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Generalized Invexity and Duality in Multiple Objective Variational Problems

R. N. MUKHERJEE AND S. K. MISHRA

Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi 221 005, India

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Multiple objective programming problems with the concept of weak minima are extended to multiple objective variational problems. A number of weak, strong, and converse duality theorems are given under a variety of generalized invexity conditions.

1. INTRODUCTION

Duality for multiobjective variational problems has been of much interest in the recent past. Hanson [2] introduced the notion of invexity to mathematical programming. Since that time, it has been shown [3, 4, 6, 8] that many results in variational problems previously established for convex functions actually hold for the wider class of invex functions.

Weir and Mond [9] considered the concept of weak minima and established duality results for multiple objective programming problems. In [9] different scalar duality results are extended to multiple objective programming problems. In this paper, we consider the concept of weak minima in the continuous case and give a complete generalization of the results of Weir and Mond [9] to multiple objective variational problems. Moreover, we relax the generalized convexity conditions to generalized invexity conditions.

2. NOTATION AND PRELIMINARIES

Throughout this paper we will follow the notations of Mond *et al.* [6] and Weir and Mond [9]. We consider the following multiple objective variational primal problem:

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(P) Minimize

$$\int_a^b f(t, x(t), \dot{x}(t))\, dt$$

subject to

$$x(a) = x_0, \qquad x(b) = x_1$$
$$g(t, x(t), \dot{x}(t)) \leq 0,$$

where f: $[a, b] \times R^n \times R^n \to R^p$ and g: $[a, b] \times R^n \times R^n \to R^m$.

For the problem (P), a point x_0 is said to be a weak minimum if there exists no other feasible point x for which

$$\int_a^b f(t, x_0(t), \dot{x}_0(t))\, \mathrm{dt} > \int_a^b f(t, x(t), \dot{x}(t))\, dt.$$

The following continuous versions of Theorems 2.1 and 2.2 of [9] will be needed in the sequel:

THEOREM 1 [9]. *Let* (P) *have a weak minimum at* $x = x_0$. *Then there exist* $\lambda \in R^P$, $y \in R^m$ *such that*

$$\lambda^T f_x(t, x_0(t), \dot{x}_0(t)) + y(t)^T g_x(t, x_0(t), \dot{x}_0(t)) = (d/dt)[\lambda^T f_{\dot{x}}(t, x_0(t), \dot{x}_0(t) + y(t)^T g_{\dot{x}}(t, x_0(t), \dot{x}_0(t))] \tag{1}$$

$$y(t)^T g(t, x_0(t), \dot{x}_0(t)) = 0; \tag{2}$$

$$(\lambda, y) \geq 0. \tag{3}$$

THEOREM 2 [9]. *Let* x_0 *be a weak minimum for* (P) *at which the Kuhn–Tucker constraint qualification is satisfied. Then there exist* $\lambda \in R^P$, $y \in R^m$ *such that*

$$\lambda^T f_x(t, x_0(t), \dot{x}_0(t)) + y(t)^T g_x(t, x_0(t), \dot{x}_0(t)) = (d/dt)[\lambda^T f_{\dot{x}}(t, x_0(t), \dot{x}_0(t)) + y(t)^T g_{\dot{x}}(t, x_0(t), \dot{x}_0(t))] \tag{4}$$

$$y(t)^T g(t, x_0(t), \dot{x}_0(t)) = 0 \tag{5}$$

$$y(t) \geq 0 \tag{6}$$

$$\lambda(t) \geq 0, \lambda^T e = 1, \tag{7}$$

where $e = (1, \ldots, 1) \in R^P$.

3. DUALITY

In relation to the primal problem (P), we consider the following dual problem:

(D) Maximize

$$\int_a^b \{f(t, u(t), \dot{u}(t)) + y(t)^T g(t, u(t), \dot{u}(t))e\}\, dt$$

subject to

$$x(a) = x_0, \qquad x(b) = x_1 \tag{8}$$

$$f_u(t, u(t), \dot{u}(t)) + y(t)^T g_u(t, u(t), \dot{u}(t))e = (d/dt)[f_{\dot{u}}(t, u(t), \dot{u}(t)) + y(t)^T g_{\dot{u}}(t, u(t), \dot{u}(t))e] \tag{9}$$

$$y \geq 0 \tag{10}$$

$$\lambda \in \Lambda, \tag{11}$$

where $\Lambda = \{\lambda \in R^P : \lambda \geq 0, \lambda^T e = 1\}$.

We shall now establish duality theorems:

THEOREM 3 (Weak Duality). *If, for all feasible* (x, u, y, λ),

(a) $\int_a^b \{f(t, \cdot, \cdot) + y(t)^T(t, ., .)e\}\, dt$ *is pseudoinvex; or*

(b) $\int_a^b \{\lambda^T f(t, ., .) + y(t)^T g(t, ., .)\}\, dt$ *is pseudoinvex, then* $\int_a^b f(t, x(t), \dot{x}(t))\, dt \nless \int_a^b \{f(t, u(t), \dot{u}(t)) + y(t)^T g(t, u(t), \dot{u}(t))e\}\, dt$.

