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ON GENERALISED CONVEX MULTI-OBJECTIVE NONSMOOTH PROGRAMMING

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Abstract

We extend the concept of V-pseudo-invexity and V-quasi-invexity of multi-objective programming to the case of nonsmooth multi-objective programming problems. The generalised subgradient Kuhn-Tucker conditions are shown to be sufficient for a weak minimum of a multi-objective programming problem under certain assumptions. Duality results are also obtained.

1. Introduction

In the differentiable case, Jeyakumar and Mond [3] defined a vector invexity that avoids the major difficulty of verifying that the inequality holds for the same function $\eta(\cdot, \cdot)$ for invex functions. Jeyakumar and Mond [3] established sufficient optimality criteria under V-pseudo-invexity and V-quasi-invexity and obtained duality results under these assumptions. This relaxation allows us to treat nonlinear fractional programming problems also. Egudo and Hanson [2] used the concept of Zhao [4] to generalise the concept of V-invexity of Jeyakumar and Mond [3] to the nonsmooth case by replacing the gradients with the gradients of Clarke [1].

In this paper we extend the concept of V-pseudo-invexity and V-quasi-invexity of Jeyakumar and Mond [3] to the nonsmooth case. Further sufficient optimality conditions and duality results have been derived for such nonsmooth multi-objective programming.

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2. Preliminaries

Egudo and Hanson [2] considered the nonlinear multi-objective programming problem:

$$\text{Minimise} \quad (f_i(x); i = 1, 2, \dots, p) \qquad \text{(P)}$$
$$\text{subject to} \quad g_j(x) \leq 0, j = 1, 2, \dots, m$$

where $f_i : R^n \to R$, $i = 1, 2, \dots, p$ and $g_j : R^n \to R$, $j = 1, 2, \dots, m$ are locally Lipschitz functions.

The generalised directional derivative of a Lipschitz function f at x in the direction d denoted by $f^0(x; d)$ (see, for example, Clarke [1]) is :

$$f^0(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} t^{-1} (f(y + td) - f(y)).$$

The Clarke generalised subgradient of f at x is denoted by

$$\partial f(x) = \left\{\xi : f^0(x; d) \geq \xi^T d, \forall d \in R^n\right\}.$$

Egudo and Hanson [2] defined invexity for locally Lipschitz functions as follows. A locally Lipschitz function $f(x)$ is invex on $X_0 \subset R^n$ if for $x, u \in X_0$ there exists a function $\eta(x, u) : X_0 \times X_0 \to R$ such that $f(x) - f(u) \geq \xi^T \eta(x, u), \forall \xi \in \partial f(u)$.

The following example is from [2].

$$f(x) = \begin{cases} 20 - x & \text{if} \quad x \leq -15 \\ 5 - 2x & \text{if} \quad -15 \leq x \leq 0 \\ 5 + 2x & \text{if} \quad 0 \leq x \leq 15 \\ 20 + x & \text{if} \quad x \geq 15. \end{cases}$$

The function $f(x)$ is regular in the sense of Clarke [1] in that $f^0(x; d) = f'(x; d)$, where $f'(x; d)$ is the directional derivative

$$f'(x; d) = \lim_{t \downarrow 0} t^{-1} (f(x + td) - f(x)).$$

It was shown in [2] that $f(x)$ is invex.

A locally Lipschitz $f(x)$ is pseudo-invex on $X_0 \subset R^n$ if for $x, u \in X_0$ there exists a function $\eta(x, u) : X_0 \times X_0 \to R$ such that $\xi^T \eta(x, u) \geq 0 \Rightarrow f(x) \geq f(u)$, $\forall \xi \in \partial f(u)$.

A locally Lipschitz $f(x)$ is quasi-invex on $X_0 \subset R^n$ if for $x, u \in X_0$ there exists a function $\eta(x, u) : X_0 \times X_0 \to R$ such that $f(x) \leq f(u) \Rightarrow \xi^T \eta(x, u) \leq 0$, $\forall \xi \in \partial f(u)$.

It is clear from the definitions that every locally Lipschitz invex function is locally Lipschitz pseudo-invex and locally Lipschitz quasi-invex. Examples can be constructed easily.

