JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **208,** 49–57 (1997)
ARTICLE NO. AY975281

Generalized Pseudoconvex Functions and Multiobjective Programming

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Submitted by E. Stanley Lee

Received June 1, 1995

In a recent work, cited in the Introduction, a concept of generalized pseudoconvexity was used to obtain optimality results in nonlinear programming. In the present work we give sufficient optimality conditions, in the context of the multiobjective programming problem under the assumptions of generalized pseudoconvexity on objective and constraint functions. An application of such a result is given for fractional programming also.

1. INTRODUCTION

In a work which appeared in 1969, Tanaka, Fukushima, and Ibaraki [11] introduced generalized pseudoconvexity and connected that with the concept of invexity and arcwise pseudoconvexity. In fact the concept of invexity existed earlier in the literature, e.g., Hanson [6] used it to give Kuhn–Tucker sufficient optimality conditions in nonlinear programming. The concept of arcwise convexity was introduced by Avriel and Zhang [1], and later Singh [10] derived some analytical properties of arcwise convex functions and used them to characterize local global minimum properties for problems in nonlinear programming. In [9] as a further extension, the concept of an arcwise pseudoconvex function was introduced. Subsequently the unification of the three concepts like generalized pseudoconvexity invexity and arcwise pseudoconvexity was done and under certain assumptions their equivalence was shown. In the same work optimality conditions for the nonlinear programming problem were given under the assumption of generalized pseudoconvexity.

0022-247X/97 $25.00

In the present work our aim is to derive sufficient optimality conditions for a multiobjective programming problem under such generalized pseudoconvexity assumptions. Applications are given in the context of fractional programs.

In the organizational set up of the paper we introduce the concepts as cited in the previous paragraph in Section 2. The unification results are mentioned thereafter. A theorem is quoted from [9] which gives equivalence of the concepts and the underlying assumption in such an equivalence result. In Section 3 a multiobjective programming problem is introduced and sufficient optimality results are given. In Section 4 such results are used to consider the multiobjective fractional programming problem. Here a standard reference is made for "efficient solutions" to replace "minimal solutions" (for the single objective problem) for the case of the multiobjective programming problem.

2. PSEUDOCONVEX FUNCTIONS AND THEIR GENERALIZATIONS

Let $f\colon R^n \to R$ be a real valued function of n-variables. For a differentiable function f the following definition is standard for pseudoconvex functions.

DEFINITION 2.1. Let $S \subset R^n$ be a convex open set. Then $f\colon S \to R$ is pseudoconvex on S, if for all $x, y \in S$

$$\nabla f(y)^T (x - y) \geq 0 \qquad \text{implies } f(x) \geq f(y).$$

In case the second inequality is replaced by $f(x) > f(y)$, for all $x \neq y$, $x, y \in S$ then f is said to be strictly pseudoconvex on S. To introduce invexity we give the following preliminaries.

Let $f\colon R^n \to R$ be a locally Lipschitz continuous function, i.e., for each $x \in R^n$ there exists $\delta > 0$ and $c > 0$ such that $\|f(y) - f(z)\| \leq |y - z|$, when ever $\|x = y\| < \delta$ and $\|z - x\| \leq \delta$. f is said to be regular in the sense of Clarke if (i) for all d, there exists the one-sided directional derivative $f'(x; d) = \lim_{t \downarrow 0}[f(y + td) - f(x)]/t$; (ii) for all d, $f'(x; d) = f^0(x; d)$, where $f^0(x; d)$ is the generalized directional derivative defined by

$$f^0(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + td) - f(y)}{t}.$$

It then follows that

$$f'(x; d) = \max\{\xi^T d \mid \xi \in \partial f(x)\}, \qquad \text{for any } x \text{ and } d, \tag{2.1}$$

where $\partial f(\cdot)$ denotes Clarke's generalized gradient [9].

DEFINITION 2.2. A function $f\colon R^n \to R$ is said to be invex if there exists a mapping $\eta\colon R^n \times R^n \to R^n$ such that, for each $x, u \in R^n$,

$$f(x) - f(u) \geq f'(u; \eta(x, u)). \tag{2.2}$$

Note that if f is differentiable then (2.2) reduces to the definition of invexity given in [5]. The result which connects global optimality and invexity is the following.

THEOREM 2.1. *The function f is invex if and only if every point u such that $0 \in \partial f(u)$ is a global minimum of f.*

DEFINITION 2.3. A function f is essentially pseudoconvex if there exists a function $\psi\colon R^n \to R$ and a diffeomorphism $T\colon R^n \to R^n$ such that $f = \psi \circ T$ and for any x and $\bar{x}$

$$\psi'(\bar{z}; z - \bar{z}) \geq 0 \qquad \text{implies that } \psi(z) \geq \psi(\bar{z}), \tag{2.3}$$

where $z = T(x)$ and $\bar{z} = T(\bar{x})$.