Proof. (a). Let x be feasible for (P) and (u, y, λ) feasible for (D). From (9), we have

$$\int_a^b \eta(t, x, u)\{f_u(t, u(t), \dot{u}(t)) + y(t)^T g_u(t, u(t), \dot{u}(t))e\}\, dt = \int_a^b \eta(t, x, u)(d/dt)[f_{\dot{u}}(t, u(t), \dot{u}(t)) + y(t)^T g_{\dot{u}}(t, u(t), \dot{u}(t))e]\, dt.$$

Suppose contrary to the result, i.e.,

$$\int_a^b \{f_i(t, x(t), \dot{x}(t)) + y(t)^T g(t, x(t), \dot{x}(t))\}\, dt < \int_a^b \{f_i(t, u(t), \dot{u}(t)) + y(t)^T g(t, u(t), \dot{u}(t))\}\, dt, \qquad \forall i = 1, 2, \ldots, p.$$

Then by the pseudoinvexity of $\int_a^b \{f(t, \cdot, \cdot) + y(t)^T g(t, ., .)e\}\, dt$, we have

$$\int_a^b \{\eta(t,x,u)[f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t))e] \\ + (d\eta/dt)(t,x,u)[f_{\dot{u}}(t,u(t),\dot{u}(t)) + y(t)^T g_{\dot{u}}(t,u(t),\dot{u}(t))e]\, dt < 0.$$

Now, by integration by parts, we have

$$\int_a^b \eta(t,x,\dot{u})[f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t))e] \\ + [f_{\dot{u}}(t,u(t),\dot{u}(t)) + y(t)^T g_{\dot{u}}(t,u(t),\dot{u}(t))\ \eta(t,x,u)]\Big|_{t=a}^{t=b} \\ - \int_a^b (d/dt)[f_{\dot{u}}(t,u(t),\dot{u}(t)) + y(t)^T g_{\dot{u}}(t,u(t),\dot{u}(t))]\eta(t,x(t),u(t))\, dt < 0.$$

$\because\ \eta(t,x,x) = 0$, we get

$$\int_a^b \eta(t,x,u)[f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t))e] \\ - \int_a^b (d/dt)[f_{\dot{u}}(t,u(t),\dot{u}(t)) + y(t)^T g_{\dot{u}}(t,u(t),\dot{u}(t))]\ \eta(t,x(t),u(t))\, dt < 0.$$

Again by (9) we have

$$\int_a^b \eta(t,x,u)[f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t))e] \\ - \int_a^b \eta(t,x,u)[f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t))\, e] = 0 < 0,$$

which is a contradiction. Thus,

$$\int_a^b f(t,x(t),x(t))\, dt \not< \int_a^b \{f(t,u(t),\dot{u}(t)) + y(t)^T g(t,u(t),\dot{u}(t))e\}\, dt.$$

The proof of part (b) proceeds in a fashion similar to that of the proof of part (a). ∎

THEOREM 4 (Strong Duality). *Let x_0 be a weak minimum for* (P) *at which the Kuhn–Tucker constraint qualification is satisfied. Then there exists* (y,λ) *such that* (x_0, y, λ) *is feasible for* (D) *and the objective values of* (P) *and* (D) *are equal. If, also,*

(a) $\int_a^b \{f(t,.,.) + y(t)^T g(t,.,.)e\}\, dt$ *is pseudoinvex; or*

(b) $\int_a^b \{\lambda^T f(t,.,.) + y(t)^T g(t,.,.)\}\, dt$ *is pseudoinvex,*

then (x_0, y, λ) *is a weak minimum for* (D).

Proof. Since x_0 is a weak minimum for (P) at which the Kuhn–Tucker constraint qualification is satisfied, then by Theorem 2.2 of [9], there exist $y \geq 0$, $\lambda \geq 0$, $\lambda^T e = 1$ such that

$$\lambda^T f_x(t, x_0(t), \dot{x}_0(t)) + y(t)^T g_x(t, x_0(t), \dot{x}_0(t)) \\ = (d/dt)[\lambda^T f_{\dot{x}}(t, x_0(t), \dot{x}_0(t)) + y(t)^T g_{\dot{x}}(t, x_0(t), \dot{x}_0(t))],$$

and $y(t)^T g(t, x_0(t), \dot{x}_0(t)) = 0$. Thus (x_0, y, λ) is feasible for (D) and clearly the objective values of (P) and (D) are equal.

If (x_0, y, λ) is not a weak maximum for (D) then there exists feasible (u^*, y^*, λ^*), for (D) such that

$$\int_a^b \{f_i(t, u^*(t), \dot{u}^*(t)) + y^*(t)^T g(t, u^*(t), \dot{u}^*(t))\}\, dt > \int_a^b \{f_i(t, x_0(t), \dot{x}_0(t)) \\ + y^*(t)^T g(t, x_0(t), \dot{x}_0(t))\}\, dt, \qquad \forall i = 1, 2, ..., p.$$

(a) Since $\int_a^b \{f(t, ., .) + y(t)^T g(t, ., .)e\}\, dt$ is pseudoinvex,

$$\int_a^b \{\eta(t, x_0, u)[f_u^i(t, u^*(t), \dot{u}^*(t)) + y^*(t)^T g_u(t, u^*(t), \dot{u}^*(t))] \\ + (d/dt)\, \eta(t, x_0, u)[f_{\dot{u}}^i(t, u^*(t), \dot{u}^*(t)) + y^*(t)^T g_{\dot{u}}(t, u^*(t), \dot{u}^*(t))]\}\, dt < 0 \\ \forall i = 1, 2, ..., p.$$

Thus,

$$\int_a^b \{\eta(t, x_0, u)[\lambda^*(t)^T f_u(t, u^*(t), \dot{u}^*(t)) + y^*(t)^T g_u(t, u(t), \dot{u}^*(t))] \\ + (d/dt)\, \eta(t, x_0, u)[\lambda^{*T} f_{\dot{u}}(t, u^*(t), \dot{u}^*(t)) + \\ + y^*(t)^T g_{\dot{u}}(t, u^*(t), \dot{u}^*(t))]\}\, dt < 0,$$

contradicting the feasibility of (u^*, y^*, λ^*). *Thus* (x_0, y, λ) *is a weak maximum for* (D). ∎

The proof of part (b) *is similar to that of part* (a).

