dt \leq 0.$$

Therefore

$$\int_a^b \eta(t,x,x^*)^T \left[ \sum_{j=1}^{m} z^{j*} g_x^j(t,x^*,\dot{x}^*) \right] dt + \eta(t,x,x^*)^T \sum_{j=1}^{m} z^{j*} g_{\dot{x}}^j(t,x^*,\dot{x}^*)\Big|_{t=a}^{t=b} - \int_a^b \eta(t,x,x^*)^T D \left[ \sum_{j=1}^{m} z^{j*} g_{\dot{x}}^j(t,x^*,\dot{x}^*) \right] dt \leq 0,$$

by integration by parts.

That is,

$$\int_a^b \eta(t,x,x^*)^T \left\{ \sum_{j=1}^{m} z^{j*}(t) g_x^j(t,x^*,\dot{x}^*) - D\left[ \sum_{j=1}^{m} z^{j*}(t) g_{\dot{x}}^j(t,x^*,\dot{x}^*) \right] \right\} dt \leq 0,$$

the integrand term is zero since $\eta(t, x, x^*) = 0$ at $t = a$ and $b$. Now, from (10), we have

$$\int_a^b \eta(t, x, x^*)^T \left\{ \sum_{i=1}^{p} y^{i*}[f_x^i(t, x^*, \dot{x}^*) + B_i(t)w^*(t) - v^* h_x^i(t, x^*, \dot{x}^*)] - D\left[ \sum_{i=1}^{p} y^{i*}[f_{\dot{x}}^i(t, x^*, \dot{x}^*) - v^* h_{\dot{x}}^i(t, x^*, \dot{x}^*)] \right] \right\} dt$$

$$= -\int_a^b \eta(t, x, x^*)^T \left\{ \sum_{j=1}^{m} z^{j*} g_x^j(t, x^*, \dot{x}^*) - D\left[ \sum_{j=1}^{m} z^{j*} g_{\dot{x}}^j(t, x^*, \dot{x}^*) \right] \right\} dt \leq 0.$$

Then

$$\int_a^b \left\{ \eta(t, x, x^*)^T \sum_{j=1}^{p} y^{i*}[f_x^i(t, x^*, \dot{x}^*) + B_i(t)w^* - v^* h_x^i(t, x^*, \dot{x}^*)] + (D\eta(t, x, x^*))^T \left\{ \sum_{i=1}^{p} y^{i*}[f_{\dot{x}}^i(t, x^*, \dot{x}^*) - v^* h_{\dot{x}}^i(t, x^*, \dot{x}^*)] \right\} \right\} dt \geq 0.$$

using integration by parts with integrated term zero. By pseudoinvexity of $\theta(x)$ we now have

$$\sum_{i=1}^{p} y^{i*} \int_a^b [f^i(t, x, \dot{x}) + \cdot^T B_i(t)w - v^* h^i(t, x, \dot{x})]\, dt$$

$$\geq \sum_{i=1}^{p} y^{i*} \int_a^b f^i(t, x^*, \dot{x}^*) + \cdot^T B_i(t)w^* - v^* h^i(t, x^*, \dot{x}^*)]\, dt.$$

Using (18), (8), and (16) on the LHS and (11) and (17) on the RHS in the above inequality, we get

$$q \geq 0 = q^*, \qquad x \in {}^C EP_{v^*},$$

and using this with Lemma 2, we have the result. ∎

THEOREM 3 (Sufficient optimality conditions). *Let* $(x^*, y^*, v^*, z^*)$ *satisfy* (10)–(18) *and at* $x^*$ *let*

$$\theta_1(x) = \sum_{i=1}^{p} y^{i*}[f^i(t, x, \dot{x}) - v^* h^i(t, x, \dot{x})]\, dt \text{ be } QIX$$

*and*

$$G(x) = \sum_{j=1}^{m} \int_a^b z^{j*} g^j(t, x, \dot{x})\, dt \text{ be } SPIX \qquad \text{for all } x \in {}^C EP_{v^*}.$$

*Then $x^*$ is optimal to* (P) *with the corresponding optimal objective value to $v^*$.*

*Proof.* From (13), (14), $x^* \in {}^C EP_{v^*}$ and from (14), $x^* \in {}^C P$. Now for any $x \in {}^C EP_{v^*}$ (and hence $x \in {}^C P$), we have as in Theorem 2,

$$\sum_{j=1}^{m} z^{j*} g^j(t, x, \dot{x}) \le \sum_{j=1}^{m} z^{j*} g^j(t, x^*, \dot{x}^*) \Rightarrow \sum_{j=1}^{m} \int_a^b z^{j*} g^j(t, x, \dot{x})\, dt$$
$$\le \sum_{j=1}^{m} \int_a^b z^{j*} g^j(t, x^*, \dot{x}^*)\, dt \le 0.$$