For $T = I$ (identity mapping) the above definition reduces to the ordinary definition of pseudoconvexity for directionally differentiable functions. Further if a function is differentiable, then finally the above definition will lead to the classical definition of a pseudoconvex function.

DEFINITION 2.4. For $u, x \in R^n$, a continuous mapping $P_{ux}\colon [0,1] \to R^n$ is called an arc from u to x if

$$P_{ux}(0) = u, \qquad P_{ux}(1) = x.$$

$\dot{P}_{ux}(t)$ exists and is continuous for any $t \in (0,1)$, and $\dot{P}_{ux}(t) \neq 0$ for any t such that $P_{ux}(t)$ is not a global minimum of f. $\dot{P}_{ux}(0)$ is defined as $\lim_{+t \downarrow 0} \dot{P}_{ux}(t)$.

DEFINITION 2.5. The function f is called arcwise pseudoconvex if for each $u, x \in R^n$, there exists an arc $P_{ux}(\cdot)$ such that

$$\begin{aligned} &f'\big(P_{ux}(\bar{t}), \dot{P}_{ux}(\bar{t})\big) \geq 0 \\ &\quad \text{implies } f\big(P_{ux}(t)\big) \geq f\big(P_{ux}(\bar{t})\big) \text{ for any } 0 \leq \bar{t} \leq t < 1. \end{aligned} \tag{2.4}$$

$L_f(\alpha) = \{x\colon f(x) \leq \alpha\}$ stands for the level set of a function $F(\cdot)$, for each $\alpha \in R$.

The following assumption is made in some of the equivalence criteria in the sequel which follows.

Assumption (A). There exists a diffeomorphic mapping $\hat{T}$ such that, for any x_1 and x_2, at least one of the arcs $P_{x_1 x_2}(\cdot)$ satisfying (2.4) is transformed into the line segment from $\hat{T}(x_1)$ to $\hat{T}(x_2)$, i.e., $\{\hat{T}(P_{x_1, x_2}(t)) | 0 \le t \le 1\} = \{(1-s)\hat{T}(x_1) + s\hat{T}(x_2) | 0 \le s \le 1\}$.

We have the equivalent characterization of various generalizations given as follows:

THEOREM 2.2. *Consider the following three statements.*

(a) *f is essentially pseudoconvex*

(b) *f is arcwise pseudoconvex*

(c) *f is invex.*

Then (a) *implies* (b), *and* (b) *implies* (c). *If Assumption* (A) *holds, then* (a) *is equivalent to* (b). *If f is continuously differentiable and has a compact level set $L_f(\alpha)$ for each α and admits a unique minimum then* (b) *is equivalent to* (c).

In the next section we introduce a multiobjective program and study the sufficient optimality conditions for such a program to have an efficient solution.

3. MULTIOBJECTIVE OPTIMIZATION PROBLEM

Consider

$$V - \min\big(f_1(x), f_2(x), \ldots, f_p(x)\big) \tag{3.1}$$

subject to $g_j(x) \le 0$, $j = 1, 2, \ldots, m$, where f_i: $R^n \to R$ $(i = 1, 2, \ldots, p)$ and g_i: $R^n \to R$, $i = 1, 2, \ldots, m$, are locally Lipschitz and regular. For each feasible x, we define the index set $I(x) \triangleq \{i\colon\ g_i(x) = 0\}$. By the optimal solution of (3.1) we mean the efficient solution. Let $L(x_1, \lambda)$ stand for the lagrangian function given as

$$L(x_1, \lambda) = \sum_{i=1}^{p} \tau_i f_i(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

for $\lambda \in R_+^m$ and $\tau_i \ge 0$ and $\sum_{i=1}^{p} \tau_i = 1$. We have the following sufficiency condition for the efficient solution of program (3.1).

THEOREM 3.1. *Let $L(x_1, \lambda)$ be invex for $\tau_i \ge 0$, $\Sigma \tau_i = 1$, and $\lambda \in R_+^m$. Let x^* be such that there exists λ^* and τ^** $(\ge 0, \Sigma \tau_i^* = 1)$

$$0 \in \partial\left(\sum_{i=1}^{p} \tau_i^* f_i(x^*)\right) + \sum_{i=1}^{m} \lambda_i^* \partial g_i(x^*) \tag{3.2}$$

$$\lambda_i^* \ge 0, \tau_i^* \ge 0, \sum \tau_i^* = 1, g_i(x^*) \le 0, \lambda_i^* g_i(x^*) = 0, \tag{3.3}$$

$i = 1, 2, \ldots, m$. *Then x^* is an efficient solution for program* (3.1).