We now consider the following dual problem in relation to the primal problem (P):

(D1) Maximize

$$\int_a^b f(t, u(t), \dot{u}(t))\, dt$$

subject to

$$x(a) = x_0, \qquad x(b) = x_1; \tag{12}$$

$$\lambda^T f_u(t, u(t), \dot u(t)) + y(t)^T g_u(t, u(t), \dot u(t)) \\ = (d/dt)[\lambda\lambda^T f_{\dot u}(t, u(t), \dot u(t)) + y(t)^T g_{\dot u}(t, u(t), \dot u(t))]; \tag{13}$$

$$y(t)^T g(t, u(t), \dot u(t)) \geq 0; \tag{14}$$

$$y \geq 0; \tag{15}$$

$$\lambda \in \Lambda. \tag{16}$$

THEOREM 5 (Weak Duality). *If, for all feasible* (x, u, y, λ),

(a) $\int_a^b f(t, ., .)\, dt$ *is pseudoinvex and* $\int_a^b y(t)^T g(t, ., .)\, dt$ *is quasi-invex; or*

(b) $\int_a^b \lambda^T f(t, ., .)\, dt$ *is a pseudoinvex and* $\int_a^b y(t)^T g(t, ., .)\, dt$ *is quasi-invex; or*

(c) $\int_a^b f(t, ., .)\, dt$ *is quasi-invex and* $\int_a^b y(t)^T g(t, ., .)\, dt$ *is strictly pseudoinvex; or*

(d) $\int_a^b {}^T f(t, ., .)\, dt$ *is quasi-invex and* $\int_a^b y(t)^T g(t, ., .)\, dt$ *is strictly pseudoinvex,*

then, $\int_a^b f(t, x(t), x(t))\, dt \nless \int_a^b f(t, u(t), u(t))\, dt$.

Proof. (a) Let x be feasible for (P) and (u, y, λ) be feasible for (D1). Suppose $\int_a^b f_i(t, x(t), \dot x(t))\, dt < \int_a^b f_i(t, u(t), \dot u(t))\, dt$, $\forall\, i = 1, 2, ..., p$. By pseudoinvexity of $\int_a^b f_i(t, x(t), \dot x(t))\, dt$, $\forall\, i = 1, 2, ..., p$, we have

$$\int_a^b \{\eta(t, x, u) f_u^i(t, u(t), \dot u(t)) + ((d/dt)\, \eta(t, x, u)) f_{\dot u}^i(t, u(t), \dot u(t))\}\, dt < 0; \\ \forall i = 1, 2, ..., p,$$

and $\because$ $\lambda \geq 0$, we have

$$\int_a^b \{\eta(t, x, u)[\lambda^T f_u(t, u(t), \dot u(t))] + (d/dt)\, \eta(t, x, u)[\lambda^T f_{\dot u}(t, u(t), \dot u(t))\}\, dt < 0. \tag{17}$$

Since $\int_a^b y(t)^T g(t, x(t), \dot x(t)) - \int_a^b y(t)^T g(t, u(t), \dot u(t)) \leq 0$, the quasi-invexity of $\int_a^b y(t)^T g(t, ., .)\, dt$ implies that

$$\int_a^b \{\eta(t, x, u) y(t)^T g_u(t, u(t), \dot u(t)) \\ + ((d/dt)\, \eta(t, x, u)) y(t)^T g_{\dot u}(t, u(t), \dot u(t))\}\, dt \leq 0. \tag{18}$$

Combining (17) and (18) gives

$$\int_a^b \{\eta(t,x,u)[\lambda(t)^T f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t))]\}\,dt$$
$$+ \int_a^b \{((d/dt)\,\eta(t,x,u))[\lambda(t)^T f_{\dot{u}}(t,u(t),\dot{u}(t))$$
$$+ y(t)^T g_{\dot{u}}(t,u(t),\dot{u}(t))\}\,dt < 0.$$

By integration by parts, we have

$$\int_a^b \eta(t,x,u)[\lambda(t)^T f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t))]\,dt$$
$$+ \eta(t,x,u)[\lambda(t)^T f_{\dot{u}}(t,u(t),\dot{u}(t)) + y(t)^T g_{\dot{u}}(t,u(t),\dot{u}(t))]\Big|_{t=a}^{t=b}$$
$$- \int_a^b \{\eta(t,x,u)(d/dt)[\lambda(t)^T f_{\dot{u}}(t,u(t),\dot{u}(t))$$
$$+ y(t)^T g_{\dot{u}}(t,u(t),\dot{u}(t))]\}\,dt < 0.$$

From (13) and $n(t, x, u) = 0$, we have

$$\int_a^b \eta(t,x,u)[\lambda(t)^T f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t))]\,dt$$
$$- \int_a^b \eta(t,x,u)[\lambda(t)^T f_u(t,u(t),\dot{u}(t)) + y(t)^T f_u(t,u(t),\dot{u}(t))]\,dt = 0 < 0,$$

which is a contradiction. Thus,

$$\int_a^b f(t,x(t),\dot{x}(t))\,dt \nless \int_a^b f(t,u(t),\dot{u}(t))\,dt.$$

(b) The proof of part (b) is similar to the proof of part (a).