Using strict-pseudoinvexity of $G(x)$ at $x^*$, we get

$$\int_a^b \eta(t, x, x^*)^T \sum_{j=1}^{m} z^{j*} g_x^j(t, x^*, \dot{x}^*) + (D\eta(t, x, x^*))^T \sum_{j=1}^{m} z^{j*} g_{\dot{x}}^j(t, x^*, \dot{x}^*) \Big\}\, dt < 0.$$

Therefore

$$\int_a^b \eta(t, x, x^*)^T \left[\sum_{j=1}^{m} z^{j*} g_x^j(t, x^*, \dot{x}^*)\right] dt + \eta(t, x, x^*)^T \sum_{j=1}^{m} z^{j*} g_{\dot{x}}^j(t, x^*, \dot{x}^*)\Big|_{t=a}^{t=b}$$
$$- \int_a^b \eta(t, x, x^*)^T D\left[\sum_{j=1}^{m} z^{j*} g_{\dot{x}}^j(t, x^*, \dot{x}^*)\right] dt < 0,$$

by integration by parts.

That is,

$$\int_a^b \eta(t, x, x^*)^T \left\{\sum_{j=1}^{m} z^{j*} g_x^j(t, x^*, \dot{x}^*) - D\left[\sum_{j=1}^{m} z^{j*} g_{\dot{x}}^j(t, x^*, \dot{x}^*)\right]\right\} dt < 0;$$

the integrand term is zero since $\eta(t, x, x^*) = 0$ at $t = a$ and $b$.

Now, from (10), we have

$$\int_a^b \eta(t,x,x^*)^T \left\{ \sum_{i=1}^{p} y^{i*}[f_x^i(t,x^*,\dot{x}^*) + B_i(t)w^* - v^* h_x^i(t,x^*,\dot{x}^*)] \right.$$

$$\left. - D\left[ \sum_{i=1}^{p} y^{i*}[f_{\dot{x}}^i(t,x^*,\dot{x}^*) - v^* h_{\dot{x}}^i(t,x^*,\dot{x}^*)] \right] \right\} dt$$

$$= -\int_a^b \eta(t,x,x^*)^T \left\{ \sum_{j=1}^{m} z^{j*} g_x^j(t,x^*,\dot{x}^*) \right.$$

$$\left. - D\left[ \sum_{j=1}^{m} z^{j*} g_{\dot{x}}^j(t,x^*,\dot{x}^*) \right] \right\} dt > 0.$$

Then

$$\int_a^b \left\{ \eta(t,x,x^*)^T \sum_{j=1}^{p} y^{i*}[f_x^i(t,x^*,\dot{x}^*) + B_i(t)w^* - v^* h_x^i(t,x^*,\dot{x}^*)] \right.$$

$$\left. + (D\eta(t,x,x^*))^T \left\{ \sum_{i=1}^{p} y^{i*}[f_{\dot{x}}^i(t,x^*,\dot{x}^*) - v^* h_{\dot{x}}^i(t,x^*,\dot{x}^*)] \right\} \right\} dt > 0,$$

using integration by parts with integrated term zero. Using quasi-invexity of $\theta(x)$ we now have

$$\theta(x^*) \leq \theta(x), \qquad \forall x \in X.$$

Hence the result. ∎

## 4. Duality

In this section, we present two different duals to ($EP_v$) and establish various duality theorems relating to them.

(D-1) Maximize

$$\int_a^b \sum_{i=1}^{p} [y^i\{f^i(t,u,\dot{u}) + u^T B_i(t)w - vh^i(t,u,\dot{u})\}]\, dt + \int_a^b \sum_{j=1}^{m} z^j g^j(t,u,\dot{u})\, dt \tag{19}$$

subject to

$$u(a) = \alpha, \qquad u(b) = \beta \tag{20}$$

$$\sum_{i=1}^{p} y^i\{f_x^i(t,u,\dot{u}) + B_i(t)w(t) - vh_x^i(t,u,\dot{u})\} + \sum_{j=1}^{m} z^j g_x^j(t,u,\dot{u})$$

$$= D\left[\sum_{j=1}^{p} y^i\{f_{\dot{x}}^i(t,u,\dot{u}) - vh_{\dot{x}}^i(t,u,\dot{u})\} + \sum_{j=1}^{m} z^j g_{\dot{x}}^j(t,u,\dot{u})\right] \tag{21}$$

$$w^T B_i(t) w \leq 1, \qquad i = 1, 2, \ldots, p \tag{22}$$

$$\sum_{i=1}^{p} y^i = 1 \tag{23}$$

$$v \in \mathbb{R}, \qquad y \in \mathbb{R}^p, \qquad z \in \mathbb{R}^m, \qquad y, z \geq 0, \qquad t \in I. \tag{24}$$