Proof. We have

$$\partial\Big(\sum \tau_i^* f_i(x)\Big) + \sum \lambda_i^* \partial g_i(x^*)$$
$$= \partial\Big(\sum \tau_i^* f_i + \sum \lambda_i^* g_i\Big)(x^*).$$

Since a nonnegative linear combination of regular functions is regular and its generalized gradient is the sum of each generalized gradient, hence from (3.2) and (3.3),

$$\begin{aligned}\sum_{i=1}^{p} \tau_i^* f_i(x^*) &= \sum_{i=1}^{p} \tau_i^* f_i(x^*) + \sum_{i=1}^{m} \lambda_i^* g_i(x^*)\\ &\le \sum_{i=1}^{p} \tau_i^* f_i(x) + \sum_{i=1}^{m} \lambda_i^* g_i(x)\\ &\le \sum_{i=1}^{p} \tau_i^* f_i(x).\end{aligned}$$

Hence from [8, Theorem 2, p. 310] it follows that x^* is an efficient solution of program (3.1).

THEOREM 3.2. *Let $L(f, \lambda)$ be essentially pseudoconvex for $\tau_i \ge 0$, $\Sigma\tau_i = 1$, and $\lambda_i \ge 0$, $i = 1, 2, \dots, m$. Then if there exist τ^* and $\lambda^* \ge 0$ such that* (3.1) *and* (3.2) *are satisfied for x^*, then x^* is an efficient solution for program* (3.1).

Proof. From Theorem 2.2 essential pseudoconvexity implies invexity. Hence the result follows from Theorem 3.1.

The following assumption is made for the sufficient optimality conditions depicted in the ensuing theorem.

Assumption (B). The functions f_1 $(i = 1, 2, \dots, p)$ and g_j $(j = 1, 2, \dots, m)$ are essentially pseudoconvex with respect to a common mapping T.

THEOREM 3.3. *Consider problem* (3.1) *where f_i $(i = 1, 2, \dots, p)$ and g_i $(1, \dots, m)$ are functions satisfying the condition that $\Sigma_{i=1}^{p} \tau_i^* f_i$ and each g_i is essentially pseudoconvex for some $\tau^* \ge 0$ (with $\Sigma_{i=1}^{p} \tau_i^* = 1$). Also the Kuhn–Tucker conditions* (3.2) *and* (3.3) *are satisfied for x^* in the feasibility domain of* (3.1). *Then x^* is an efficient solution for program* (3.1).

Proof. Let x be an arbitrary feasible solution of (3.1). Since $\sum_{i=1}^{p}\tau_i^* f_i + \sum_{i=1}^{m}\lambda_i^* g_i$ is regular we have

$$\left(\sum_{i=1}^{p}\tau_i^* f_i + \sum_{i=1}^{m}\lambda_i^* g_i\right)(x^*; x - x^*)$$
$$= \max\left\{\xi^T(x - x^*) \,|\, \xi \in \partial\left(\sum_{i=1}^{p}\tau_i^* f_i(x^*)\right)\right.$$
$$\left. + \sum_{i=1}^{m}\lambda_i^* \partial g_i(x^*)\right\}. \tag{3.4}$$

However, since $0 \in \partial(\sum_{i=1}^{p}\tau_i^* f_i(x^*))$

$$+ \sum_{i=1}^{m}\lambda_i^* \partial g_i(x^*)$$

it follows from (3.4) that there exists $\xi_0^* \in \partial(\sum \tau_i^* f_i(x^*))$ and $\xi_i^* \in \partial g_i(x^*)$, $i = 1, \ldots, m$, such that

$$\left(\xi_0^* + \sum_{i=1}^{m}\lambda_i \xi_i^*\right)(x - x^*) \geq 0. \tag{3.5}$$

On the other hand since g_i is regular for each i,

$$g_i'(x^*; x - x^*) = \max\left\{\xi_i^T(x - x^*) \,|\, \xi_i \in \partial g_i(x^*)\right\}. \tag{3.6}$$