(c) Since $\int_a^b f$ is quasi-invex

$$\int_a^b [\eta(t,x,u)^T f_u^i(t,u^*(t),\dot{u}^*(t)) + ((d/dt)\,\eta(t,x,u)) f_{\dot{u}}^i(t,u^*(t),\dot{u}^*(t))]\,dt \leq 0,$$
$$\forall i = 1, 2, \ldots, p.$$

Since $\lambda \geq 0$, we have

$$\int_a^b \{\eta(t,x,u)\lambda^*(t)^T f_u(t,u^*(t),\dot{u}^*(t))\}\,dt$$
$$+ \int_a^b ((d/dt)\,\eta(t,x,u))\lambda^{*T} f_{\dot{u}}(t,u^*(t),\dot{u}^*(t))\}\,dt \leq 0,$$

$\because \int_a^b y^*(t)^T g(t, x_0(t), \dot{x}_0(t))\,dt \leq \int_a^b y^*(t)^T g(t, u^*(t), \dot{u}^*(t))\,dt$ and $\int_a^b y^*(t)^T g(t, ., .)\,dt$ is strictly pseudoinvex, we have

$$\int_a^b \{\eta(t, x_0, u^*) y^*(t)^T g_u(t, u^*(t), \dot{u}^*(t)) + ((d/dt)\, \eta(t, x_0, u^*)) y^*(t)^T g_{\dot{u}}(t, u^*(t), \dot{u}^*(t))\}\, dt < 0.$$

$$\int_a^b \{\eta(t, x, u^*)[\lambda^*(t)^T f_u(t, u^*(t), \dot{u}^*(t)) + y^*(t)^T g_u(t, u^*(t), \dot{u}^*(t))] + ((d/dt)\, \eta(t, x, u^*))[\lambda^*(t)^T f_{\dot{u}}(t, u^*(t), \dot{u}^*(t)) + t^*(t)^T g_{\dot{u}}(t, u^*(t), \dot{u}^*(t))]\}\, dt < 0.$$

Now, integration by parts and Eq. (9) give a contradiction as in part (a).

(d) Proof of part (d) is very similar to the proof of part (c). ∎

THEOREM 6 (Strong Duality). *Let x_0 be a weak minimum for* (P) *at which the Kuhn–Tucker constraint qualification is satisfied. Then there exists* (y, λ) *such that* (x_0, y, λ) *is feasible for* (D1) *and the objective values of* (P) *and* (D1) *are equal. If, also,*

(a) $\int_a^b f(t, ., .)\, dt$ *is pseudoinvex and* $\int_a^b y(t)^T g(t, ., .)\, dt$ *is quasi-invex; or*

(b) $\int_a^b (t)^T f(t, ., .)\, dt$ *is pseudoinvex and* $\int_a^b y(t)^T g(t, ., .)\, dt$ *is quasi-invex; or*

(c) $\int_a^b f(t, ., .)\, dt$ *is quasi-invex and* $\int_a^b y(t)^T g(t, ., .)\, dt$ *is strictly pseudoinvex; or*

(d) $\int_a^b \lambda(t)^T f(t, ., .)\, dt$ *is quasi-invex and* $\int_a^b y(t)^T g(t, ., .)$ *is strictly pseudoinvex;*

then (x_0, y, λ) *is a weak maximum for* (D1).

Proof. Since x_0 is a weak minimum for (P) at which the Kuhn–Tucker constraint qualification is satisfied, then by Theorem 2, there exist $y \geq 0$, $\lambda \geq 0$, $\lambda' e = 1$ such that

$$\lambda^T f_x(t, x_0(t), \dot{x}_0(t)) + y(t)^T g_x(t, x_0(t), \dot{x}_0(t)) = (d/dt)[\lambda^T f_{\dot{x}}(t, x_0(t), \dot{x}_0(t)) + y(t)^T g_{\dot{x}}(t, x_0(t), \dot{x}_0(t))]$$

and

$$y(t)^T g(t, x_0(t), \dot{x}_0(t)) = 0.$$

Thus (x_0, y, λ) is feasible for (D1) and clearly the objective values of (P) and (D1) are equal.

If (x_0, y, λ) is not a weak maximum for (D1) there exists a feasible (u^*, y^*, λ^*) for (D1) such that

$$\int_a^b f_i(t, u^*(t), \dot{u}^*(t))\, dt > \int_a^b f_i(t, x_0(t), \dot{x}_0(t))\, dt, \qquad \text{for all } i = 1, 2, ..., p.$$

(a) Since $\int_a^b f(t, ., .)\, dt$ *is pseudoinvex*

$$\int_a^b \{\eta(t, x^\circ, u^*) f_u^i(t, u^*(t), \dot{u}^*(t)) + ((d/dt)\, \eta(t, x^\circ, u^*))\, f_{\dot{u}}^i(t, u^*(t), \dot{u}^*(t))\}\, dt < 0,$$

for all $i = 1, 2, ..., p$. Since $\lambda^* \geq 0$,

$$\int_a^b \{\eta(t, x^\circ, u^*)\lambda^*(t)^T f_u(t, u^*(t), \dot{u}^*(t)) + ((d/dt)\, \eta(t, x^\circ, u^*))\lambda^*(t)^T f_{\dot{u}}(t, u(t), \dot{u}(t))\}\, dt < 0. \tag{19}$$

Note that

$$\int_a^b y^*(t)^T g(t, x_0(t), \dot{x}_0(t))\, dt \leq \int_a^b y^*(t)^T g(t, u^*(t), \dot{u}(t))\, dt$$

and since $\int_a^b y^*(t)^T g(t, ., .)\, dt$ is quasi-invex, we have

$$\int_a^b \{\eta(t, x^\circ, u)\, y^*(t)^T g_u(t, u^*(t), \dot{u}^*(t)) + ((d/dt)\, \eta(t, x^\circ, u))\, y^*(t)^T g_{\dot{u}}(t, u^*(t), \dot{u}^*(t))\}\, dt \leq 0. \tag{20}$$

Combining (19) and (20), we get

$$\int_a^b \{\eta(t, x^\circ, u^*)[\lambda^*(t)^T f_u(t, u^*(t), \dot{u}^*(t)) + y^*(t)^T g_u(t, u^*(t), \dot{u}^*(t))] + ((d/dt)\, \eta(t, x^\circ, u^*))[\lambda^*(t)^T f_{\dot{u}}(t, u^*(t), \dot{u}^*(t)) + y^*(t)^T + y^*(t)^T g_{\dot{u}}(t, u^*(t), \dot{u}^*(t))]\}\, dt < 0,$$

which contradicts the feasibility of (u^*, y^*, λ^*).