(D-2) Maximize

$$\int_a^b \sum_{i=1}^{p} y^i\{f^i(t,u,\dot{u}) + u^T B_i(t) w - vh^i(t,u,\dot{u})\}\,dt \tag{25}$$

subject to

$$u(a) = \alpha, \qquad u(b) = \beta \tag{26}$$

$$\sum_{i=1}^{p} y^i\{f_x^i(t,u,\dot{u}) + B_i(t)w - vh_x^i(t,u,\dot{u})\} + \sum_{j=1}^{m} z^j g_x^j(t,u,\dot{u})$$

$$= D\left[\sum_{i=1}^{p} y^i\{f_{\dot{x}}^i(t,u,\dot{u}) - vh_{\dot{x}}^i(t,u,\dot{u})\} + \sum_{j=1}^{m} z^j g_{\dot{x}}^j(t,u,\dot{u})\right] \tag{27}$$

$$\int_a^b z^j g^j(t,u,\dot{u})\,dt \geq 0, \qquad j = 1, 2, \ldots, m \tag{28}$$

$$w^T B_i(t) w \leq 1, \qquad \forall i = 1, 2, \ldots, p, t \in I \tag{29}$$

$$\sum_{i=1}^{p} y^i = 1 \tag{30}$$

$$v \in \mathbb{R}, \qquad y \in \mathbb{R}^p, \qquad z \in \mathbb{R}^m, \qquad y, z \geq 0, \qquad t \in I. \tag{31}$$

We now prove duality theorems relating ($EP_v$) and (D-1).

THEOREM 4 (Weak Duality). *Let*

$$\int_a^b \sum_{i=1}^{p} y^i\{f^i(t,u,\dot u) + \cdot^T B_i(t)w(t) - vh^i(t,u,\dot u)\}\,dt$$

*and*

$$\int_a^b \sum_{j=1}^{m} z^j g^j(t,u,\dot u)\,dt$$

*be invex with respect to* $\eta(t,x,u)$. *Then the infimum of* $(EP_v)$ *is greater than or equal to the supremum of* (D-1).

*Proof.* Let $x$ be feasible for $(\mathrm{EP}_v)$ and $(u, y, w)$ be feasible for (D-1). Now

$$\int_a^b \sum_{i=1}^{p} y^i\{f^i(t,x,\dot x) + x^T B_i(t)x(t) - vh^i(t,x,\dot x)\}\,dt$$

$$- \int_a^b \sum_{i=1}^{p} y^i\{f^i(t,u,\dot u) + u^T B_i(t)w - vh^i(t,u,\dot u)\}\,dt$$

$$- \int_a^b \sum_{j=1}^{m} z^j g^j(t,u,\dot u)\,dt$$

$$\geq \int_a^b \left\{(\eta(t,x,u))^T\left[\sum_{i=1}^{p} y^i\{f_x^i(t,u,\dot u) + B_i(t)w(t) - vh_x^i(t,u,\dot u)\right\}\right.$$

$$\left. + \sum_{j=1}^{m} z^j g_x^j(t,u,\dot u)\right]$$

$$+ (D\eta(t,x,u))^T\left[\sum_{i=1}^{p} y^i\{f_{\dot x}^i(t,u,\dot u) - vh_{\dot x}^i(t,u,\dot u)\}\right.$$

$$\left. + \sum_{j=1}^{m} z^j g_{\dot x}^j(t,u,\dot u)\right\}\,dt$$

$$+ \int_a^b \left[\sum_{i=1}^{p} y^i\{x^T B_i(t)x - x^T B_i(t)w\} - \sum_{j=1}^{m} z^j g^j(t,x,\dot x)\right] dt$$

(by invexity assumptions),

$$= \int_a^b (\eta(t,x,u))^T\left[\sum_{i=1}^{p} y^i\{f_x^i(t,u,\dot u) + B_i(t)w - vh_x^i(t,u,\dot u)\}\right.$$

$$\left. + \sum_{j=1}^{m} z^j g_x^j(t,u,\dot u)\right] dt$$

$$+ (\eta(t,x,u))^T \left[ \sum_{i=1}^{p} y^i\{f_{\dot{x}}^i(t,u,\dot{u}) - vh_{\dot{x}}^i(t,u,\dot{u})\} \right.$$

$$\left. + \sum_{j=1}^{m} z^j g_{\dot{x}}^j(t,u,\dot{u}) \right|_{t=a}^{t=b}$$

$$- \int_a^b (\eta(t,x,u))^T D \left[ \sum_{i=1}^{p} y^i\{f_{\dot{x}}^i(t,u,\dot{u}) - vh_{\dot{x}}^i(t,u,\dot{u})\} \right.$$

$$\left. + \sum_{j=1}^{m} z^j g_{\dot{x}}^j(t,u,\dot{u}) \right] dt$$

$$+ \int_a^b \left[ \sum_{j=1}^{p} y^i\{(x^T B_i(t) x)^{1/2} - x^T B_i(t) w\} - \sum_{j=1}^{m} z^j g^j(t,x,\dot{x}) \right] dt$$