If $g_i(x^*) = 0$, then the feasibility of x implies $g_i(x) \leq g_i(x^*) = 0$. Since g_i is essentially pseudoconvex, then it can be seen that

$$g_i'(x^*; x - x) \leq 0. \tag{3.7}$$

Thus, if $g_i(x^*) = 0$ then it follows from (3.6) and (3.7) that

$$\xi_i^{*T}(x - x) \leq 0. \tag{3.8a}$$

However, $\lambda_i^* \geq 0$ for all i, and in particular $\lambda_i^* = 0$ whenever $g_i^*(x^*) < 0$. Therefore by (3.5) and (3.8a) we must have

$$\xi_0^{*T}(x - x^*) \geq 0. \tag{3.8b}$$

Now from the definition of $\sum_{i=1}^{p}\tau_i f_i'(x^*; x - x)$ it follows that

$$\sum_{i=1}^{p}\tau_i f_i'(x^*, x - x^*) \geq 0.$$

Also from Definition 2.3, it follows that

$$\sum_{i=1}^{p} \tau_i f_i(x) \geq \sum_{i=1}^{p} \tau_i f_i(x^*).$$

Since x was arbitrary it follows that x^* is an efficient solution of (3.1).

For the general case when $T \neq I$, we proceed as follows.

The following lemma is proved in [9].

LEMMA 3.4. *Under Assumption* (B), *i.e.*, f_i *and* g_i *are essentially pseudoconvex with respect to a common mapping* T, *the feasible region of* (3.1) *is connected.*

We have the following theorem.

THEOREM 3.5. *Suppose Assumption* (B) *is satisfied. Then if for* x^* *in the feasible domain of* (3.1) *the Kuhn–Tucker conditions are satisfied then* x^* *is an efficient solution of* (3.1).

Proof. Let ψ_0^i, ψ_j $(i = 1, \dots, p;\ j = 1, \dots, m)$ be such that $f_i = \psi_0^i \circ T$ $(i = 1, \dots, p)$ and $g_i = \psi_i \circ T$ $(j = 1, \dots, m)$. Then consider the program

$$\begin{aligned} &\text{minimize } \sum_{i=1}^{p} \tau_i^* \psi_0^i(z) \\ &\text{subject to } \psi_i(z) \leq 0, \qquad i = 1, 2, \dots, m, \end{aligned} \tag{3.9}$$

and let $z^* = T(x^*)$. Since $\nabla T(x)$ is nonsingular, conditions (3.2) hold if and only if

$$0 \in \partial\left(\sum_{i=1}^{p} \tau_i^* \psi_0^i(z^*) \right) + \sum_{j=1}^{m} \lambda_i^* \psi_i(z^*)$$

$$\tau_i^* \geq 0,\ \sum \tau_i^* = 1,\ \psi_i(z^*) \leq 0,\ \lambda_i^* \psi_i(z^*) = 0;\ i = 1, 2, \dots, m.$$

Then by the proof of Theorem 3.4 it follows that z^* is efficient for the corresponding multiobjective program with ψ-functions. From which it easily follows that x^* is an efficient solution of multiobjective program (3.1). Finally observe that the minimum set of (3.9) is convex from the pseudoconvexity of ψ_0^i and ψ_j $(j = 1, 2, \dots, m)$. In the same manner in which Lemma 3.4 is derived it can now be shown that the efficiency set of (3.1) is connected (via the connectedness of the minimum set of the scalarized program).

4. APPLICATIONS: FRACTIONAL PROGRAMMING

We consider a multiobjective program with fractional objectives of the type

$$V\text{-minimize}\left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \dots, \frac{f_p(x)}{g_p(x)}\right) \tag{4.1}$$

subject to $h(x) \leq 0$, where $f_i\colon R^n \to R$ $(i = 1, 2, \dots, p)$ and $g_i\colon R^n \to R$ $(i = 1, 2, \dots, p)$ are real valued locally Lipschitz functions and $h\colon R^n \to R^m$ is a vector valued locally Lipschitz function over R^n. Also f_i, g_i and h are regular. The usual assumption is that $g_i > 0$ $(i = 1, 2, \dots, p)$. One can see that (4.1) is equivalent to a parametric program of the type

$$V\text{-minimize}\left(f_1(x) - v_1^* g_1(x) \dots f_p(x) - v_p^* g_p(x)\right) \tag{4.2}$$

subject to $h(x) \leq 0$, where $(v_1^*, v_2^*, \dots, v_p^*)$ is some preassigned parameter vector (see [1]).

Under the assumption as in Section 3 for a Lagrangian

$$L(x_1 \lambda) = \sum \tau_i\left(f_i(x) - v_i^* g_i(x)\right) + \lambda^T h, \quad \text{for } \lambda \in R_+^m \tau_i \geq 0, \ \sum \tau_i = 1,$$

we can give sufficient optimality conditions for efficiency points of program (4.2). Further, Assumption (B) will lead to sufficient optimality conditions like Theorem 3.4 and optimality conditions for the parametric program (4.2), which in turn gives appropriate optimality conditions for program (4.1) via equivalence of (4.1) and (4.2).

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