3. Generalised invex vector functions

In the differentiable case Jeyakumar and Mond [3] defined vector invexity thus: (P) is said to be V-invex if there exist $\eta : X_0 \times X_0 \to R^n$ and $\alpha_i, \beta_j : X_0 \times X_0 \to R^+\backslash\{0\}$ such that

$$f_i(x) - f_i(u) - \alpha_i(x, u)\nabla f_i(u)\eta(x, u) \geq 0,$$
$$g_j(x) - g_j(u) - \beta_j(x, u)\nabla g_j(u)\eta(x, u) \geq 0.$$

Jeyakumar and Mond [3] further extended V-invexity to V-pseudo-invexity and V-quasi-invexity.

Using the results of Zhao [4], Egudo and Hanson [2] generalised the V-invexity concept of Jeyakumar and Mond [3] to the nonsmooth case by replacing the gradients ∇f_i and ∇g_j with the generalised gradients of Clarke [1]. Hence (P) is said to be V-invex if there exist $\eta : X_0 \times X_0 \to R^n$ and $\alpha_i, \beta_j : X_0 \times X_0 \to R^+\backslash\{0\}$ such that

$$f_i(x) - f_i(u) - \alpha_i(x, u)\xi_i\eta(x, u) \geq 0, \quad \forall \xi_i \in \partial f_i(u),$$
$$g_j(x) - g_j(u) - \beta_j(x, u)\zeta_j\eta(x, u) \geq 0, \quad \forall \zeta_j \in \partial g_j(u).$$

The following example is a V-invex nonsmooth multi-objective programming problem. Consider the multi-objective problem

$$V\text{-minimise} \quad \left(\left|\frac{2x_1 - x_2}{x_1 + x_2}\right|, \frac{x_1 + 2x_2}{x_1 + x_2}\right)$$

subject to $x_1 - x_2 \leq 0$, $1 - x_1 \leq 0$, $1 - x_2 \leq 0$, $\alpha_i(x, u) = 1$ for $i = 1, 2$, $\beta_j(x, u) = (x_1 + x_2)/3$ for $j = 1, 2$ and

$$\eta_i(x, u) = \left(\frac{3(x_1 - 1)}{x_1 + x_2}, \frac{2(x_2 - 2)}{x_1 + x_2}\right)^{\mathrm{T}}.$$

As we can see the generalised directional derivative of $f_1(x) = \left|\frac{2x_1 - x_2}{x_1 + x_2}\right|$ is

$$\begin{aligned} f^0(x; d) &= \limsup_{\substack{y_1 \to x_1 \\ t \downarrow 0}} t^{-1} \left[\left|\frac{2(y_1 + td) - x_2}{y_1 + td + x_2}\right| - \left|\frac{2y_1 - x_2}{y_1 + x_2}\right|\right] \\ &= \limsup_{\substack{y_1 \to x_1 \\ t \downarrow 0}} t^{-1} \left[\frac{3tdx_2}{(y_1 + x_2 + td)(y_1 + x_2)}\right] \quad \left(\text{if } \frac{2x_1 - x_2}{x_1 + x_2} \geq 0\right) \\ &= \frac{3dx_2}{(x_1 + x_2)^2}. \end{aligned}$$

If we take $x_1 = 1$ and $x_2 = 2$ (that is, for an efficient solution $(1, 2)$) then $f^0(x; d) = 2d/3$.

If $y_2 \to x_2$, then $f^0(x; d) = -d/3$. Thus $(2d/3, -d/3) \in \partial f_1(u)$. It is easy to see that $(-2/9, 1/9) \in \partial f_2(u)$. At these particular points we can easily see that the above program is V-invex for the nonsmooth case.

We now extend V-invexity as in Egudo and Hanson [2] to V-pseudo-invexity and V-quasi-invexity.