(b) The proof of part (b) follows the lines of the proof of part (a).

(c) Since $\int_a^b f(t, ., .)\, dt$ is quasi-invex, we have

$$\int_a^b \{\eta(t, x^\circ, u^*) f_u^i(t, u^*(t), \dot{u}^*(t)) + ((d/dt)\, \eta(t, x^\circ, u^*)) f_{\dot{u}}^i(t, u^*(t), \dot{u}^*(t))\}\, dt \leq 0,$$

for all $i = 1, 2, .., p$. Since $\lambda \geq 0$, we have

$$\int_a^b \{\eta(t, x^\circ, u^*)\eta^*(t)^T f_u(t, u^*(t), \dot{u}^*(t)) \\ + ((d/dt)\, \eta(t, x^\circ, u^*))\lambda^*(t)^T f_{\dot{u}}(t, u^*(t), u^*(t))\}\, dt \leq 0. \tag{21}$$

Since $\int_a^b y^*(t)^T g(t, x_0(t), \dot{u}^*(t))\, dt \leq \int_a^b y^*(t)^T g(t, u^*(t), \dot{u}^*(t))\, dt \geq 0$, $(\because \int_a^b y^{*T} g(t, x_0(t), \dot{x}_0(t))\, dt \leq 0$, $\int_a^b y^*(t)^T g(t, u^*(t), \dot{u}^*(t))\, dt \geq 0$, and $\int_a^b y^*(t)^T g(t, ., .)\, dt$ is strictly pseudoinvex, we have

$$\int_a^b \{\eta(t, x_0, u^*) y^*(t)^T g_u(t, u^*(t), \dot{u}^*(t)) \\ + (d/dt)\, \eta(t, x_0, u^*) y^*(t)^T g_{\dot{u}}(t, u^*(t), \dot{u}^*(t))\}\, dt < 0.) \tag{22}$$

Now combining (21) and (22), we have

$$\int_a^b \{\eta(t, x^\circ, u^*)[\lambda^*(t)^T f_u(t, u^*(t), \dot{u}^*(t)) + y^*(y)^T g_u(t, u^*(t), \dot{u}^*(t))] \\ ((d/dt)\, \eta(t, x_0, u^*))[\lambda^*(t)^T f_{\dot{u}}(t, u^*(t), \dot{u}^*(t))]\}\, dt < 0$$

Now, integration by parts and Eq. (9) give a contradiction, as in part (a).

(d) The proof of part (d) follows the lines of the proof of part (c). ∎

In a similar manner, we now state the continuous form of a general dual for the multiple objective variational optimization problem. We shall consider the case, similar to that of [9], where the primal problem has equality as well as inequality constraints. Consider the problem:

(PE) minimize

$$\int_a^b f(t, x(t), \dot{x}(t))\, dt$$

subject to

$$x(a) = x_0, \qquad x(b) - x_1$$
$$g(t, x(t), x(t)) \leq 0;$$
$$h(t, x(t), x(t)) = 0;$$

where f: $I \times R^n \times R^m \to R^k$, g: $I \times R^n \times R^n \to R^m$, h: $I \times R^n \times R^n \to R^l$ are all differentiable.

Let $M = \{1, 2, ..., m\}$, $L = \{1, 2, ..., 1\}$, $I_\alpha \subseteq M$, $\alpha = 0, 1, ..., \nu$ with $I_\alpha \cap I_\beta = \emptyset$, $\alpha \neq \beta$, and $\cup_{\alpha=0}^{\nu} I_\alpha = M$ and $J_\alpha \subseteq L$, $\alpha = 0, ..., \nu$ with $J_\alpha \cap J_\beta = \emptyset$, $\alpha \neq \beta$, and $\cup_{\alpha=0}^{\nu} J_\alpha = L$.

Note that any particular I_α or J_α may be empty. Thus if M has ν_1 disjoint subsets and L has ν_2 disjoint subsets, $\nu = \text{Max}[\nu_1, \nu_2]$. So, if $\nu_1 > \nu_2$, then J_α, $\alpha > \nu_2$ is empty.

In relation to (PE) consider the problem:

(DE) maximize

$$\int_a^b \{f(t, u(t), \dot{u}(t)) + \sum_{j\in I_0} y_j g_j(t, u(t), \dot{u}(t))e + \sum_{j\in J_0} z_j h_j(t, u(t), \dot{u}(t))e\}\, dt$$

subject to

$$x(a) = x_0, \qquad x(b) = x_1$$

$$\lambda(t)^T f_u(t, u(t), \dot{u}(t)) + y(t)^T g_u(t, u(t), \dot{u}(t)) + z(t)^T h_u(t, u(t), \dot{u}(t)) = (d/dt)[\lambda(t)^T f_{\dot{u}}(t, u(t), \dot{u}(t)) + y(t)^T g_{\dot{u}}(t, u(t), \dot{u}(t)) + z(t)^T h_{\dot{u}}(t, u(t), u(t))],$$

$$\sum_{j\in I_\alpha} y_i g_i(t, u(t), \dot{u}(t)) + \sum_{j\in J_\alpha} z_j h_j(t, u(t), \dot{u}(t)) \geq 0, \qquad \alpha = 1, 2, ..., \nu,$$

$$y \geq 0,$$

$$\lambda \in \Lambda.$$

The following weak and strong duality theorems are stated without proof. They may be established in a manner very similar to that of Theorems 3, 4, 5, and 6.