(by integration by parts)

$$= \int_a^b \left[ \sum_{i=1}^{p} y^i\{(x^T B_i(t) x)^{1/2} - x^T B_i(t) w\} - \sum_{j=1}^{m} z^j g^j(t,x,\dot{x}) \right] dt$$

by (21) and since at $t = a$ and $b$, $x = u$ gives $\eta(t, x, u) = 0$,

$$\geq \int_a^b \sum_{i=1}^{p} y^i(t)\{x^T B_i(t) x - x^T B_i(t) w\}\, dt, \qquad \text{by (5), (23), (24)}$$

$$\geq \int_a^b \sum_{i=1}^{p} y^i\{(x^T B_i(t) x)^{1/2} (w^T B_i(t) w)^{1/2} - x^T B_i(t) w\}\, dt, \qquad \text{by (22)}$$

$$\geq 0, \qquad \text{by (23) and Schwarz inequality.}$$

Therefore, $\inf(\text{EP}_v) \geq \sup(\text{D-1})$, by (8) and (23). ∎

THEOREM 5 (Weak Duality). *Let*

$$\int_a^b \sum_{i=1}^{p} y^i\{f^i(t,u,\dot{u}) + \cdot^T B_i(t) w - vh^i(t,u,\dot{u})\}\, dt$$

*be pseudoinvex and* $\int_a^b \sum_{j=1}^{m} z^j g^j(t,u,\dot{u})\, dt$

*be quasi-invex with respect to the same* $\eta(t,x,u)$. *Then the* $\inf(\text{EP}_v)$ *is greater than or equal to the supremum of* (D-1).

*Proof.* Let $x$ be feasible for $(\mathrm{EP}_v)$ and $(u, y, w)$ be feasible for (D-1). Then (5), (9), (13), and (14) imply

$$\int_a^b \sum_{j=1}^m z^j g^j(t, x, \dot{x})\,dt \le \int_a^b \sum_{j=1}^m z^j g^j(t, u, \dot{u})\,dt.$$

Thus, quasi-invexity of

$$\int_a^b \sum_{j=1}^m z^j g^j(t, \cdot, \cdot)\,dt \text{ gives}$$

$$\int_a^b \left\{(\eta(t,x,u))^T\left[\sum_{j=1}^m z^j g_x^j(t,u,\dot{u}) + (D\eta(t,x,u))^T \right.\right.$$
$$\left.\left. - \sum_{j=1}^m z^j g_{\dot{x}}^j(t,u,\dot{u})\right\}dt \le 0.\right.$$

Therefore

$$\int_a^b (\eta(t,x,u))^T\left[\sum_{j=1}^m z^j g_x^j(t,u,u)\right]dt + (\eta(t,x,u))^T \sum_{j=1}^m z^j g_{\dot{x}}^j(t,u,\dot{u})\Big|_{t=a}^{t=b}$$
$$-\int_a^b (\eta(t,x,u))^T D\left[\sum_{j=1}^m z^j g_{\dot{x}}^j(t,u,\dot{u})\right]dt \le 0, \qquad \text{by integration by parts.}$$

That is,

$$\int_a^b (\eta(t,x,u))^T\left\{\sum_{j=1}^m z^j g_x^j(t,u,\dot{u}) - D\left[\sum_{j=1}^m z^j g_{\dot{x}}^j(t,u,\dot{u})\right]\right\}dt \le 0;$$

the integrated term is zero since $\eta(t, x, u) = 0$ at $t = a$ and $t = b$. Now, from (27) we have

$$\int_a^b (\eta(t,x,u))^T\left\{\sum_{i=1}^p y^i[f_x^i(t,u,\dot{u}) + B_i(t)w - vh_x^i(t,u,\dot{u})\right]$$
$$- D\left[\sum_{i=1}^p y^i[f_{\dot{x}}^i(t,u,\dot{u}) - vh_{\dot{x}}^i(t,u,\dot{u})]\right]\Bigg\}dt$$
$$= -\int_a^b (\eta(t,x,u))^T\left\{\sum_{j=1}^m z^j g_x^j(t,u,\dot{u}) - \mathrm{D}\left[\sum_{j=1}^m z^j g_{\dot{x}}^j(t,u,\dot{u})\right]\right\}dt.$$