A vector function $f : X_0 \to R^p$ is said to be V-pseudo-invex if there exist functions $\eta : X_0 \times X_0 \to R^p$ and $\alpha_i : X_0 \times X_0 \to R_+ \backslash \{0\}$ such that for each $x, u \in X_0$,

$$\sum_{i=1}^{p} \xi_i \eta(x, u) \geq 0 \Rightarrow \sum_{i=1}^{p} \alpha_i(x, u) f_i(x) \geq \sum_{i=1}^{p} \alpha_i(x, u) f(u), \quad \forall \xi_i \in \partial f_i(u).$$

The vector function f is said to be V-quasi-invex if there exist functions $\eta : X_0 \times X_0 \to R^p$ and $\beta_i : X_0 \times X_0 \to R_+ \backslash \{0\}$ such that for each $x, u \in X_0$,

$$\sum_{i=1}^{p} \beta_i(x, u) f_i(x) \leq \sum_{i=1}^{p} \beta_i(x, u) f_i(u)$$
$$\Rightarrow \sum_{i=1}^{p} \zeta_i \eta(x, u) \leq 0, \quad \forall \zeta_i \in \partial f_i(u).$$

It is apparent from the definitions that every V-invex function of Egudo and Hanson [2] is V-pseudo-invex and V-quasi-invex as defined above.

Recall from Jeyakumar and Mond [3] that $u \in X_0$ is said to be a (global) weak minimum of a vector function $f : X_0 \to R^p$ if there exists no $x \in X^0$ for which $f_i(x) < f_i(u)$, $i = 1, \ldots, p$.

4. Sufficiency and duality

In this section we show that the subgradient Kuhn-Tucker conditions are sufficient for a weak minimum in (P) when generalised V-invexity is present.

THEOREM 4.1. *Let (u, τ, λ) satisfy the Kuhn-Tucker conditions that*

$$0 \in \sum_{i=1}^{p} \tau_i \partial f_i(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u), \quad \lambda_j g_j(u) = 0, \quad j = 1, 2, \ldots, m,$$
$$\tau_i \geq 0, \quad \tau^T e > 0, \quad y_i \geq 0.$$

If $(\tau_1 f_1, \ldots, \tau_p f_p)$ is V-pseudo-invex and $(\lambda_1 g_1, \ldots, \lambda_m g_m)$ is V-quasi-invex in nonsmooth sense, and u is feasible in (P), then u is a global weak minimum of (P).

PROOF. Since $0 \in \sum_{i=1}^{p} \tau_i \partial f_i(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u)$, there exist $\xi_i \in \partial f_i(u)$ and $\zeta_j \in \partial g_j(u)$ such that

$$\sum_{i=1}^{p} \tau_i \xi_i + \sum_{j=1}^{m} \lambda_j \zeta_j = 0.$$

Suppose that u is not a global weak minimum point. Then, following the lines of proof of Theorem 3.1 of Jeyakumar and Mond [3], the V-pseudo-invexity conditions yield $\sum_{i=1}^{p} \tau_i \xi_i \eta(x_0, u) < 0$. Thus, we have $\sum_{j=1}^{m} \lambda_j \zeta_j \eta(x_0, u) > 0$. Then, V-quasi-invexity yields $\sum_{j=1}^{m} \beta_j(x_0, u)\lambda_j g_j(x_0) > \sum_{j=1}^{m} \beta_j(x_0, u)\lambda_j g_j(u)$. Since x_0 is feasible for (P), that is, $\lambda_j g_j(x_0) \leq 0$, and $\lambda_j g_j(u) = 0$, $j = 1, 2, \ldots, \lambda_j > 0$, $\beta_j > 0$. This contradicts the previous inequality.

For the problem (P), consider a corresponding Mond-Weir dual problem.

$$\text{Maximise} \quad (f_i(u) : i = 1, 2, \ldots, p) \tag{D}$$

$$\text{subject to} \quad 0 \in \sum_{i=1}^{p} \tau_i \partial f_i(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u), \quad \lambda_j g_j(u) \geq 0, \quad j = 1, \ldots, m.$$

$$\tau_i \geq 0, \quad \sum_{i=1}^{p} \tau_i = 1, \quad \lambda_j \geq 0.$$

THEOREM 4.2 (Weak Duality). *Let X be feasible in* (P) *and let* (u, τ, λ) *be feasible in* (D). *If* $(\tau_1 f_1, \ldots, \tau_p f_p)$ *is V-pseudo-invex and* $(\lambda_1 g_1, \ldots, \lambda_m g_m)$ *is V-quasi-invex as in Theorem* 4.1, *then* $(f_1(x), \ldots, f_p(x))^{\mathrm{T}} - (f_1(u), \ldots, f_p(u))^{\mathrm{T}} \notin -\operatorname{int} R_+^p$.