THEOREM 7 (Weak Duality). *If, for all feasible* (x, u, y, z, λ),

(a) $\int_a^b \{f(t, ., .) + \sum_{i\in I_0} y_i(t) g_i(t, ., .)e^+ \sum_{j\in J_0} z_j(t) h_j(t)e\}\, dt$ *is pseudoconvex and* $\int_a^b \{\sum_{i\in I_0} y_j(t) g_i(t, ., .) + \sum_{j\in J_0} z_j(t) h_j(t, ., .)\}\, dt$, $\alpha = 1, 2, ..., \nu$, *is quasi-invex; or*

(b) $\int_a^b \{\lambda(t)^T f(t, ., .) + \sum_{i\in I_0} y_i(t) g_i(t, ., .) + \sum_{j\in J_0} z_j(t) h_j(t, ., .)\}\, dt$ *is pseudoinvex and* $\int_a^b \sum_{i\in I_\alpha} y_i(t) g_j(t, ., .) + \sum_{j\in J_\alpha} z_j(t) h_j(t, ., .)\}\, dt$, $\alpha = 1, 2, ..., \nu$, *is quasi-invex; or*

(c) $I_0 \neq M$ *and* $J_0 \neq L$, $\int_a^b \{f(t, ., .) + \sum_{i\in I_0} y_i(t) g_i(t, ., .)e + \sum_{j\in J_0} z_j(t) h_j(t, ., .)e\}\, dt$ *is quasi-invex and* $\int_a^b \{\sum_{i\in I_\alpha} y_i(t) g_i(t, ., .) + \sum_{j\in J_\alpha} z_j(t) h_j(t, ., .)\}\, dt$, $\alpha = 1, 2, ..., \nu$, *is strictly pseudoinvex; or*

(d) $I_0 \neq M$ *and* $J_0 \neq L$, $\int_a^b \{\lambda(t)^T f(t, ., .) + \sum_{i\in I_0} y_i(t) g_i(t, ., .) + \sum_{j\in J_0} z_j(t) h_j(t, ., .)\}\, dt$ *is quasi-invex and* $\int_a^b \{\sum_{i\in I_\alpha} y_i(t) g_i(t, ., .) + \sum_{j\in J_\alpha} z_j(t) h_j(t, ., .)\}\, dt$, $\alpha = 1, 2, ..., \nu$, *is strictly pseudoinvex*,

then

$$\int_a^b f(t, x(t), \dot{x}(t)\, dt \nless \int_a^b \{f(t, u(t), \dot{u}(t)) + \sum_{i \in I_0} y_i(t) g_i(t, u(t), \dot{u}(t)) e$$

$$+ \sum_{j \in J_0} z_j(t) h_j(t, u(t), \dot{u}(t))\}\, dt.$$

THEOREM 8 (Strong Duality). *Let X_0 be a weak minimum for* (PE) *at which the Kuhn–Tucker constraint qualification is satisfied. Then there exist (y, z, λ) such that (x_0, y, z, λ) is feasible for* (DE) *and the objective values of* (PE) *and* (DE) *are equal. If also the assumption* (a), (b), (c) *or* (d) *of Theorem 7 is satisfied, then (x_0, y, z, λ) is a weak maximum for* (DE).

4. CONVERSE DUALITY

THEOREM 9. *Let (x_0, y_0, λ_0) be a weak maximum of* (D1). *Assume the $n \times n$ Hessian matrix*

$$\begin{pmatrix} \lambda^T f_{xx} + y^T g_{xx} & \lambda^T f_{xx} + y^T g_{xx} \\ \lambda^T f_{xx} + y^T g_{xx} & \lambda^T f_{xx} + y^T g_{xx} \end{pmatrix} \tag{23}$$

is positive or negative and the vectors $f_x(t, x_0(t), x_0(t)) + (d/dt) f_x(t, x_0(t), x_0(t))$, $i = 1, \ldots, k$, are linearly independent. If for all feasible (x, u, y, λ)

(a) *$\int_a^b f(t, ., .)\, dt$ is pseudoinvex and $\int_a^b y(t)^T g(t, ., .)\, dt$ is quasi-invex; or*

(b) *$\int_a^b (t)^T f(t, ., .)\, dt$ is pseudoinvex and $\int_a^b y(t)^T g(t, ., .)\, dt$ is quasi-invex; or*

(c) *$\int_a^b f(t, ., .)\, dt$ is quasi-invex and $\int_a^b y(t)^T g(t, ., .)\, dt$ is strictly pseudoinvex; or*

(d) *$\int_a^b (t)^T f(t, ., .)\, dt$ is quasi-invex and $\int_a^b y(t)^T g(t, ., .)\, dt$ is strictly pseudoinvex,*

then x_0 is a weak minimum for (P).

Proof. Since (x_0, y_0, λ_0) is a weak maximum for (D), then by Theorem 1 there exist $\tau \in R^p$, $\nu \in R^n$, $p \in R$, $s \in R^m$, $w \in R^p$ such that

$$\begin{aligned} u(t)^T f_x(t, x_0(t), \dot{x}_0(t)) &+ (\partial/\partial x)\lambda(t)^T[\lambda_0(t)^T f_x(t, x_0(t), \dot{x}_0(t)) \\ &+ y_0(t)^T g_x(t, x_0(t), \dot{x}_0(t))] + p y_0(t)^T g_x(t, x_0(t), \dot{x}_0(t)) \\ &= (d/dt)[\lambda(t)^T f_{\dot{x}}(t, x_0(t), \dot{x}_0(t)) + (\partial/\partial \dot{x})\lambda(t)^T[\lambda_0(t)^T f_{\dot{x}}(t, x_0(t), \\ &\quad \dot{x}_0(t)) + y_0(t)^T g_{\dot{x}}(t, x_0(t), \dot{x}_0(t))] + p y_0(t)^T g_{\dot{x}}(t, x_0(t), \dot{x}_0(t)) \end{aligned} \tag{24}$$

$$g_x(t, x_0(t), \dot{x}_0(t))^T \nu + p g(t, x_0(t), \dot{x}_0(t)) + s = 0 \tag{25}$$