Therefore,

$$\int_a^b (\eta(t,x,u))^T \left\{ \sum_{i=1}^p y^i [f_x^i(t,u,\dot u) + B_i(t)w - vh_x^i(t,u,\dot u)] - D\left[ \sum_{i=1}^p y^i \{ f_{\dot x}^i(t,u,\dot u) - vh_{\dot x}^i(t,u,\dot u)\} \right] \right\} dt \geq 0.$$

Then

$$\int_a^b \left\{ (\eta(t,x,u))^T \left\{ \sum_{i=1}^p y^i [f_x^i(t,u,\dot u) + B_i(t)w - vh_x^i(t,u,\dot u)] \right\} + (D\eta(t,x,u))^T \left\{ \sum_{i=1}^p y^i [f_{\dot x}^i(t,u,\dot u) - vh_{\dot x}^i(t,u,\dot u)] \right\} \right\} dt \geq 0,$$

using integration by parts, with integrated term zero. Pseudoinvexity of

$$\int_a^b \sum_{i=1}^p y^i \{ f^i(t,\cdot,\cdot) + \cdot^T B_i(t)w - vh^i(t,\cdot,\cdot)\}\, dt$$

gives

$$\int_a^b \sum_{i=1}^p y^i [f^i(t,x,\dot x) + x^T B_i(t)w - vh^i(t,x,\dot x)]\, dt \geq \int_a^b \sum_{i=1}^p y^i [f^i(t,u,\dot u) + u^T B_i(t)w - vh^i(t,u,\dot u)]\, dt.$$

But,

$$x^T B_i(t)w \leq (x^T B_i(t)x)^{1/2} (w^T B_i(t)w)^{1/2}, \qquad \text{for } i = 1, \ldots, p,$$

by the Schwarz inequality

$$\leq (x^T B_i(t)x)^{1/2}, \qquad \text{by (29).}$$

Therefore, $\inf(\mathrm{EP}_v) \geq \sup$ (D-2) by (8). ∎

THEOREM 6 (Strong Duality). *Let $x^*$ be an optimal solution of* ($\mathrm{EP}_V$) *with the normal condition satisfied at $x^*$. If the objective and constraint functionals satisfy the invexity conditions of Theorem* 1; *or if the invexity conditions of Theorem* 2 *are satisfied then there exist $y^*$ and $w^*$ such that $(x^*, y^*, w^*)$ is optimal for* (D-1) *or for* (D-2), *respectively. In either case,* *the objective value of the dual is equal to that of the primal with each of the objective values equal to zero.*

*Proof.* Since $x^*$ is optimal for $(EP_v)$ therefore, by Theorem 1, there exist $y^* \in \mathbb{R}^p$, $z^* \in \mathbb{R}^m$ such that $(x^*, v^*, q^*, y^*, w^*)$ satisfies (10)–(18). From (10), (15), and (17) we see that $(x^*, y^*, z^*) \in C_{(D\text{-}1)}$. Also (11), (12), (15), and (17) yield

$$\begin{aligned}\min q = q^* = 0 &= \sum_{i=1}^{p} y^{i*} \int_a^b [f^i(t, x^*, \dot{x}^*) + x^{*T} B_i(t) w^* - v^* h^i(t, x, \dot{x}^*)] \\ &\quad + \sum_{j=1}^{m} z^{j*} \int_a^b g^j(t, x, \dot{x}^*)\, dt \\ &= \max \left[ \sum_{i=1}^{p} y^i \int_a^b [f^i(t, x, \dot{x}) + x^T B_i(t) x - v h^i(t, x, \dot{x})]\, dt \right. \\ &\quad \left. + \sum_{j=1}^{m} z^j \int_a^b g^j(t, x, \dot{x})\, dt \right].\end{aligned}$$

Then, if the invexity conditions of Theorem 4 are satisfied $(x^*, u^*, q^*, y^*, w^*)$ is optimal for (D-1) with the objective value equal to zero, by weak duality, and if those of Theorem 5 are satisfied $(x^*, v^*, q^*, y + w^*)$ is optimal for (D-2) by weak duality. ∎

We now give strict converse duality results for both dual problems.