PROOF. From the feasibility conditions, and $\beta_j(x, u) > 0$, we have

$$\sum_{j=1}^{m} \beta_j(x, u)\lambda_j g_j(x) \leq \sum_{j=1}^{m} \beta_j(x, u)\lambda_j g_j(u).$$

Then, by V-quasi-invexity, we have $\sum_{j=1}^{m} \zeta_j \eta(x, n) \leq 0$, $\forall \zeta_j \in \partial g_j(u)$. Since

$$0 \in \sum_{i=1}^{p} \tau_i \partial f_i(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u),$$

there exist $\xi_i \in \partial f_i(u)$ and $\zeta_j \in \partial g_j(u)$ such that $\sum_{i=1}^{p} \tau_i \xi_i + \sum_{j=1}^{m} \lambda_j \zeta_j(u) = 0$. This implies that

$$\sum_{i=1}^{p} \tau_i \xi_i \eta(x, n) + \sum_{j=1}^{m} \lambda_j \zeta_j \eta(x, u) = 0.,$$

Thus,

$$\sum_{i=1}^{p} \tau_i \xi_i \eta(x,u) \geq 0, \quad \forall \xi_i \in \partial f_i(u).$$

The conclusion now follows from the V-pseudo-invexity condition since $\tau e = 1$ and $\alpha(x,u) > 0$.

THEOREM 4.3 (Strong Duality). *Let x^0 be a weak minimum of* (P) *at which a constraint qualification is satisfied. Then there exist $\tau^0 \in R^p$, $\lambda^0 \in R^m$ such that (x^0, τ^0, λ^0) is feasible in* (D). *If weak duality holds between* (P) *and* (D), *then (x^0, τ^0, λ^0) is a weak minimum of* (D).

PROOF. From Kuhn-Tucker necessary conditions (see, for example, Theorem 6.1.3 of Clarke [1]), there exist $\tau \in R^p$, $\lambda \in R^m$ such that

$$0 \in \sum_{i=1}^{p} \tau_i \partial f_i(x^0) + \sum_{j=1}^{m} \lambda_j \partial g_j(x^0),$$

$\tau_i \geq 0$, $\tau \neq 0$, $\lambda_j \geq 0$, $\lambda_j g_j(x^0) = 0$, $j = 1, 2, \ldots, m$. Now since $\tau_i \geq 0$, $\tau \neq 0$ we can scale the τ_i's and λ_j's as

$$\tau_i^0 = \tau_i \Big/ \left(\sum_{i=1}^{p} \tau_i\right) \quad \text{and} \quad \lambda_j^0 = \lambda_j \Big/ \left(\sum_{i=1}^{p} \tau_i\right).$$

Now we have (x^0, τ^0, λ^0) that is feasible in (D).

If (x^0, τ^0, λ^0) is not a weak maximum of (D), then there exists a feasible (u, τ, λ) for (D) such that

$$\left(f_1(u), \ldots, f_p(u)\right)^{\mathrm{T}} - \left(f_1(x^0), \ldots, f_p(x^0)\right)^{\mathrm{T}} \in \operatorname{int} R_+^p.$$

Since x^0 is feasible in (P), this contradicts weak duality (Theorem 4.2).

5. Nonsmooth multi-objective fractional programming

In this section we apply the results of the previous section to study nonsmooth fractional multi-objective problems.

In the differentiable case, Jeyakumar and Mond [3] considered the fractional programming problem,

$$V\text{-minimize} \quad \left(\frac{p_1(x)}{q_1(x)}, \ldots, \frac{p_r(x)}{q_r(x)}\right) \qquad \text{(FI)}$$

subject to $x \in X_0$, $g(x) \leq 0$, where $p_i : X_0 \to R$, $q_i : X_0 \to R$ and $g : X_0 \to R^m$. It is assumed that $p_i(x) \geq 0$, for each x on the feasible set $\Delta = \{x \in X_0 : g(x) \leq 0\}$, $q_i(x) > 0$, for each $x \in \Delta$. The problem (FI) is said to be a V-invex fractional problem if the functions p, q and g satisfy

$$x, u \in \Delta \Rightarrow \begin{cases} p_i(x) - p_i(u) & \geq \gamma_i(x,u) p_i'(u) \eta(x,u) \\ q_i(x) - q_i(u) & \geq \gamma_i(x,u) q_j'(u) \eta(x,u) \\ g_j(x) - g_j(u) & \geq \beta_j(x,u) g_j'(u) \eta(x,u) \end{cases}$$

with $\eta : X_0 \times X_0 \to R^n$, $\gamma_i, \beta_j : X_0 \times X_0 \to R_+ \backslash \{0\}$.