$$f_x(t, x_0(t), \dot{x}_0(t))^T \nu + \omega = 0 \tag{26}$$

$$p y_0(t)^T g(t, x_0(t), \dot{x}_0(t)) = 0 \tag{27}$$

$$s^T y_0(t) = 0 \tag{28}$$

$$w^T \lambda_0(t) = 0 \tag{29}$$

$$(\tau, s, p, w) > 0 \tag{30}$$

$$(\tau, \nu, s, p, w) = 0. \tag{31}$$

Since $\lambda_0 \in \Lambda$, (29) gives $w = 0$; (26) then gives

$$\nu(t)^T(t, x_0(t), \dot{x}_0(t)) = 0. \tag{32}$$

Multiplying (25) by y_0 and using (27) and (28) give

$$\nu(t)^T y_0(t)^T g_x(t, x_0(t), \dot{x}_0(t)) = 0. \tag{33}$$

Multiplying (24) by $\nu(t)^T$ and using (32) and (31) give

$$\nu(t)^T \begin{pmatrix} \lambda_0^T f_{xx} + y_0^T g_{xx} & \lambda_0^T f_{\dot{x}x} + y_0^T f_{x\dot{x}} \\ \lambda_0^T f_{x\dot{x}} + y_0^T g_{\dot{x}x} & \lambda_0^T f_{\dot{x}\dot{x}} + y_0^T g_{\dot{x}\dot{x}} \end{pmatrix} \nu = 0.$$

Since (23) is assumed positive or negative definite, $\nu = 0$. Since $\nu = 0$, (24) and the equality constraint (13) of (D1) give

$$(\tau - p\lambda_0(t))^T [f_x(t, x_0(t), \dot{x}_0(t)) + (d/dt) f_{\dot{x}}(t, x_0(t), \dot{x}_0(t))] = 0.$$

By the linear independence of $f_x(t, x_0(t), \dot{x}_0(t)) + (d/dt) f_{\dot{x}}(t, x_0(t), \dot{x}_0)(t))$, $i = 1, 2, \ldots, k$, it follows that

$$\tau = p\lambda_0.$$

Since $\lambda_0 \geq 0$, $\tau \Rightarrow 0 \Rightarrow p \Rightarrow 0$ and then by (25), $s = 0$, giving (, $\nu, s, p, w) = 0$, contradicting (31). Thus, $\tau = 0$ and $p > 0$. Since $\nu = 0$, $p > 0$, and $s \geq 0$, (25) gives $g(t, x_0(t), \dot{x}_0(t)) \leq 0$ and (27) gives $y^T(t) g(t, x_0(t), \dot{x}_0(t)) = 0$. Thus x_0 is feasible for (P). That x_0 is a weak minimum for (P) then follows under assumption (a), (b), (c), or (d) from weak duality, Theorem 5. ∎

As in [9], a more general converse duality result may be established for (PE) and (DE). The proof follows in a fashion similar to that of the proof of Theorem 9.

THEOREM 10. *Let* $(x_0, y_0, z_0, \lambda_0)$ *be a weak maximum of* (DE). *Assume the* $n \times n$ *Hessian matrix*

$$\begin{pmatrix} \lambda_0^T f_{xx} + y_0^T g_{xx} + z_0^T h_{xx} & \lambda_0^T f_{\dot{x}x} + y_0^T g_{x\dot{x}} + z_0^T h_{x\dot{x}} \\ \lambda_0^T f_{x\dot{x}} + y_0^T g_{\dot{x}x} + z_0^T h_{\dot{x}x} & \lambda_0^T f_{\dot{x}\dot{x}} + y_0^T g_{\dot{x}\dot{x}} + z_0^T h_{\dot{x}\dot{x}} \end{pmatrix}$$

is positive or negative definite and that the set

$$\{f_x^i(t, x_0(t), \dot{x}_0(t)) + (d/dt) f_{\dot{x}}^i(t, x_0(t), \dot{x}_0(t)), \qquad i = 1, ..., k,$$
$$\sum_{i \in I} \{(y_0^i g_x^i(t, x_0(t), \dot{x}_0(t)) + (d/dt)(y_0^i g_{\dot{x}}^i(t, x_0(t), \dot{x}_0(t))\}$$
$$+ \sum_{j \in J} \{z_0^j h_x^j(t, x_0(t), \dot{x}_0(t)) + (d/dt)(z_0^j h_{\dot{x}}^j(t, x_0(t), \dot{x}_0(t))\}$$

is linearly independent whenever $I_0 \neq M$ *or* $J_0 \neq L$.

If the assumption (a), (b), (c), *or* (d) *of Theorem* 7 *holds, then* x_0 *is a weak minimum for* (PE).

In the case $I_0 = M$ and $L = \phi$ this result simplifies slightly.

THEOREM 11. *Let* (x_0, y_0, λ_0) *be a weak maximum of* (D). *Assume the* $n \times n$ *Hessian matrix*

$$\begin{pmatrix} \lambda_0^T f_{xx} + y_0^T g_{xx} & \lambda_0^T f_{\dot{x}x} + y_0^T f_{x\dot{x}} \\ \lambda_0^T f_{x\dot{x}} + y_0^T g_{\dot{x}x} & \lambda_0^T f_{\dot{x}\dot{x}} + y_0^T g_{\dot{x}\dot{x}} \end{pmatrix}$$

is positive or negative definite. If the assumption (a) *or* (b) *of Theorem* 3 *holds, then* x_0 *is a weak minimum of* (P).

We now turn our attention to strict converse duality.