THEOREM 7 (Strict Converse Duality). *For*

$$v^* = \min_{x \in C_p} \max_{i \le i \le p} \left[ \int_a^b [f^i(t, x, \dot{x}) + x^T B_i(t) w]\, dt \Big/ \int_a^b h^i(t, x, \dot{x})\, dt \right],$$

*let* $(x^*, q^*)$ *be a normal optimal solution of* $(EP_{V^*})$. *Let* $(u, y, z)$ *be* (D-1)-*optimal. For all feasible solutions of* $(EP_{V^*})$ *and* (D-1) *let*

$$\theta(\cdot) = \int_a^b \sum_{i=1}^{p} y^i [f^i(t, \cdot, \cdot) + \cdot^T B_i(t) w - v^* h^i(t, \cdot, \cdot)]\, dt$$

*be strictly invex and*

$$G(\cdot) = \int_a^b \sum_{j=1}^{m} z^j g^j(t, \cdot, \cdot)\, dt$$

*be invex, both with respect to the same* $\eta$ *as that in Theorem* 4. *Then* $u = x^*$, *i.e.*, $(u, q^*)$ *is* $(EP_{V^*})$*-optimal with each of the objective values equal to zero.*

*Proof.* Suppose that $u \neq x^*$. Since $(x^*, q^*)$ is an optimal solution of $(EP_{V^*})$ and is normal, it follows from Theorem 4 that there exist $y^* \in R^p$, $z^* \in R^m$ such that $(x^*, y^*, z^*)$ is an optimal solution of (D-1). Since $(u, y^*, z^*)$ is also an optimal solution of (D-1), it follows that

$$q^* = 0 = \int_a^b \sum_{i=1}^{p} y^i[f^i(t, u, \dot{u}) + u^T B_i(t)w - v^* h^i(t, u, \dot{u})]\, dt$$

$$+ \int_a^b \sum_{j=1}^{m} z^j g^j(t, u, \dot{u})\, dt$$

$$= \int_a^b \sum_{i=1}^{p} y^{i*}[f^i(t, x^*, \dot{x}^*) + x^{*T} B_i(t)w^* - v^* h^i(t, x^*, \dot{x}^*)]\, dt$$

$$+ \int_a^b \sum_{j=1}^{m} z^{j*} g^j(t, x^*, \dot{x}^*)\, dt.$$

Now, strict invexity of $\theta(\cdot)$ implies

$$\int_a^b \sum_{i=1}^{p} y^i[f^i(t, x^*, \dot{x}^*) + x^{*T} B_i(t)w - v^* h^i(t, x^*, \dot{x})]\, dt$$

$$- \int_a^b \sum_{i=1}^{p} y^i[f^i(t, u, \dot{u}) + u^T B_i(t)w - v^* h^i(t, u, \dot{u})]\, dt$$

$$> \int_a^b \left[ \eta(t, x^*, u) T \left\{ \sum_{i=1}^{p} y_i[f_x^i(t, u, \dot{u}) + B_i(t)w - v^* h_x^i(t, u, \dot{u})] \right\}\right.$$

$$\left. + (D\eta(t, x^*, u))^T \left\{ \sum_{i=1}^{p} y^i[f_{\dot{x}}^i(t, u, \dot{u}) - v^* h_{\dot{x}}^i(t, u, \dot{u})] \right\} \right] dt$$

and invexity of $G(\cdot)$ implies

$$\int_a^b \sum_{j=1}^{m} z^j g^j(t, x^*, \dot{x}^*)\, dt - \int_a^b \sum_{j=1}^{m} z^j g^j(t, u, \dot{u})\, dt$$

$$\geq \int_a^b [\eta(t, x^*, u)^T \left\{ \sum_{j=1}^{m} z^{j*} g_x^j(t, u, \dot{u}) \right\}$$

$$+ (D\eta(t, x^*, u))^T \left\{ \sum_{j=1}^{m} z^{j*} g_{\dot{x}}^j(t, u, \dot{u}) \right\} ]\, dt.$$

Adding these two inequalities gives

$$\int_a^b \left\{ \sum_{i=1}^p y^i[f^i(t,x^*,\dot{x}^*) + x^{*T} B_i(t) w - v^* h^i(t,x^*,\dot{x}^*)]\, dt \right.$$

$$+ \sum_{j=1}^m z^j g^j(t,x^*,\dot{x}^*) - \sum_{i=1}^p y^i[f^i(t,u,\dot{u}) + u^T B_i(t) w$$

$$\left. - v^* h^i(t,u,\dot{u})] - \sum_{j=1}^m z^j g^j(t,u,\dot{u}) \right\} dt$$

$$> \int_a^b [(\eta(t,x^*,u))^T \left\{ \sum_{i=1}^p y^i[f_x^i(t,u,\dot{u}) + B_i(t) w - v^* h_x^i(t,u,\dot{u})] \right.$$

$$\left. + \sum_{j=1}^m z^j g_x^j(t,u,\dot{u}) \right\}$$

$$+ (D\eta(t,x^*,u))^T \left\{ \sum_{i=1}^p y^i[f_{\dot{x}}^i(t,u,\dot{u}) - v^* h_{\dot{x}}^i(t,u,\dot{u})] \right.$$

$$\left. \left. + \sum_{j=1}^m z^j g_{\dot{x}}^j(t,u,\dot{u}) \right\} \right] dt$$