Following Egudo and Hanson [2] we can generalise (FI) to the nonsmooth case by replacing p_i', q_i' and g_j' with the generalised gradients of Clarke. Hence (FI) is said to be V-invex nonsmooth fractional if there exists $\eta : X_0 \times X_0 \to R^n$ and $\gamma_i, \beta_j : X_0 \times X_0 \to R_+ \backslash \{0\}$ such that for all $x, u \in \Delta$

$$\begin{aligned} p_i(x) - p_i(u) &\geq \gamma_i(x,u) \xi_i \eta(x,u), & \forall \xi_i \in \partial p_i(u), \\ q_i(x) - q_i(u) &\leq \gamma_i(x,u) \zeta_i \eta(x,u), & \forall \zeta_i \in \partial q_i(u), \\ g_j(x) - g_j(u) &\geq \beta_j(x,u) \mu_j \eta(x,u), & \forall \mu_j \in \partial g_j(u). \end{aligned} \tag{FI$'$}$$

We need the following proposition from Clarke [1] in order to prove the main Theorem of this section.

PROPOSITION 5.1. (Clarke [1]). *Let f_1, f_2 be Lipschitz near x, and suppose $f_2(x) \neq 0$. Then f_1/f_2 is Lipschitz near x, and*

$$\partial \left(\frac{f_1}{f_2} \right)(x) \subset \frac{f_2(x) \partial f_1(x) - f_1(x) \partial f_2(x)}{(f_2(x))^2}.$$

If in addition $f_1(x) \geq 0$, $f_2(x) > 0$ and if f_1 and $-f_2$ are regular at x, then equality holds and f_1/f_2 is regular at x.

In the next theorem, we assume that p_1 and p_2 are regular.

THEOREM 5.1. *Consider the problem* (FI). *Let $u \in \Delta$. Assume that there exist (τ, λ) such that $\tau \geq 0$, $\tau \neq 0$, $\lambda \geq 0$,*

$$0 \in \sum_{i=1}^{r} \tau_i \partial \left(\frac{p_i}{q_i} \right)(u) + \sum_{j=1}^{m} \lambda_i \partial g_j(u)$$

and $\lambda_j g_j(u) = 0$, $j = 1, 2, \ldots, m$. Then u is a global weak minimum for (FI)$'$.

PROOF. The proof follows the lines of the proof of Theorem 4.1 of Jeyakumar and Mond [3] with appropriate changes in $(p_i/q_i)'$. Proposition 5.1 plays a crucial role in this proof.

For a V-invex nonsmooth multi-objective fractional programming problem (FI)$'$, the weak and strong duality properties hold with the following dual problem:

$$\begin{aligned} V\text{-maximise} \quad & \left(\frac{p_1(u)}{q_1(u)}, \dots, \frac{p_r(u)}{q_r(u)}\right) \\ \text{subject to} \quad & 0 \in \sum_{i=1}^{r} \tau_i \partial \left(\frac{p_i}{q_i}\right)(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u) \\ & \lambda_j g_j \geq 0, \quad 1, 2, \dots, m \\ & \lambda_j \geq 0, \quad \tau \geq 0, \quad \tau e = 1. \end{aligned}$$

6. Conclusion

The Kuhn-Tucker subgradient conditions are shown to be sufficient for a weak minimum of a multi-objective programming problem when generalised invexity (V-pseudo-invexity/V-quasi-invexity) is present. Weak and strong duality theorems have been established. We use the results of Section 4 to extend Egudo and Hanson [2] to the fractional case in Section 5. If $p = 1$, then our result extends the results on invexity used in Zhao [4] for the case of nonsmooth programming to pseudo-invexity and quasi-invexity.

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