THEOREM 12. *Let* x_0 *be a weak minimum for* (P) *and* (x_0, y_0, λ_0) *be a weak maximum for* (D1) *such that*

$$\int_a^b \lambda_0^T f(t, x_0(t), \dot{x}_0(t))\, dt \leq \int_a^b \lambda^T f(t, u_0(t), u_0(t))\, dt.$$

Assume that

(a) $\int_a^b \lambda_0(t)^T f(t, ., .)\, dt$ *is strictly pseudoinvex at* u_0 *and* $\int_a^b y_0(t)^T g(t, ., .)\, dt$ *is quasi-invex at* u_0; *or*

(b) $\int_a^b \lambda_0(t)^T f(t, ., .)\, dt$ *is quasi-invex at* u_0 *and* $\int_a^b y_0(t)^T g(t, ., .)\, dt$ *is strictly pseudoinvex at* u_0;

then $x_0 = u_0$; *that is,* u_0 *is a weak minimum for* (P).

Proof. (a) We assume $x_0 \neq u_0$. Since x_0 and (u_0, y_0, λ_0) are feasible for (P) and (D1), respectively, $\int_a^b y_0(t)^T g(t, x_0, \dot{x}_0(t))\, dt \leq \int_a^b y_0(t)^T g(t, u_0(t), \dot{u}_0(t))\, dt$ and quasi-invexity of $\int_a^b y_0(t)^T g(t, ., .)\, dt$ implies

$$\int_a^b \{\eta(t, x_0, u_0) y_0(t)^T g_{u_0}(t, u_0(t), \dot{u}_0(t)) + ((d/dt)\eta(t, x_0, u_0)) y_0(t)^T g_{\dot{u}_0}(t, u_0(t), \dot{u}_0(t))\}\, dt \leq 0. \quad (34)$$

Since $\int_a^b \lambda_0(t)^T f(t, x_0(t), \dot{x}_0(t))\, dt \leq \int_a^b \lambda_0(t)^T f(t, u_0(t), \dot{u}_0(t))\, dt$ and $\int_a^b \lambda_0(t)^T f(t, ., .)\, dt$ is strictly pseudoinvex, we have

$$\int_a^b \{\eta(t, x^\circ, u^\circ) \lambda_0(t)^T f_{u_0}(t, u_0(t), \dot{u}_0(t)) + ((d/dt)\eta(t, x_0, u_0)) \lambda_0(t)^T f_{\dot{u}_0}(t, u_0(t), \dot{u}_0(t))\}\, dt < 0. \quad (35)$$

Now combining (34) and (35), we have

$$\int_a^b \{\eta(t, x_0, u_0)[\lambda_0(t)^T f_{u_0}(t, u_0(t), \dot{u}_0(t)) + y_0(t)^T g_{u_0}(t, u_0(t), \dot{u}_0(t))] + ((d/dt)\eta(t, x_0, u_0))[\lambda_0(t)^T f_{\dot{u}_0}(t, u_0(t), \dot{u}_0(t)) + y_0(t)^T g_{\dot{u}_0}(t, u_0(t), \dot{u}_0(t))]\} \leq 0.$$

Now the proof follows similarly to that of part (a) of the proof of Theorem 9.

THEOREM 13. *Let* x_0 *be a weak minimum for* (CPE) *and* $(u_0, y_0, z_0, \lambda_0)$ *be a weak maximum for* (CDE) *such that*

$$\int_a^b \lambda_0(t)^T f(t, x_0(t), \dot{x}_0(t)) \leq \int_a^b \{\lambda_0(t)^T f(t, u_0(t), \dot{u}_0(t)) + \sum_{i \in I} y_{0_i}(t) \times g_i(t, u_0(t), \dot{u}_0(t)) + \sum_{j \in J_0} z_{0_j}(t) \times h_j(t, u_0(t), \dot{u}_0(t))\}\, dt.$$

If

(a) $\int_a^b \{\lambda_0(t)^T f(t, ., .) + \Sigma_{i \in I} y_{0_i}(t) g_i(t, ., .) + \Sigma_{j \in J} z_{0_j}(t) h_j(t, ., .)\}\, dt$ *is strictly pseudoinvex at* u_0 *and each* $\int_a^b \{\Sigma_{i \in I_\alpha} y_{0_i}(t) g_i(t, ., .) + \Sigma_{j \in J_\alpha} z_{0_j}(t) h_j(t, ., .)\}\, dt$, $\alpha = 1, 2, \ldots$, *is quasi-invex at* u_0; *or*

(b) $\int_a^b \{\lambda_0(t)^T f(t, ., .) + \sum_{i\in I} y_{0_i}(t) g_i(t, ., .) + \sum_{j\in J_0} z_{0_j}(t) h_j(t, ., .)\}$ *is quasi-invex at* u_0 *each* $\int_a^b \{\sum_{i\in I_\alpha} y_{0_i}(t) g_i(t, ., .) + \sum_{j\in J_\alpha} z_{0_j}(t) h_j(t, ., .)\}\, dt$, $\alpha = 1, 2, \ldots, \nu$, *is strictly pseudoinvex at* u_0,

then $x_0 = u_0$; that is, u_0 is a weak minimum for (P).

COROLLARY 14. *Let* x_0 *be a weak minimum for* (P) *and* (u_0, y_0, λ_0) *be a weak maximum for* (D) *such that*

$$\int_a^b \lambda_0(t)^T f(t, x_0(t), \dot{x}_0(t))\, dt \leq \int_a^b \{\lambda_0(t)^T f(t, u_0(t), \dot{u}_0(t)) + y_0(t)^T g(t, u_0(t), \dot{u}_0(t))\}\, dt.$$

If $\int_a^b \{\lambda_0(t)^T f(t, ., .) + y_0(t)^T g(t, ., .)\}\, dt$ *is strictly pseudoinvex at* u_0, *then* $x_0 = u_0$; *that is,* u_0 *is a weak minimum for* (P).

These strict converse duality results give continuous analogues of the multiple objective scalar programming theorems of Weir and Mond [9].

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