$$= \int_a^b (\eta(t,x^*,u))^T \left\{ \sum_{i=1}^p y^i[f_x^i(t,u,\dot{u}) + B_i(t) w - v^* h_x^i(t,u,\dot{u})] \right.$$

$$\left. + \sum_{j=1}^m z^j g^j(t,u,\dot{u}) \right\} dt$$

$$+ \sum_{i=1}^p y^i[f_{\dot{x}}^i(t,u,\dot{u}) - v^* h_{\dot{x}}^i(t,u,\dot{u})] \eta(t,x^*,u) \Big|_{t=a}^{t=b}$$

$$- \int_a^b (\eta(t,x^*,u))^T D \left[ \sum_{i=1}^p y^{i*} \{ f_{\dot{x}}^i(t,u,\dot{u}) - v^* h_{\dot{x}}^i(t,u,\dot{u}) \} \right.$$

$$\left. + \sum_{j=1}^m z^j g_{\dot{x}}^j(t,u,\dot{u}) \right] dt \qquad \text{by integration by parts}$$

$= 0$ by (21) and $\eta(t,x^*,u) = 0$ at $t = a$ and $b$.

That is,

$$\int_a^b \sum_{i=1}^p y^i[f^i(t,x^*,\dot{x}^*) + x^{*T} B_i(t) w - v^* h^i(t,x^*,\dot{x}^*)]\, dt$$

$$+\int_a^b \sum_{j=1}^m z^j g^j(t, x^*, \dot{x}^*)\, dt$$

$$-\int_a^b \sum_{i=1}^p y^i[f^i(t, u, \dot{u}) + u^T B_i(t) w - v^* h^i(t, u, \dot{u})]\, dt$$

$$-\int_a^b \sum_{j=1}^m z^j g^j(t, u, \dot{u})\, dt > 0,$$

so

$$\int_a^b \sum_{i=1}^p y^i[f^i(t, x^*, \dot{x}^*) + x^{*T} B_i(t) w - v^* h^i(t, x^*, \dot{x}^*)]\, dt$$

$$+\int_a^b \sum_{j=1}^m z^j g^j(t, x^*, \dot{x}^*)\, dt$$

$$> \int_a^b \sum_{i=1}^p y^{i*}[f^i(t, x^*, \dot{x}^*) + x^{*T} B_i(t) w^* - v^* h^i(t, x^*, \dot{x}^*)]\, dt$$

$$+\int_a^b \sum_{j=1}^m z^{j*} g^j(t, x^*, \dot{x}^*)\, dt.$$

This gives

$$\int_a^b \left[\sum_{i=1}^p x^{*T} B_i(t) w + \sum_{j=1}^m z^j g^j(t, x^*, \dot{x}^*)\right] dt$$

$$> \int_a^b \left[\sum_{i=1}^p x^{*T} B_i(t) w^* + \sum_{j=1}^m z^{j*} g^j(t, x^*, \dot{x}^*)\right] dt.$$

But

$$z^{j*} g^j(t, x^*, \dot{x}^*) = 0, \qquad t \in I, j = 1, \ldots, m$$

by Proposition 1, and $z^j g^j(t, x^*, \dot{x}^*) \leq 0$ by (2) and (18). Therefore,

$$\int_a^b \sum_{i=1}^p x^{*T} B_i(t) w\, dt > \int_a^b \sum_{i=1}^p x^{*T} B_i(t) w^*\, dt.$$

Now, Proposition 1 also gives

$$x^{*T} B_i(t) w^* = (x^{*T} B_i(t) x^*)^{1/2}, \qquad \forall i = 1, ..., p.$$

But,

$$\begin{aligned} x^{*T} B_i(t) w &\leq (x^{*T} B_i(t) x^*)^{1/2} (w^T B_i(t) w)^{1/2} && \text{(by Schwarz inequality)} \\ &\leq (x^{*T} B_i(t) x^*)^{1/2}, && \text{by (22).} \end{aligned}$$

This implies

$$\int_a^b \sum_{i=1}^p (x^{*T} B_i(t) x^*)^{1/2}\, dt > \int_a^b \sum_{i=1}^p (x^{*T} B_i(t) x^*)^{1/2}\, dt,$$

a contradiction. Therefore $u = x^*$. ∎

THEOREM 8 (Strict Converse Duality). *For*

$$v^* = \min_{x \in C_p} \max_{1 \leq i \leq p} \left[ \frac{\int_a^b [f^i(t, x, \dot{x}) + x^T B_i(t) w]\, dt}{\int_a^b h^i(t, x, \dot{x})\, dt} \right].$$

*Let* $(x^*, g^*)$ *be a normal optimal solution of* $(\text{EP}_{v^*})$. *Let* $(u, v, z)$ *be* (D-2)-*optimal. For all feasible solutions of* $(\text{EP}_{v^*})$ *and* (D-2) *let* $\theta(\cdot)$ *be SPIX and* $G(\cdot)$ *be QIX, both with respect to the same* $\eta$ *as that in Theorem* 5. *Then* $u = x^*$; *i.e.,* $(u, q^*)$ *is* $(\text{EP}_{v^*})$-*optimal with each of the objective values equal to zero.*

*Proof.* Assume $u \neq x^*$. By Theorem 6, there exist $y^* \in \mathbb{R}^p$, $z^* \in \mathbb{R}^m$, $w^*$ such that $(x^*, y^*, w^*)$ is optimal for (D-2). Thus

$$\begin{aligned} q^* = 0 &= \int_a^b \sum_{i=1}^p y^{i*} [f^i(t, x^*, \dot{x}^*) + x^{*T} B_i(t) w^* - v h^i(t, x^*, \dot{x}^*)]\, dt \\ &= \int_a^b \sum_{i=1}^p y^i [f^i(t, u, \dot{u}) + u^T B_i(t) w - v h^i(t, u, \dot{u})]\, dt \end{aligned}$$

and

$$\int_a^b \sum_{j=1}^m z^j g^j(t, u, \dot{u})\, dt \geq 0$$

and

$$\int_a^b \sum_{j=1}^m z^j g^j(t, x^*, \dot{x}^*) \leq 0.$$

Therefore

$$\int_a^b \sum_{j=1}^m z^j g^j(t, x^*, \dot{x}^*)\, dt \leq \int_a^b \sum_{j=1}^m z^j g^j(t, u, \dot{u})\, dt.$$

by quasi-invexity of $G(\cdot)$, we have

$$\int_a^b \left\{ (\eta(t, x^*, u))^T \sum_{j=1}^m z^j g_x^j(t, u, \dot{u}) + (D\eta(t, x^*, u))^T \times \sum_{j=1}^m z^j g_{\dot{x}}^j(t, u, \dot{u}) \right\} dt \leq 0.$$

Then, using integration by parts, and since $\eta = 0$ at $a$ and $b$

$$\int_a^b \eta(t, x^*, u)^T \left[ \sum_{j=1}^m z^j g_x^j(t, u, \dot{u}) - D\left\{ \sum_{j=1}^m z^j g_{\dot{x}}^j(t, u, \dot{u}) \right\} \right] dt \leq 0.$$

This inequality along with (27) gives

$$\int_a^b \eta(t, x^*, u)^T \left[ \sum_{i=1}^p y^i\{f_x^i(t, u, \dot{u}) + B_i(t)w - vh_x^i(t, u, \dot{u})\} - D\left\{ \sum_{i=1}^p y^i\{f_{\dot{x}}^i(t, u, \dot{u}) - vh_{\dot{x}}^i(t, u, \dot{u})\} \right\} \right] dt \geq 0.$$

Again, using integration by parts,

$$\int_a^b \eta(t, x^*, u)^T \left[ \sum_{i=1}^p y^i\{f_x^i(t, u, \dot{u}) + B_i(t)w - vh_x^i(t, u, \dot{u})\} \right] + (D\eta(t, x^*, u))^T \left[ \sum_{i=1}^p y^i\{f_{\dot{x}}^i(t, u, \dot{u}) - vh_{\dot{x}}^i(t, u, \dot{u})\} \right] dt \geq 0.$$

Strict pseudoinvexity of $\theta(x)$ gives

$$\int_a^b \sum_{i=1}^p y^i[f^i(t, x^*, \dot{x}^*) + x^{*T} B_i(t)w - v^* h^i(t, x^*, \dot{x}^*)]\, dt$$
$$> \int_a^b \sum_{i=1}^p y^i[f^i(t, u, \dot{u}) + u^T B_i(t)w - v h^i(t, u, \dot{u})]\, dt.$$

Then

$$\int_a^b \sum_{i=1}^p y^i[x^{*T} B_i(t)w]\, dt > \int_a^b \sum_{i=1}^p y_i[x^{*T} B_i(t)w^*]\, dt,$$

since

$$\int_a^b \sum_{i=1}^p y^{i*}[f^i(t, x^*, \dot{x}^*) + x^{*T} B_i(t)w^* - v h^i(t, x^*, \dot{x}^*)]\, dt$$
$$= \int_a^b \sum_{i=1}^p y^i[f^i(t, u, \dot{u}) + u^T B_i(t)w - v h^i(t, u, \dot{u})]\, dt.$$

But, by the argument used in the proof of Theorem 7, this yields a contradiction. Hence $u = x^*$. ▌

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